HOMOGENIZATION OF A VISCOELASTIC MATRIX IN LINEAR FRICTIONAL CONTACT

ROBERT P. GILBERT, ALEXANDER PANCHENKO, AND XUMING XIE

ABSTRACT. The paper is devoted to study of acoustic wave propagation in a partially consolidated composite material containing loose particles. Friction of particles against the consolidated part of the material causes mechanical energy dissipation. This situation is modelled by assuming that the medium has a periodic microstructure changing rapidly on the small scale \( \varepsilon \). Each of the periodic microscopic cells is composed of a viscoelastic matrix containing a rigid particle in frictional contact with the matrix. We use the methods of two-scale convergence to obtain effective acoustic equations for the homogenized material. The effective equations are history-dependent and contain the body force term, reminiscent of the well known Stokes drag force.

1. INTRODUCTION

We consider a model for the propagation of acoustic waves in a partially consolidated material, where the material properties change rapidly on a small scale characterized by a parameter \( \varepsilon \). Our model is suggested by a work of Buckingham [4, 5, 6] where the dissipation of acoustic energy in an unconsolidated material is caused primarily by the rubbing of grains against one another.

Some mathematical results on frictional contact can be found in [10, 18, 17, 11]. The authors of these articles study the deformation of a body coming into frictional contact with an absolutely rigid foundation. It is assumed that the material properties of the body are given by the Kelvin-Voight viscoelastic constitutive equations [18, 17]. Moreover, the friction is assumed to be dry, and the friction law is given by a version of Coulomb’s law

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known as normal compliance. The contact conditions of Coulomb type are formulated as
equalities involving tangential and normal forces on the contact surface.

Unfortunately, the results mentioned cannot be directly extended to the case of dry
friction between two elastic bodies. Analysis of the lubricated friction presents an even
more challenging problem. Consequently, we consider a simplified problem where a partially
consolidated medium is modeled as a poroelastic material that consists of a connected
viscoelastic matrix whose pores contain rigid particles. Particles come into frictional contact
with the matrix, but particle-particle contacts are prohibited. A further simplification is
the assumption that the material has a periodic structure.

The microscale problem with Coulomb type contact conditions is formulated as a vari-
ational inequality. Averaging this inequality would probably lead to another variational
inequality or even more general inclusion. In order to guarantee that the effective model
has the form of an equation, in [14] we approximated Coulomb contact conditions by non-
linear equations. Using Taylor expansions, we constructed a family of microscopic models
of increasing complexity, and averaged these models using formal two-scale asymptotic ex-
pansions and homogenization.

In this paper, we present a rigorous analysis of the simplest model from [14] that corre-
sponds to completely linearized contact conditions. The method of two-scale convergence is
used to derive the effective equations. We show that the resulting effective equations have
not only memory terms but also the so-called drag force terms which were present in our
work [14]. The drag force is the macroscopic body force, reminiscent of the classical Stokes
drag force.

Homogenization in acoustics of composite materials with consolidated solid phase was
addressed in [7], [21] by using formal expansions, and rigorously in [12], [15] for periodic
geometry, and in [13] for more general irregular geometries. However, this paper seems to
be first where acoustic homogenization of a non-consolidated material is treated rigorously.
The paper is organized as follows: In section 2, we describe the geometry of the medium and formulate the microscale problem (problem 1). Section 3 provides the proof of existence and uniqueness of solution to problem 1 using the standard Galerkin’s method. In section 4, we use the method of two scale convergence to derive the effective equation whose coefficients contain solutions of several cell problems. In section 5, we prove existence and uniqueness of solutions to these cell problems.

2. Formulation of Problem 1

Let $U$ be a bounded domain in $\mathbb{R}^n$ containing a large number of identical, periodically arranged cells. First we define the unit cell $Y = [0,1]^n$. Let $Y_p$ (rigid particle part) be a closed subset of $Y$ and $Y_s = Y/Y_p$ be the viscoelastic solid part. Let $\Gamma$ be the interface of $Y_s$ and $Y_r$, assumed to be smooth. For any set $D \subset \mathbb{R}^n$, define $eD = \{ x : e^{-1}x \in D \}$, and $D^k = D + k$ for $k \in \mathbb{Z}^n$. Next, define

$$U_e = \bigcup \{ eY^k \cup eY_s \}$$

$$\Gamma_e = \bigcup \{ e\Gamma^k \cup \Gamma \}$$

Now we describe the problem that we are interested in. Let $u^e(t,x)$ be the displacements of the matrix, and $v^e = \dot{u}^e = \partial_t u$ be the velocity vector. The equation of motion is given by

$$\ddot{v} = \text{div} \sigma^e + f \text{ in } U_e$$

We assume that the stress tensor $\sigma^e$ satisfies Kelvin-Voight constitutive equation of linear visco-elasticity:

$$\sigma^e = A^e e(u^e) + B^e e(v^e)$$

where

$$e(u)_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$
The coefficients in the constitutive equations are

\[ A^\varepsilon(x) = (A_{ijkl}(x, x / \varepsilon)), \quad B^\varepsilon(x) = (B_{ijkl}(x, x / \varepsilon)). \]

We assume that \( A(x, y) \) and \( B(x, y) \) are smooth, periodic in \( y \) and satisfy the usual symmetry

\begin{align}
A_{ijkl} &= A_{ijlk} = A_{jikl} = A_{klij}, \\
B_{ijkl} &= B_{ijlk} = B_{jikl} = B_{klij},
\end{align}

and ellipticity conditions

\begin{align}
A_{ijkl} h_{ij} h_{kl} &\geq \lambda_1 h_{ij} h_{ij}, \\
B_{ijkl} h_{ij} h_{kl} &\geq \lambda_2 h_{ij} h_{ij},
\end{align}

where \( \lambda_1 \) and \( \lambda_2 \) are positive constants independent of \( \varepsilon \).

On the boundary \( \partial U \), we assume a homogeneous Dirichlet condition

\[ u^\varepsilon|_{\partial U} = 0 \]

To describe the contact and friction condition on \( \Gamma^\varepsilon \), we introduce the following notation.

Let \( \mathbf{n} = (n_1, n_2, \ldots, n_n) \) be the unit normal on \( \Gamma \) pointing into \( Y_p \), we denote the normal and tangential components of the displacement as

\[ u_n = u \cdot \mathbf{n}, \quad u_T = u - u_n \mathbf{n}, \]

and the normal and tangential components of the stress tensor as

\[ \sigma_n = \sigma_{ij} n_i n_j, \quad \sigma_T = \sigma \cdot \mathbf{n} - \sigma_n \mathbf{n}. \]

We impose the following linear contact and friction condition on \( \Gamma_\varepsilon \):

\begin{align}
\sigma_n^\varepsilon &= -\varepsilon g_\varepsilon(x) - \varepsilon h_\varepsilon(x) u_n^\varepsilon, \\
\sigma_T^\varepsilon &= -\varepsilon v_T^\varepsilon,
\end{align}
where \( g_\epsilon(x) = g\left(\frac{x}{\epsilon}\right), h_\epsilon(x) = h\left(\frac{x}{\epsilon}\right) \). We assume that \( g(y) \) and \( h(y) \) are smooth, positive and periodic in \( y \).

