2.6 37 \((3x+y)dx + x\,dy = 0\)

\text{rewrite as } y' = -\frac{(3x+y)}{x} = -3 - \frac{y}{x}

\text{now, let } v = \frac{y}{x}, \quad y = x\cdot v

y' = xv' + v

so becomes \(xv' + v = -3 - v\)

\(xv' = -3 - 2v\)

\(\int \frac{1}{3+2v}\,dv = \int \frac{-1}{x}\,dx\)
\[ \frac{1}{2} \ln |3+2v| = -\ln |x| + C \]

\[ \sqrt{3+2v} = k \cdot x \]

\[ 13 + 2v = \frac{M}{x^2} \]

\[ 2v = \frac{D}{x^2} - 3 \]

\[ v = \frac{B}{x^2} - \frac{3}{2} \]

So \[ y = x \cdot v = \frac{B}{x} - \frac{3}{2} x \]
Recall: a function $f(x,y)$ which is built from $x$s and $y$s using polynomial, rational, trig, inverse trig, exponential, log, root functions, and operations $+, -, \cdot, \div, o$, is continuous on its domain.

**Example:** $f(x,y) = \frac{\ln(x+y)}{x-3}$ is continuous on the region $\{(x,y) \mid x \neq 3 \text{ and } x+y > 0\}$. 
**Theorem 7.6** Existence of solutions

Suppose the function \( f(t, x) \) is defined and continuous on the rectangle \( R \) in the \( tx \)-plane. Then given any point \((t_0, x_0) \in R\), the initial value problem

\[
x' = f(t, x) \quad \text{and} \quad x(t_0) = x_0
\]

has a solution \( x(t) \) defined in an interval containing \( t_0 \). Furthermore, the solution will be defined at least until the solution curve \( t \to (t, x(t)) \) leaves the rectangle \( R \).

\[\text{Ex. 2.7 (5)} \quad \text{Given: } x' = \frac{t}{x+1}, \quad x(0) = 0.\]
The \( f(t,x) \) in Thm 7.6 is
\[
f(t,x) = \frac{t}{x+1}.
\]
This is continuous at all \((t,x)\) except those where \(x = -1\):

Since \( f \) is cont. on \(-\infty < t < \infty, -1 < x < \infty\), and \((0,0)\) is in this rect., This FVP \( A \) is guaranteed to have a solution.
Suppose \( f(t, x) \) and \( \frac{df}{dx} \) are both cont. on a rectangle \( R \) (in the tx plane), and let
\[
M = \max_{(t, x) \in R} \left| \frac{df}{dx} (t, x) \right|.
\]
If \( (t_0, x_0) \) \& \( (t_0, y_0) \) are both in \( R \), and
\[
x'(t) = f(t, x(t)) \quad \text{and} \quad x(t_0) = x_0
\]
\[
y'(t) = f(t, y(t)) \quad \text{and} \quad y(t_0) = y_0,
\]
then as long as \( (t, x(t)) \neq (t, y(t)) \)
stay in $\mathbb{R}$,

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}$$

$$s, \frac{df}{dx}$$

cont.

$$M = \max_{\text{over } \mathbb{R}} \left| \frac{df}{dx} \right|$$

As a consequence of 7.15, with $y_0 = x_0$, we get:
Thm 7.16: If $f$ and $\frac{df}{dx}$ are both cont. in a rectangle $R$, and $(t_0, x_0) \in R$, then the IVP

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a unique solution which remains unique as long it stays in $R$.

Back to 2.7 (5): $x' = \frac{t}{x+1}$, $x(0) = 0$.

$$\frac{df}{dx} = \frac{d}{dx} \left( \frac{t}{x+1} \right) = \frac{-t}{(x+1)^2}.$$
Note that \( f \) and \( f_x \) are both cont. here:

So, this \( \neq \) VP (with \( x(0)=0 \)) will have a unique solution, for as long as that solution stays off of \( x=-1 \).