The integral on the left contains the expression $y'(t)\,dt$. This is inviting us to change the variable of integration to $y$, since when we do that, we use the equation $dy = y'(t)\,dt$. Making the change of variables leads to

$$\int h(y)\,dy = \int g(t)\,dt. \quad (2.37)$$

Notice the similarity between (2.36) and (2.37). Equation (2.36), which has no meaning by itself, acquires a precise meaning when both sides are integrated. Since this is precisely the next step that we take when solving separable equations, we can be sure that our method is valid.

We mention in closing that the objects in (2.36), $h(y)\,dy$ and $g(t)\,dt$, can be given meaning as formal objects that can be integrated. They are called differential forms, and the special cases like $dy$ and $dt$ are called differentials. The basic formula connecting differentials $dy$ and $dt$ when $y$ is a function of $t$ is

$$dy = y'(t)\,dt,$$

the change-of-variables formula in integration. These techniques will assume greater importance in Section 2.6, where we will deal with exact equations. The use of differential forms is very important in the study of the calculus of functions of several variables and especially in applications to geometry and to parts of physics.

**EXERCISES**

In Exercises 1–12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

1. $y' = xy$
2. $x^2 y' = 2y$
3. $y' = e^{x-y}$
4. $y' = (1 + y^2)e^x$
5. $y' = xy + y$
6. $y' = ye^x - 2e^x + y - 2$
7. $y' = x / (y + 2)$
8. $y' = xy / (x - 1)$
9. $x^2 y' = y \ln y - y'$
10. $x y' - y = 2x^2 y$
11. $y^2 y' = x + 2y$
12. $y' = (2xy + 2x) / (x^2 - 1)$

In Exercises 13–18, find the exact solution of the initial value problem. Indicate the interval of existence.

13. $y' = y/x$, $y(1) = -2$
14. $y' = -2t(1 + y^2) / y$, $y(0) = 1$
15. $y' = (\sin x) / y$, $y(\pi/2) = 1$
16. $y' = e^{x+y}$, $y(0) = 0$
17. $y' = (1 + y^2)$, $y(0) = 1$
18. $y' = x / (1 + 2y)$, $y(-1) = 0$

In Exercises 19–22, find exact solutions for each given initial condition. State the interval of existence in each case. Plot each exact solution on the interval of existence. Use a numerical solver to duplicate the solution curve for each initial value problem.

19. $y' = x/y$, $y(0) = 1$, $y(0) = -1$
20. $y' = -x/y$, $y(0) = 2$, $y(0) = -2$
21. $y' = y - y$, $y(0) = 3$, $y(0) = 1$
22. $y' = (y^2 + 1) / y$, $y(1) = 2$
23. Suppose that a radioactive substance decays according to the model $N' = N_0 e^{-t}$. Show that the half-life of the radioactive substance is given by the equation

$$T_{1/2} = \frac{\ln 2}{\lambda}. \quad (2.38)$$

24. The half-life of $^{238}\text{U}$ is $4.47 \times 10^7$ yr.

(a) Use equation (2.38) to compute the decay constant $\lambda$ for $^{238}\text{U}$.

(b) Suppose that 1000 mg of $^{238}\text{U}$ are present initially. Use the equation $N = N_0 e^{-t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.

25. Tritium, $^3\text{H}$, is an isotope of hydrogen that is sometimes used as a biochemical tracer. Suppose that 100 mg of $^3\text{H}$ decays to 80 mg in 4 hours. Determine the half-life of $^3\text{H}$.

26. The isotope Technetium 99m is used in medical imaging. It has a half-life of about 6 hours, a useful feature for radioisotopes that are injected into humans. The Technetium, having such a short half-life, is created artificially on scene by harvesting from a more stable isotope, $^{99}\text{Mo}$. If 10 g of $^{99m}\text{Tc}$ are “harvested” from the Molybdenum, how much of this sample remains after 9 hours?

27. The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 500 mg of $^{131}\text{I}$, how much of the isotope will be left after 20 days?