The initial conditions are

\[
(2.10) \quad \mathbf{v}^\epsilon(x, 0) = 0, \quad \mathbf{u}^\epsilon(x, 0) = 0.
\]

In order to give the variational formulation of the problem, let us introduce the following Banach spaces:

\[
(2.11) \quad H_\epsilon = \{ \mathbf{u} \in [H^1(U_\epsilon)]^n, \mathbf{u}|_{\partial U} = 0 \}, \quad V = [L^2(U_\epsilon)]^n
\]

The variational form of the problem \((2.1) (2.8) \) and \((2.9) \) can then be expressed as follows:

**Problem 1:** Find \( \mathbf{u}^\epsilon \in L^\infty([0, T], H_\epsilon), \partial_t \mathbf{u} \in L^\infty([0, T], V) \cap L^2([0, T], H_\epsilon), \partial_{tt} \mathbf{u} \in L^\infty([0, T], H'_\epsilon) \) so that \( \mathbf{u}^\epsilon|_{t=0} = 0, \partial_t \mathbf{u}^\epsilon|_{t=0} = 0 \) and \( \mathbf{u} \) satisfies

\[
(2.12) \quad \int_{U_\epsilon} \partial_{tt} \mathbf{u}^\epsilon \cdot \phi + \int_{U_\epsilon} [Ae(\mathbf{u}^\epsilon)] : e(\phi) + \int_{U_\epsilon} [Be(\partial_t \mathbf{u}^\epsilon)] : e(\phi) = \int_{\Gamma_\epsilon} \mathbf{e} \cdot \mathbf{n} \cdot \phi + \int_{\Gamma_\epsilon} e_{\partial_t \mathbf{u}^\epsilon} \cdot \mathbf{n} \cdot \phi + \int_{\Gamma_\epsilon} e_{\partial \mathbf{u}^\epsilon} \cdot \mathbf{n} \cdot \phi + \int_{\Gamma_\epsilon} e_{\partial \mathbf{u}^\epsilon} \cdot \mathbf{n} \cdot \phi
\]

where we assume \( \mathbf{f} \in L^2([0, T], [L^2(U)]^n) \).

3. **Existence and uniqueness of a solution to Problem 1**

We shall use Galerkin’s method to demonstrate that a solution to Problem 1 exists for each \( \epsilon > 0 \). For simplicity of notation, we suppress \( \epsilon \) dependence in this section. Since \( H_\epsilon \) is separable, there exists a complete system of functions \( \{ \phi^k(x) \} \) in \( H_\epsilon \). For each \( m \), we shall look for approximate solutions \( \mathbf{u}_m(x, t) \) of the form

\[
(3.1) \quad \mathbf{u}_m(x, t) = \sum_{i=1}^{m} g_{tm}(t) \phi^i(x).
\]
Substituting this expression into (2.12) we determine the \( g_{im}(t) \)

\[
\sum_{l=1}^{m} \left( \int_{U_{e}} \phi^{l} \cdot \phi^{j} \right) g^{l}_{im} + \sum_{l=1}^{m} \int_{U_{e}} [Ae(\phi^{l})] : e(\phi^{l}) g^{l}_{lm} + \sum_{l=1}^{m} \int_{U_{e}} [Be(\phi^{l})] : e(\phi^{l}) g^{l}_{lm} + \int_{\Gamma_{e}} e \{ \phi^{i} \cdot \phi^{j} - (\phi^{i} \cdot n)(\phi^{j} \cdot n) \} g^{i}_{lm} + e \sum_{l=1}^{m} \int_{\Gamma_{e}} h_{e}(\phi^{i} \cdot n)(\phi^{j} \cdot n) g^{l}_{lm}
\]

(3.2)

\[
= \left( \int_{U_{e}} f(t) \cdot \phi^{j} - e \int_{\Gamma_{e}} g_{e}(\phi^{j} \cdot n) \right), \quad j = 1, 2, \ldots, m.
\]

Since the matrix \( \left( \int_{U_{e}} \phi^{i} \cdot \phi^{j} \right) \) is nonsingular, the above can be written as

\[
(3.3) \quad g^{\prime\prime}_{jm} + \sum_{l=1}^{m} \alpha_{ji} g^{l}_{im} + \sum_{l=1}^{m} \beta_{ji} g_{lm} = \sum_{l=1}^{m} \gamma_{ji} \left( \int_{U_{e}} f(t) \cdot \phi^{j} - e \int_{\Gamma_{e}} g(\phi^{j} \cdot n) \right), \quad 1 \leq j \leq m,
\]

where \((\alpha_{ji}), (\beta_{ji})\) and \((\gamma_{ji})\) are constant matrices.

The initial conditions for the coefficients \( g_{jm}(t) \) are

\[
(3.4) \quad g_{jm}(0) = g^{\prime}_{jm}(0) = 0, \quad 1 \leq j \leq m.
\]

The system (3.3) and (3.4) has a unique solution \( g_{jm} \) on the interval \([0, T] \). Since \((f(t), \phi) = \int_{U_{e}} f(t) \cdot \phi\) and \((g, \phi) = \int_{\Gamma_{e}} g(\phi \cdot n)\) are square integrable with respect to \( t \), so are the \( g_{jm}, g_{jm}^{\prime}, g_{jm}^{\prime\prime} \); hence,

\[
(3.5) \quad u_{m} \in L^{2}([0, T], H_{e}), \partial_{t} u_{m} \in L^{2}([0, T], H_{e}), \partial_{tt} u_{m} \in L^{2}([0, T], H_{e})
\]

By (3.2), \( u_{m}(x, t) = \sum_{l=1}^{m} g_{lm} \phi^{l} \) satisfies

\[
\int_{U_{e}} \partial_{t} u_{m} \cdot \phi^{j} + \sum_{l=1}^{m} \int_{U_{e}} [Ae(u_{m})] : e(\phi^{l}) + \sum_{l=1}^{m} \int_{U_{e}} [Be(\partial_{t} u_{m})] : e(\phi^{l}) + \int_{\Gamma_{e}} e \partial_{t} u_{m} \cdot \phi^{j} - \int_{\Gamma_{e}} e(\partial_{t} u_{m} \cdot n)(n \cdot \phi^{j}) + \sum_{l=1}^{m} \int_{\Gamma_{e}} e g_{e}(n \cdot \phi^{l}) + \int_{\Gamma_{e}} e h_{e}(n \cdot u_{m})(n \cdot \phi^{j})
\]

(3.6)

\[
= \int_{U_{e}} f \cdot \phi^{j}, \quad j = 1, 2, \ldots, m.
\]

Therefore, we have the following Lemma:
Lemma 3.1. There exists a solution $u_m(x, t)$ to (3.6) such that $u_m(x, t)$ satisfies (3.5) and $u_m(x, 0) = \partial_t u_m(x, 0) = 0$.

We introduce the notation:

\begin{equation}
(u, \phi) = \int_{U_e} u \cdot \phi,
\end{equation}

\begin{equation}
\|u\|^2 = \|u\|_{L^2}^2 = \int_{U_e} |u|^2.
\end{equation}

Multiplying (3.6) by $g'_{mj}$ and summing, we obtain:

\begin{equation}
\begin{align*}
\int_{U_e} \partial_t u_m \cdot \partial_t u_m + \int_{U_e} [A e(u_m)] : e(\partial_t u_m) + \int_{U_e} [B e(\partial_t u_m)] : e(\partial_t u_m) \\
+ \int_{\Gamma_e} e \partial_t u_m \cdot \partial_t u_m - \int_{\Gamma_e} e(\partial_t u_m \cdot n)^2 + \int_{\Gamma_e} eg(n \cdot \partial_t u_m) + \int_{\Gamma_e} e(h_e(n \cdot u_m) (n \cdot \partial_t u_m)
\end{align*}
\end{equation}

\begin{equation}
= \int_{U_e} f \cdot \partial_t u_m.
\end{equation}

Lemma 3.2. Let $u_m$ be the solution indicated in Lemma 3.1, then

\begin{equation}
\sup_{t \in [0, T]} \| \nabla u_m(t) \|^2 \leq C \left( \int_0^T \| f(t) \|^2 dt + \int_0^T \| g(t) \|_{L^2(\Gamma)}^2 dt \right)
\end{equation}

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\sup_{t \in [0, T]} \| \partial_t u_m(t) \|^2 \leq C \left( \int_0^T \| f(t) \|^2 dt + \int_0^T \| g(t) \|_{L^2(\Gamma)}^2 dt \right)
\end{equation}

\begin{equation}
\int_0^T \| \nabla \partial_t u_m(t) \|^2 dt \leq C \left( \int_0^T \| f(t) \|^2 dt + \int_0^T \| g(t) \|_{L^2(\Gamma)}^2 dt \right)
\end{equation}

Proof. Let us define

\[ q(t) = \| \partial_t u_m \|^2 + \int_{U_e} [A e(u_m)] : e(u_m) + \frac{1}{2} \epsilon \int_{\Gamma_e} h_e(u_m \cdot n)^2. \]

Since

\[ (\partial_t u \cdot \partial_t u) - (\partial_t u \cdot n)^2 \geq 0, \]

and

\[ \epsilon \int_{\Gamma_e} g n \cdot \partial_t u_m \leq \frac{1}{2} \epsilon \int_{\Gamma_e} g^2 + \frac{1}{2} \epsilon \int_{\Gamma_e} |\partial_t u_m|^2, \]
(3.8) implies

\[
\frac{1}{2} \frac{dg}{dt} + \int_{U_\varepsilon} [Be(\partial_t u_m)] : e(\partial_t u_m) - \frac{1}{2} \varepsilon \int_{\Gamma_\varepsilon} |\partial_t u_m|^2 \leq (f, \partial_t u_m) + \frac{1}{2} \varepsilon \int_{\Gamma_\varepsilon} g^2.
\]

Ellipticity of $B$ implies

\[
\int_{U_\varepsilon} [Be(\partial_t u_m)] : e(\partial_t u_m) \geq K_1 \|\nabla \partial_t u_m\|
\]

where $K_1 > 0$ is a constant depending only on $\lambda_2$.

Applying trace theorem on $U_\varepsilon$ and rescaling we obtain

\[
\frac{1}{2} \varepsilon \int_{\Gamma_\varepsilon} |\partial_t u_m|^2 \leq K_2 \left(\|\partial_t u_m\|^2 + \varepsilon^2 \|\nabla \partial_t u_m\|^2\right),
\]

where $K_2 > 0$ is a constant independent of $\varepsilon$.

For $\varepsilon$ small enough, $\varepsilon^2 K_2 \leq K_1/2$, hence

\[
\int_{U_\varepsilon} [Be(\partial_t u_m)] : e(\partial_t u_m) - \frac{1}{2} \varepsilon \int_{\Gamma_\varepsilon} |\partial_t u_m|^2 \geq (K_1 - \varepsilon^2 K_2) \|\nabla \partial_t u_m\|^2 - K_2 \|\partial_t u_m\|^2 \\
\geq \frac{1}{2} K_1 \|\nabla \partial_t u_m\|^2 - K_2 g(t)
\]

Using $(f(t), \partial_t u_m) \leq \frac{1}{2} (\|f(t)\|^2 + \|\partial_t u_m\|^2)$ (3.8) we have

\[
\frac{dg}{dt} - (2K_2 + 1)g(t) + K_1 \|\nabla \partial_t u_m\|^2 \leq \|f(t)\|^2 + \|g(t)\|^2_{L^2(\Gamma)},
\]

so that

\[
q(t) \leq e^{(2K_2 + 1)t} \int_0^T e^{-(2K_2 + 1)s} \left(\|f(s)\|^2 + \|g(s)\|^2_{L^2(\Gamma)}\right) ds,
\]

Integrating (3.16) with respect to $t$ over $[0, T]$:

\[
q(T) + K_1 \int_0^T \|\nabla \partial_t u_m\|^2 dt \leq (2K_2 + 1) \int_0^T q(t) dt + \int_0^T \left(\|f(t)\|^2 + \|g(t)\|^2_{L^2(\Gamma)}\right) dt
\]

The lemma follows from (3.17), (3.18) and the ellipticity of tensor $A$. 
Lemma 3.3. Let $u_m$ be as in Lemma 3.1, then

$$
\sup_{t \in [0,T]} \|u_m(t)\|^2 \leq C \left( \int_0^T \|f(t)\|^2 dt + \int_0^T \|g(t)\|^2_{L^2(\Gamma)} dt \right).
$$

Proof. The Lemma follows from Lemma 3.2 and $u_m = \int_0^t \partial_t u_m dt$.

Lemma 3.4. There exists a weak solution $u \in L^\infty([0,T],H_e)$ to Problem 1.

Proof. By Lemma 3.2 and Lemma 3.3,

$$
\{u_m\} \text{ is bounded in } L^\infty([0,T],H_e),
$$

hence there exists an element $u \in L^\infty([0,T],H_e)$ such that for a subsequence which we still denote by $\{u_m\}$

$$
u_m \text{ converges to } u \text{ in the weak* topology of } L^\infty([0,T],H_e),
$$

that is

$$
\int_0^T (u_m(t) - u(t), \phi(t))_{H_e} dt \to 0 \text{ for any } \phi \in L^1([0,T],H_e).
$$

Let $\gamma$ be the trace operator on $\Gamma_e$. By the trace theorem (see (3.14)),

$$
\{\gamma u_m\} \text{ is bounded in } L^\infty([0,T],[L^2(\Gamma_e)]^n),
$$

hence there exists an element $u^* \in L^\infty([0,T],[L^2(\Gamma_e)]^n)$ such that for a subsequence which we still denote by $\{u_m\}$

$$
\gamma u_m \text{ converges to } u^* \text{ in the weak* topology of } L^\infty([0,T],[L^2(\Gamma_e)]^n),
$$

that is

$$
\int_0^T (\gamma u_m(t) - u^*(t), \phi(t))_{L^2(\Gamma_e)} dt \to 0 \text{ for any } \phi \in L^1([0,T],[L^2(\Gamma_e)]^n).$$
For \( \psi \in C[0, T], \phi(x) \in [C^\infty(U_c)]^n \), using integration by parts, we obtain

\[
(3.26) \quad \int_{U_c} \nabla u_m \cdot \phi = - \int_{U_c} u_m \cdot \nabla \phi + \int_{\Gamma_c} \gamma u_m (\phi \cdot n).
\]

Multiplying the above by \( \psi \) and integrating we have

\[
(3.27) \quad \int_0^T \psi dt \int_{U_c} \nabla u_m \cdot \phi = - \int_0^T \psi dt \int_{U_c} u_m \cdot \nabla \phi + \int_0^T \psi dt \int_{\Gamma_c} \gamma u_m (\phi \cdot n),
\]

and letting \( m \to \infty \) in above equation yields

\[
(3.28) \quad \int_0^T \psi dt \int_{U_c} \nabla \cdot \phi = - \int_0^T \psi dt \int_{U_c} u \cdot \nabla \phi + \int_0^T \psi dt \int_{\Gamma_c} u^*(\phi \cdot n).
\]

Integrating by parts once more we have

\[
(3.29) \quad \int_0^T \psi dt \int_{\Gamma_c} (u^* - \gamma u) (\phi \cdot n) = 0 \text{ for all } \psi \text{ and } \phi;
\]

it follows that

\[
(3.30) \quad u^* = \gamma u.
\]

In order to pass the limit in (3.6), let us consider a scalar function \( \psi(t) \in C^2[0, T] \) satisfying \( \psi(T) = \psi'(T) = 0 \). Multiplying (3.6) by \( \psi \) and integrating by parts we obtain

\[
(3.31) \quad \int_0^T \psi''(t) dt \int_{U_c} u_m \cdot \phi^j + \int_0^T \psi(t) dt \int_{U_c} [Ae(u_m)] : e(\phi^j)
\]

\[- \int_0^T \psi'(t) dt \int_{U_c} [Be(u_m)] : e(\phi^j) - \int_0^T \psi'(t) dt \int_{\Gamma_c} e u_m \cdot \phi^j + \int_0^T \psi'(t) dt \int_{\Gamma_c} e (u_m \cdot n)(n \cdot \phi^j)
\]

\[+ \int_0^T \psi(t) dt \int_{\Gamma_c} \epsilon g(n \cdot \phi^j) + \int_0^T \psi(t) dt \int_{\Gamma_c} \epsilon h_c(n \cdot u_m)(n \cdot \phi^j)
\]

\[= \int_0^T \psi(t) dt \int_{U_c} f \cdot \phi^j, j = 1, 2, \ldots, m \]
Letting \( m \to \infty \) in the above equation results in

\[
(3.32) \quad \int_0^T \psi''(t) dt \int_{U_c} u \cdot \phi^j + \int_0^T \psi(t) dt \int_{U_c} [Ae(u)] : e(\phi^j)
- \int_0^T \psi'(t) dt \int_{U_c} [B\epsilon(u)] : e(\phi^j) - \int_0^T \psi'(t) dt \int_{U_c} \epsilon(u \cdot \phi^j) + \int_0^T \psi'(t) dt \int_{U_c} \epsilon(u \cdot n)(n \cdot \phi^j)
+ \int_0^T \psi(t) dt \int_{U_c} e\epsilon(n \cdot \phi^j) + \int_0^T \psi(t) dt \int_{U_c} \epsilon\epsilon_{\epsilon}(n \cdot u)(n \cdot \phi^j)
= \int_0^T \psi(t) dt \int_{U_c} f \cdot \phi^j, j = 1, 2, \ldots
\]

Clearly as (3.32) is a weak form of Problem 1, we obtain the lemma.

\[\Box\]

**Theorem 3.5.** There exists a unique solution \( u \in L^\infty([0, T], H_c) \), \( \partial_t u \in L^\infty([0, T], V) \cap L^2([0, T], H_c), \partial_t u \in L^\infty([0, T], H_c') \) to Problem 1. Furthermore \( u \) satisfies the estimates

\[
(3.33) \quad \sup_{t \in [0, T]} \| u(t) \|_{H_c}^2 \leq C \left( \int_0^T \| f(t) \|^2 dt + \int_0^T \| g(t) \|_{L^2(V)}^2 dt \right)
\]

\[
(3.34) \quad \sup_{t \in [0, T]} \| \partial_t u(t) \|^2 \leq C \left( \int_0^T \| f(t) \|^2 dt + \int_0^T \| g(t) \|_{L^2(V)}^2 dt \right)
\]

\[
(3.35) \quad \int_0^T \| \nabla \partial_t u(t) \|^2 dt \leq C \left( \int_0^T \| f(t) \|^2 dt + \int_0^T \| g(t) \|_{L^2(V)}^2 dt \right)
\]

where \( C \) is positive constant independent of \( \epsilon \).

**Proof.** By Lemma 3.2,

\[
(3.36) \quad \text{the sequence } \{ \partial_t u_m \} \text{ is bounded in } L^2([0, T], H_c);
\]
hence, there exists a subsequence, which we also denote by \( \{ \partial_t u_m \} \), that converges weakly to some \( v \in L^2([0,T], H_e) \). Furthermore,

\[
(3.37) \quad \text{the sequence } \{ \partial_t u_m \} \text{ converges to } v \text{ for the weak topology of } L^2([0,T], H_e),
\]

so for any \( \phi \in C^1([0,T], H_e) \), we have

\[
(3.38) \quad \lim_{m \to \infty} \int_0^T (\partial_t u_m, \phi) dt = \int_0^T (v, \phi) dt.
\]

On the other hand, using integration by parts we find

\[
(3.39) \quad \lim_{m \to \infty} \int_0^T (\partial_t u_m, \phi) dt = - \lim_{m \to \infty} \int_0^T (u_m, \partial_t \phi) dt = - \int_0^T (u, \partial_t \phi) dt;
\]

hence,

\[
(3.40) \quad \int_0^T (v, \phi) dt = - \int_0^T (u, \partial_t \phi) dt.
\]

Therefore,

\[
(3.41) \quad v = \partial_t u.
\]

Also, by Lemma 3.2,

\[
(3.42) \quad \text{the sequence } \{ \partial_t u_m \} \text{ is bounded in } L^\infty([0,T], V);
\]

hence, there exists a subsequence, which we also denote by \( \{ \partial_t u_m \} \), which converges in weak-* topology to some \( v^* \in L^\infty([0,T], V) \). Steps similar to (3.38) -(3.41) lead to \( \partial_t u = v^* \), so we have \( \partial_t u \in L^\infty([0,T], V) \).

Since \( u \in L^\infty([0,T], H_e) \), \( \partial_t u \in L^\infty([0,T], V) \), it follows that \( A e(u) \in L^\infty([0,T], V) \), \( B e(\partial_t u) \in L^\infty([0,T], V) \), and we conclude from (2.12) that \( \partial_{tt} u \in L^\infty([0,T], H_e') \).

Using the same steps as in Lemma 3.2, we obtain (3.33)-(3.35).
4. Convergence and Effective equations

4.1. Two-scale convergence. We are going to use the notion of two-scale convergence which was introduced by Nguetseng [20] and Allaire [2]. Let \( C^\infty_\#(Y) \) denote the space of those \( C^\infty \) functions periodic on \( Y \).

**Definition 4.1.** \( \{u^\epsilon(x,t)\} \subset [L^2([0,T] \times U)]^n \) two-scale converges to \( u(t,x,y) \in [L^2([0,T] \times U \times Y)]^n \) iff for any \( \phi(t,x,y) \in [C^\infty([0,T] \times U, C^\infty_\#(Y))]^n \),

\[
\lim_{\epsilon \to 0} \int_0^T \int_U u^\epsilon(t,x) \cdot \phi(t,x,\frac{x}{\epsilon}) dx dt = \int_0^T \int_Y u(t,x,y) \cdot \phi(t,x,y) dy dt dx.
\]

One can prove the following theorem (see [20],[2]):

**Theorem 4.2.** (i) If \( \{u^\epsilon(x,t)\} \) is a bounded sequence in \( [L^2([0,T], L^2(U))]^n \), then there exists \( u_0(t,x,y) \in [L^2([0,T] \times U, L^2_\#(Y))]^n \) such that a subsequence of \( \{u^\epsilon(t,x)\} \) two-scale converges to \( u_0(t,x,y) \) in the sense of Definition 4.1.

(ii) If \( \{u^\epsilon(x,t)\} \) is a bounded sequence in \( [L^2([0,T], H^1(U))]^n \), then there exist \( u_0(t,x) \in [L^2([0,T], H^1(U))]^n \) and \( u_1(t,x,y) \in [L^2([0,T] \times U, H^1_\#(Y))]^n \) such that a subsequence of \( \{u^\epsilon(t,x)\} \) two-scale converges to \( u_0(t,x) \) and a subsequence of \( \nabla_x u^\epsilon \) two-scale converges to \( \nabla_x u_0 + \nabla_y u_1 \) in the sense of Definition 4.1.

(iii) If \( \{u^\epsilon(x,t)\} \) and \( \{\epsilon \nabla_x u^\epsilon(x,t)\} \) are bounded sequence in \( [L^2([0,T], L^2(U))]^n \), then there exists \( u_0(t,x,y) \in [L^2([0,T] \times U, H^1_\#(Y))]^n \) such that a subsequence of \( \{u^\epsilon(t,x)\} \) and \( \{\epsilon \nabla_x u^\epsilon(x,t)\} \) two-scale converges to \( u_0(t,x,y) \) and \( \nabla_y u_0(t,x,y) \) in the sense of Definition 4.1.

**Proof.** The proof is a simple adaption of that in [20] and [2].

Definition 4.3. \( \{ u^\varepsilon(t, x) \} \subset [L^2([0, T] \times T^\varepsilon)]^n \) two-scale converges to \( u(t, x, y) \in [L^2([0, T] \times U, L^2(T))]^n \) iff for any \( \phi(t, x, y) \in [C^\infty([0, T] \times U, C_\#^\infty(Y))]^n \),

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{T^\varepsilon} u^\varepsilon(t, x) \cdot \phi(t, x, \frac{x}{\varepsilon}) d\sigma dt = \int_0^T \int_U \int_Y u(t, x, y) \cdot \phi(t, x, y) dt dx d\sigma_y.
\]

One can prove:

Theorem 4.4. Let \( \{ u^\varepsilon(x, t) \} \) be a sequence in \( [L^2([0, T], L^2(T^\varepsilon))]^n \) such that

\[
\varepsilon \int_{T^\varepsilon} |u^\varepsilon(x)|^2 \leq C
\]

where \( C \) is a positive constant independent of \( \varepsilon \). Then there exists \( u_0(t, x, y) \in [L^2([0, T] \times U, L^2(\#(T))]^n \) such that a subsequence of \( \{ u^\varepsilon(t, x) \} \) two-scale converges to \( u_0(t, x, y) \) in the sense of Definition 4.3.

Proof. The proof is a simple adaption of that in [19] and [3].

Two scale convergence can handle homogenization problem on perforated domains conveniently without requiring any sophisticated extensions such as used in [1]. We only need to use the trivial extension by zero in the hole \( Y_p \) in the following lemma.

Lemma 4.5. Let \( \{ u^\varepsilon(x, t) \} \subset L^2([0, T], H^\varepsilon) \) be as in Theorem 3.5, denote by \( \hat{u}^\varepsilon \) the extension by zero in \( U - U^\varepsilon \), then

(i) there exists \( u_0(t, x) \in [L^2([0, T], H^1(U))]^n \) so that up to a subsequence, \( \hat{u}^\varepsilon(t, x) \) two-scale converges to \( u_0(t, x)\chi(y) \) in the sense of both Definition 4.1 and Definition 4.3. More precisely,

\[
\lim_{\varepsilon \to 0} \int_0^T \int_{U^\varepsilon} u^\varepsilon(t, x) \cdot \phi(t, x, \frac{x}{\varepsilon}) dx dt = \int_0^T \int_U \int_{Y_s} u_0(t, x) \cdot \phi(t, x, y) dt dx dy,
\]

\[
\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{T^\varepsilon} u^\varepsilon(t, x) \cdot \phi(t, x, \frac{x}{\varepsilon}) d\sigma dt = \int_0^T \int_U \int_Y u_0(t, x) \cdot \phi(t, x, y) dt dx d\sigma_y,
\]

for any \( \phi(t, x, y) \in [C^\infty([0, T] \times U, C_\#^\infty(Y))]^n \), where \( \chi(y) \) is the characteristic function on \( Y_s \).
(ii) There exists \( u_1(t, x, y) \in [L^2([0,T] \times U, H^1_0(\mathbb{Y}_0))]^n \) such that a subsequence of extension of \( \nabla_x u^\varepsilon \) two-scale converges to \( \chi(y)[\nabla_x u_0 + \nabla_y u_1] \).

Proof. Using the same steps as in the proof of Thm 2.9 in [2], we get (4.3) and part (ii). (4.4) follows from proposition 2.6 in [3].

Now we rewrite (3.29) as follows.

\[
\int_0^T \psi'(t)dt \int_{U^c} u^\varepsilon \cdot \phi + \int_0^T \psi(t)dt \int_{U^c} [A^e e(u^\varepsilon)] : e(\phi) \\
- \int_0^T \psi'(t)dt \int_{U^c} [B^e e(u^\varepsilon)] : e(\phi) - \int_0^T \psi(t)dt \int_{\Gamma^e} \epsilon u^\varepsilon \cdot \phi + \int_0^T \psi'(t)dt \int_{\Gamma^e} e(u^\varepsilon \cdot n) (n \cdot \phi) \\
+ \int_0^T \psi(t)dt \int_{\Gamma^e} \epsilon g^e (n \cdot \phi) + \int_0^T \psi(t)dt \int_{\Gamma^e} \epsilon h^e (n \cdot u^\varepsilon) (n \cdot \phi) \\
= \int_0^T \psi(t)dt \int_{U^c} \mathbf{f} \cdot \phi.
\]

If we replace the test function by \( \phi(x) + \epsilon \phi_1(x, \frac{x}{\varepsilon}) \), then applying lemma 4.5, we obtain

\[
\int_0^T \psi'(t)dt \int_{U^c} u^\varepsilon \cdot [\phi(x) + \epsilon \phi_1(x, \frac{x}{\varepsilon})] \rightarrow \int_0^T \psi'(t)dt \int_{U^c} u_0 \cdot \phi,
\]

\[
\int_0^T \psi(t)dt \int_{U^c} [A^e e_x(u^\varepsilon)] : e_x([\phi(x) + \epsilon \phi_1(x, \frac{x}{\varepsilon})]) \\
\rightarrow \int_0^T \psi(t)dt \int_{U} \int_{Y_0} A[e_x(u_0) + e_y(u_1)] : (e_x(\phi) + e_y(\phi_1)),
\]

\[
\int_0^T \psi'(t)dt \int_{U^c} [B^e e_x(u^\varepsilon)] : e_x([\phi(x) + \epsilon \phi_1(x, \frac{x}{\varepsilon})]) \rightarrow \\
\int_0^T \psi'(t)dt \int_{U^c} \int_{Y_0} B[e_x(u_0) + e_y(u_1)] : (e_x(\phi) + e_y(\phi_1)),
\]
\[
\int_0^T \psi'(t) dt \int_{\Gamma} \epsilon \frac{\partial u}{\partial t} \cdot [\phi(x) + \epsilon \phi_1(x, \frac{x}{\epsilon})] \rightarrow \int_0^T \psi'(t) dt \int_U \int_{\Gamma}(u_0 \cdot \phi),
\]

\[
\int_0^T \psi'(t) dt \int_{\Gamma} \epsilon(u^* \cdot n)(n \cdot [\phi(x) + \epsilon \phi_1(x, \frac{x}{\epsilon})]) \rightarrow \int_0^T \psi'(t) dt \int_U \int_{\Gamma}(u_0 \cdot n)(\phi \cdot n),
\]

\[
\int_0^T \psi(t) dt \int_{\Gamma} \epsilon_g(n \cdot [\phi(x) + \epsilon \phi_1(x, \frac{x}{\epsilon})]) \rightarrow \int_0^T \psi(t) dt \int_U \int_{\Gamma} g(\phi \cdot n),
\]

\[
\int_0^T \psi(t) dt \int_{\Gamma} \epsilon_h(n \cdot u^*)(n \cdot [\phi(x) + \epsilon \phi_1(x, \frac{x}{\epsilon})]) \rightarrow \int_0^T \psi(t) dt \int_U \int_{\Gamma} h(u_0 \cdot n)(\phi \cdot n),
\]

\[
\int_0^T \psi(t) dt \int_{\Gamma} f \cdot [\phi(x) + \epsilon \phi_1(x, \frac{x}{\epsilon})] \rightarrow \int_0^T \psi(t) dt \int_U f \cdot \phi.
\]

**Lemma 4.6.** Let \(\{u_0(t, x), u_1(t, x, y)\}\) be as in Lemma 4.5. Then \(\{u_0(t, x), u_1(t, x, y)\}\) satisfy the following equations in distributional sense.

\[
\frac{d^2}{dt^2} \int_U u_0 \phi + \int_U \int_{Y_*} A[e_x(u_0) + e_y(u_1)] : e_x(\phi) + \frac{d}{dt} \int_U \int_{Y_*} B[e_x(u_0) + e_y(u_1)] : e_x(\phi)
\]

\[
+ \int_U \int_{\Gamma} \partial_t u_0 \cdot \phi - \int_U \int_{\Gamma} \partial_t u_0 \cdot n(\phi \cdot n) + \int_U \int_{\Gamma} g(\phi \cdot n) + \int_U \int_{\Gamma} h(u_0 \cdot n)(\phi \cdot n)
\]

\[
= \int_U f \cdot \phi
\]

\[
\int_U \int_{Y_*} A[e_x(u_0) + e_y(u_1)] : e_y(\phi_1) + \int_U \int_{Y_*} B[e_x(\partial_t u_0) + e_y(\partial_t u_1)] : e_y(\phi_1) = 0
\]

\[
u_0(0) = \partial_t u_0(0) = 0, u_1(0) = 0
\]
Proof. Let $\epsilon \to 0$ in (4.5) and use (4.6)-(4.13):

\begin{align}
(4.17) \int_0^T \psi''(t) \int_U u_0 \phi + \int_0^T \psi(t) \int_U \int_{Y_s} A[e_x(u_0) + e_y(u_1)] : e_x(\phi) \\
- \int_0^T \psi'(t) \int_U \int_{Y_s} B[e_x(u_0) + e_y(u_1)] : e_x(\phi) + \int_0^T \psi(t) \int_U \int_{Y_s} A[e_x(u_0) + e_y(u_1)] : e_y(\phi) \\
- \int_0^T \psi'(t) \int_U \int_{Y_s} B[e_x(u_0) + e_y(u_1)] : e_y(\phi) - \int_0^T \psi'(t) \int_U \int_{\Gamma} u_0 \cdot \phi \\
+ \int_0^T \psi'(t) \int_U \int_{\Gamma} u_0 \cdot n(\phi \cdot n) + \int_0^T \psi(t) \int_U \int_{\Gamma} g \phi \cdot n = \int_U f \cdot \phi.
\end{align}

We are now able to establish the lemma by setting $\phi_1 = 0$ and $\phi = 0$ respectively. \qed

Lemma 4.7. The system (4.14)-(4.16) has a unique solution.

Proof. It is sufficient to prove that for $f = 0$, $g = 0$ we have only the trivial solution $u_0 = u_1 = 0$. Set $\phi = \partial_t u_0, \phi_1 = \partial_t u_1$ as test function in (4.14) and (4.15):

\begin{align}
(4.18) \int_U \partial_{tt} u_0 \partial_t u_0 + \int_U \int_{Y_s} A(e_x(u_0) + e_y(u_1)) : e_x(\partial_t u_0) + \frac{d}{dt} \int_U \int_{Y_s} B(e_x(u_0) + e_y(u_1)) : e_x(\partial_t u_0) \\
+ \int_U \int_{\Gamma} \partial_t u_0 \cdot \partial_t u_0 - \int_U \int_{\Gamma} (\partial_t u_0 \cdot n)^2 + \int_U \int_{\Gamma} e h(u_0 \cdot n)(\partial_t u_0 \cdot n) = 0
\end{align}

\begin{align}
(4.19) \int_U \int_{Y_s} A(e_x(u_0) + e_y(u_1)) : e_y(\partial_t u_1) + \int_U \int_{Y_s} B(e_x(\partial_t u_0) + e_y(\partial_t u_1)) : e_y(\partial_t u_1) = 0
\end{align}

Adding (4.18) and (4.19) and integrating in time we obtain

\begin{align}
(4.20) \int_U (\partial_t u_0)^2 + \int_U \int_{Y_s} A e_x(u_0) : e_x(u_0) + 2 \int_0^t \int_U \int_{Y_s} B(e_x(\partial_t u_0) + e_y(\partial_t u_1)) : (e_x(\partial_t u_0) + e_y(\partial_t u_1)) \\
+ \int_0^t \int_U \int_{\Gamma} \partial_t u_0 \cdot \partial_t u_0 - \int_0^t \int_U \int_{\Gamma} (\partial_t u_0 \cdot n)^2 + \frac{1}{2} \int_U \int_{\Gamma} e h(u_0 \cdot n)^2 = 0.
\end{align}

Note that

\begin{align*}
\int_0^t \int_U \int_{\Gamma} \partial_t u_0 \cdot \partial_t u_0 - \int_0^t \int_U \int_{\Gamma} (\partial_t u_0 \cdot n)^2 \geq 0,
\end{align*}

so that (4.20) implies $\partial_t u_0 = 0, \partial_t u_1 = 0$, using initial condition, we have $u_0 \equiv 0, u_1 \equiv 0$. \qed
4.2. Derivation of the effective equation of $u_0$. Since system (4.14)-(4.16) has a unique solution, we seek $u_1(t,x,y)$ in the form

$$
(4.21) \quad u_1(t,x,y) = \sum_{ij} \left\{ M^{ij}(y) e_x(u_0)_{ij}(t,x) + \int_0^t K^{ij}(y, t-s)(e_x(u_0))_{ij}(x,s)ds \right\},
$$

where the vectors $M^{ij}, K^{ij}$ are to be specified. To derive equations for $M^{ij}(y), K^{ij}(y,t)$, we substitute (4.21) into (4.15):

$$
\begin{align*}
\int_U \int_Y A \left\{ e_x(u_0) + \sum_{ij} e_y(M^{ij})(y)e_x(u_0)_{ij}(t,x) \\
+ \int_0^t e_y(K^{ij})(y, t-s)(e_x(u_0))_{ij}(x,s)ds \right\} : e_y(\phi_1) \\
+ \int_U \int_Y B \left\{ e_x(\partial_t u_0) + \sum_{ij} e_y(M^{ij})(y)e_x(\partial_t u_0)_{ij}(t,x) + e_y(K^{ij})(y, 0)e_x(u_0)_{ij} \\
+ \int_0^t e_y(\partial_t K^{ij})(y, t-s)(e_x(u_0))_{ij}(x,s)ds \right\} : e_y(\phi_1) = 0.
\end{align*}
$$

(4.22)

Collecting $e_x(\partial_t u_0)$ terms, and choosing $M^{ij}$ so that

$$
\int_U \int_Y B \left( e_x(\partial_t u_0) + \sum_{ij} e_y(M^{ij})(y)e_x(\partial_t u_0)_{ij}(t,x) \right) : e_y(\phi_1) = 0,
$$

(4.23)

we have

$$
\int_Y B \left\{ \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) + e_y(M^{ij}) \right\} : e_y(\phi_1) = 0 \text{ for any } \phi_1 \in [H^1_\#(Y)]^n,
$$

(4.24)

where $\{e_i\}$ are basis vector in $\mathbb{R}^n$.

i.e $M^{ij}$ is determined by the following linear elastic system:

$$
\text{div}_y \left\{ B \left( \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) + e_y(M^{ij}) \right) \right\} = 0,
$$

$$
B \left( \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) + e_y(M^{ij}) \right) \cdot n| = 0 \text{ on } \Gamma,
$$

(4.25)

$M^{ij}(y) \in [H^1_\#(Y_s)]^n$. 

Collecting $e_x(u_0)$ terms in (4.22) and choosing $K^{ij}(0)$ so that

\begin{equation}
\int_U \int_{\mathcal{Y}_x} \left\{ A e_x(u_0) + A \sum_{ij} e_y(M^{ij})(y) e_x(u_0)_{ij}(t, x) + B \sum_{ij} e_y(K^{ij}(0)) e_x(u_0)_{ij} \right\} : e_y(\phi_1) = 0
\end{equation}

we have

\begin{equation}
\int_{\mathcal{Y}_x} \left\{ \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) + A e_y(M^{ij}) + B e_y(K^{ij}(0)) \right\} : e_y(\phi_1) = 0 \quad \text{for any } \phi_1 \in [H^1_\#(\mathcal{Y})]^n
\end{equation}

i.e $K^{ij}(0)$ is determined by the following linear elastic system:

\begin{equation}
\text{div } y \left\{ A \left( \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) \right) + A e_y(M^{ij}) + B e_y(K^{ij}(0)) \right\} = 0
\end{equation}

\begin{equation}
\left\{ A \left( \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) \right) + A e_y(M^{ij}) + B e_y(K^{ij}(0)) \right\} \cdot n = 0 \quad \text{on } \Gamma,
\end{equation}

$K^{ij}(0, y) \in [H^1_\#(\mathcal{Y})]^n$, $M^{ij}(y)$ is given by (4.26).

Collecting terms containing convolutions in time in (4.22) and choosing $K^{ij}(t, y)$ we obtain

\begin{equation}
\int_U \int_{\mathcal{Y}_x} \int_0^t \sum_{ij} \left\{ A e_y(K^{ij})(y, t - s) (e_x(u_0))_{ij}(x, s) + B e_y(\partial_t K^{ij})(y, t - s) (e_x(u_0))_{ij}(x, s) ds \right\} : e_y(\phi_1) = 0,
\end{equation}

which implies

\begin{equation}
\int_{\mathcal{Y}_x} \left[ A e_y(K^{ij})(y, t) + B e_y(\partial_t K^{ij})(y, t) \right] : e_y(\phi_1) = 0,
\end{equation}

i.e $K^{ij}(t, y)$ is determined by the following system
\[
\text{div}_y \left\{ B e_y (\partial_t K^{ij}) + A e_y (K^{ij}) \right\} = 0,
\]
\[
\left\{ B e_y (\partial_t K^{ij}) + A e_y (K^{ij}) \right\} \cdot \mathbf{n} = 0 \text{ on } \Gamma,
\]
\[
K^{ij}(t, y) \in L^2([0, T], [H^1_0(Y)]^n),
\]
\[
K^{ij}(0, y) \text{ is given by (4.28).}
\]

The well-posedness of the above cell problems will be studied in next section. Assuming now that the cell problems are solvable, we proceed to derive the effective equations. Substituting (4.21) into (4.14) we have

\[
\frac{d^2}{dt^2} \int_U u_0 \phi + \int_U \int_{Y_s} A e_x (u_0) : e_x (\phi)
\]
\[
+ \int_U \int_{Y_s} B[e_x (\partial_t u_0)] : e_x (\phi) + \int_U \int_{Y_s} B \sum_{ij} \left\{ e_y (M^{ij}(y)) e_x (u_0)_{ij}(t, x) + \int_0^t e_y (K^{ij}(y, t - s)) (e_x (u_0))_{ij}(x, s) ds \right\} : e_x (\phi)
\]
\[
+ e_y (K^{ij}(0))(e_x (u_0))_{ij} + \int_0^t e_y (\partial_t K^{ij}(y, t - s)) (e_x (u_0))_{ij}(x, s) ds \right\} : e_x (\phi)
\]
\[
+ \int_U \int_{\Gamma} \partial_t u_0 \cdot \phi - \int_U \int_{\Gamma} (\partial_t u_0 \cdot \mathbf{n}) (\phi \cdot \mathbf{n}) + \int_U \int_{\Gamma} h(u_0 \cdot \mathbf{n}) (\phi \cdot \mathbf{n}) + \int_U \int_{\Gamma} g \phi \cdot \mathbf{n}
\]
\[
= \int_U f \cdot \phi.
\]

Next, define the symmetric tensors \( A, B \) and \( C \) by

\[
A_{ijkl} = \int_{Y_s} A_{ijkl} + \int_{Y_s} \sum_{mn} \frac{1}{2} A_{klmn} \left( \frac{\partial M^{ij}_n}{\partial y_m} + \frac{\partial M^{ij}_m}{\partial y_n} \right) + \frac{1}{2} B_{klmn} \left( \frac{\partial K^{ij}_n(0)}{\partial y_m} + \frac{\partial K^{ij}_m(0)}{\partial y_n} \right),
\]
\[
B_{ijkl} = \int_{Y_s} B_{ijkl} + \int_{Y_s} \sum_{mn} \frac{1}{2} B_{klmn} \left( \frac{\partial M^{ij}_n}{\partial y_m} + \frac{\partial M^{ij}_m}{\partial y_n} \right),
\]
\begin{equation}
C_{ijkl}(t) = \int_\gamma \sum_{mn} \left\{ \frac{1}{2} A_{klmn} \left( \frac{\partial K_{hi}^{ij}}{\partial y_n} + \frac{\partial K_{mj}^{ij}}{\partial y_n} \right) + \frac{1}{2} B_{klmn} \left( \frac{\partial (\partial_t K_{hi}^{ij})}{\partial y_n} + \frac{\partial (\partial_t K_{mj}^{ij})}{\partial y_n} \right) \right\},
\end{equation}

and let

\begin{equation}
N_{ij} = |\Gamma| \delta_{ij} - \int_\gamma (n \otimes n)_{ij}, \quad H_{ij} = \int_\gamma h(n \otimes n)_{ij}, \quad f_1 = \int_\gamma g n.
\end{equation}

From (4.32), we obtain the effective equation for \(u_0\):

\begin{equation}
\partial_t u_0 - \text{div} \left[ A e(u_0) + B e(\partial_t u_0) \right] - \text{div} \int_0^t C(t-s)e(u_0)ds + N \cdot \partial_t u_0 + H \cdot u_0 = f - f_1.
\end{equation}

The above calculations prove

**Theorem 4.8.** Let \(u_0(t,x)\) be as in Lemma 4.5, then \(u_0(t,x)\) satisfies equation (4.37) and (4.16).

From the effective equation (4.37), we find the effective stress tensor

\begin{equation}
\sigma_0 = A e(u_0) + B e(\partial_t u_0) - \int_0^t C(t-s)e(u_0)ds
\end{equation}

and the effective drag force

\begin{equation}
D_0 = N \cdot \partial_t u_0 + H \cdot u_0.
\end{equation}

5. **Well-posedness for cell problems and effective problem**

**Lemma 5.1.** The problem (4.25) is uniquely solvable.

**Proof.** Since \(B\) is positive definite, the lemma follows from Korn’s inequality and the Lax-Milgram lemma.

\[ \square \]

**Lemma 5.2.** The problem (4.28) is uniquely solvable.

**Proof.** Since \(B\) is positive definite, the lemma follows from Korn’s inequality and the Lax-Milgram lemma.

\[ \square \]

**Lemma 5.3.** The problem (4.31) is uniquely solvable.
Proof. Let $\mathcal{K}^{ij}(\gamma) = \mathcal{L}K^{ij}$, where $\mathcal{L}$ is Laplace transform. For properties of Laplace transform of distributions, we refer to [9]. Then $\mathcal{K}^{ij}$ satisfies
\begin{equation}
\text{div}_y \left\{ (B\gamma + A)e_y(K^{ij}) - Be_y(K^{ij}_0) \right\} = 0.
\end{equation}

Since $B\gamma + A$ is positive definite for $\gamma > 0$, Korn’s inequality and the Lax-Milgram lemma imply that (5.1) is uniquely solvable, so the lemma follows. \hfill \square

Lemma 5.4. The tensor $B$ defined in (4.34) is positive definite.

Proof. Let $\Lambda$ be any symmetric matrix, then
\[ \Lambda = \sum_{ij} \lambda_{ij} \frac{e_i \otimes e_j + e_j \otimes e_i}{2}. \]

Let
\[ M^\lambda = \sum_{ij} \lambda_{ij} M^{ij}. \]

Then
\[ B\Lambda : \Lambda = \int_{Y^*} B(\Lambda + e_y(M^\lambda)) : \Lambda \]
\[ = \int_{Y^*} B(\Lambda + e_y(M^\lambda)) : (\Lambda + e_y(M^\lambda)) - \int_{Y^*} B(\lambda + e_y(M^\lambda)) : e_y(M^\lambda) \]
\[ = \int_{Y^*} B(\Lambda + e_y(M^\lambda)) : (\Lambda + e_y(M^\lambda)) \]
\[ \geq \lambda_2 |\Lambda + e_y(M^\lambda)|^2, \]
so the lemma follows. \hfill \square

Lemma 5.5. Let $\check{C}(\gamma)$ be the Laplace transform of $C(t)$, then $\mathcal{A} + \check{C}(\gamma)$ is positive definite for $\gamma > 0$.

Proof. Let
\[ \check{K}^\lambda = \sum_{ij} \lambda_{ij} \check{K}^{ij}, \]
where $\hat{K}^{ij}$ is the Laplace transform of $K^{ij}$.

Then

\[(A + \hat{\mathcal{Q}}) \Lambda : \Lambda \]

\[= \int_{Y_s} [A(\Lambda + e_y(M^\lambda)) : \Lambda + \int_{Y_s} [Ae_y(\hat{K}^\lambda) + \gamma Be_y(\hat{K}^\lambda)] : \Lambda + \int_{Y_s} [\gamma Be_y(\hat{K}^\lambda)] : \Lambda \]

\[= \int_{Y_s} [A(\Lambda + e_y(M^\lambda) + e_y(\hat{K}^\lambda))] : (\Lambda + e_y(M^\lambda) + e_y(\hat{K}^\lambda)) + \int_{Y_s} [\gamma Be_y(\hat{K}^\lambda)] : e_y(\hat{K}^\lambda) \]

\[+ \int_{Y_s} [\gamma Be_y(\hat{K}^\lambda)] : (\Lambda + e_y(M^\lambda) + e_y(\hat{K}^\lambda)) \]

\[= \int_{Y_s} [A(\Lambda + e_y(M^\lambda) + e_y(\hat{K}^\lambda))] : (\Lambda + e_y(M^\lambda) + e_y(\hat{K}^\lambda)) + \int_{Y_s} [\gamma Be_y(\hat{K}^\lambda)] : e_y(\hat{K}^\lambda) \]

\[\geq \lambda_1 |\Lambda + e_y(M^\lambda) + e_y(\hat{K}^\lambda)|^2 + \gamma \lambda_2 |e_y(\hat{K}^\lambda)|^2, \]

so the lemma follows. \(\square\)

**Lemma 5.6.** The homogenized equation (4.37) with initial condition $u_0(0) = u'_0(0) = 0$ has a unique solution.

**Proof.** Taking Laplace transform in (4.37), we have

\[(5.2) \quad \text{div} [A + \gamma \mathcal{B} + \hat{\mathcal{Q}}(\gamma)] \hat{u}_0 - \gamma^2 \hat{u}_0 - \gamma (N - H) \cdot \hat{u}_0 = -(f - f_1). \]

By the above lemmas, $A + \gamma \mathcal{B} + \hat{\mathcal{Q}}(\gamma)$ is positive definite. Thus, Korn's inequality and the Lax-Milgram lemma imply that (5.2) is uniquely solvable, so the lemma follows. \(\square\)

**References**


DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DE 19716

E-mail address: gilbert@math.udel.edu

DEPARTMENT OF MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WA 99164

E-mail address: panchenko@math.wsu.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DE 19716

E-mail address: xie@math.udel.edu