A prototype homogenization model for acoustics of granular materials

By Robert P. Gilbert[1], Alexander Panchenko[2] and Xuming Xie[1]

[1] *Department of Mathematical sciences*
University of Delaware, Newark DE 19617 USA
[2] *Department of Mathematics, Washington State University*
Pullman WA 99164 USA

This paper introduces a homogenization approach to modelling acoustic vibrations of composite materials with internal friction. The model medium studied in the paper consists of a consolidated viscoelastic solid matrix with a large number of periodically arranged pores containing rigid solid particles. The particles are in frictional contact with the matrix. At the length scale of particles, the frictional forces are modelled initially by the Coulomb's law with normal compliance. These inequality-type conditions are approximated by non-linear equations. The resulting microscale problem is averaged using formal two-scale homogenization. The effective acoustic equations are in general non-linear and history dependent, and contain both the effective stress and the effective drag force. The constitutive equations for the effective quantities are obtained explicitly for three different approximate models of contact conditions.

Keywords: Acoustics of granular materials, homogenization, effective equations

1. Introduction

Many problems in geophysics and engineering involve propagation of acoustic waves in granular materials such as soils or marine sediments. Existing theories of granular media fall mainly into two categories: statistical and phenomenological. The statistical approach is based on the analogy between a granular flow and a dense gas. The asymptotic expansions of the kinetic theory are used to derive effective equations (see, for instance, Jenkins & Savage (1983)). This approach is relatively well-developed, but produces good results only for fast rarefied flows with instantaneous inter-particle collisions.

In the phenomenological theories of Goodman & Cowin (1972), Harris (2001), Kirchner (2002), the equations of mass, momentum and energy balance are supplemented with the balance equations for additional state variables. These variables are difficult to access experimentally, so there is no consensus as to which variables adequately represent the state of a granular material. Without the benefit of direct measurements, it is also difficult to specify the required constitutive laws for the extra variables.

A notable representative of phenomenological theories which do not use extra variables is hypoplasticity, proposed by Kolymbas (1987). It seeks to model slow deformations of highly packed granular materials, in which grain-to-grain interactions play a dominant role. In hypoplastic models, the constitutive equation for the stress $\mathbf{T}$ takes the form of a differential equation

$$\dot{\mathbf{T}} = \mathbf{A}(\mathbf{T}, c)e(\mathbf{v}) + \mathbf{b}(\mathbf{T}, c) \parallel e(\mathbf{v}) \parallel, \tag{1.1}$$

where $\mathbf{A}, \mathbf{b}$ are nonlinear functions of the stress and the solids volume fraction c, $e(\mathbf{v})$ is the strain rate (symmetric part of the velocity gradient), and $\parallel e(\mathbf{v}) \parallel = \sqrt{e(\mathbf{v})_{ij} e(\mathbf{v})_{ij}}$. Furthermore, $\dot{\mathbf{T}}$ denotes Jaumann stress rate defined by

$$\dot{\mathbf{T}} = \frac{d\mathbf{T}}{dt} + \mathbf{TW} - \mathbf{WT}.$$

Here d/dt denotes the material time derivative, and $\mathbf{W}$ is the skew part of the velocity gradient. Nonlinear equations such as (1.1) present serious mathematical difficulties. It is also thought (Hill (2000)) that hypoplastic models are not applicable to small acoustic deformations.

In Schaeffer (1992), the constitutive equations are of the form $\frac{d\mathbf{T}}{dt} = \mathbf{A}(\mathbf{T})\nabla\mathbf{v}$. The function $\mathbf{A}$, which is different for loading and unloading, plays the role of a yield criterion, so that for large stresses the governing system of equations becomes ill-posed. The ill-posedness is thought to be responsible for the formation of shear bands observed experimentally in steady shearing of granular materials.

It seems that acoustics of highly packed granular materials cannot be modelled within the statistical framework because of high concentrations and prolonged contacts, while the phenomenological theories referenced above are complicated and may be inapplicable to wave propagation.

Recently, a linear phenomenological theory of wave propagation in unconsolidated solid-fluid mixtures was proposed by Buckingham (1997, 1998, 2000). Buckingham's theory is based on the assumption that the grain-to-grain shearing is the primary mechanism for energy dissipation. Postulating a viscoelastic stress-strain relation for a pair of grains in frictional contact, Buckingham obtained linear viscoelastic acoustic equations. The main feature of his theory is the slow time decay of the memory kernels (power-type instead of the exponential). Buckingham's theory correctly predicts dependence of attenuation on frequency. However, results of recent experiments with water-saturated sand in Isakson *at al.* (2003) suggest that part of the material behaves as a solid frame, while Buckingham's theory assumes that a sediment is completely unconsolidated. For a partially consolidated medium, elasticity of the frame must play an important role. This leads to a possibility that some terms might be missing in Buckingham's equations.

Several authors (Sunchez-Palencia (1980), Burridge & Keller (1981), Levy (1985), Gilbert & Mikelic (2002), Gilbert & Panchenko (2003)) obtained effective acoustic equations of porous media using homogenization (Sanchez-Palencia (1980), Bakhvalov & Panasenko (1989)). The homogenization approach allows one to derive effective equations for composite materials from the so-called microscopic balance equations, satisfied locally in each phase, and conditions on the interface between the phases. In Sunchez-Palencia (1980), Burridge & Keller (1981), Gilbert & Mikelic (2002) and Gilbert & Panchenko (2003) it was assumed, often implicitly, that the interface between the solid and fluid phases is stationary and the solid phase is connected. The stationary interface assumption is consistent with typical in acoustics linearization of the microscopic equations. † It is also convenient for technical reasons, since homogenization of the moving interface problems is much more difficult. When the solid phase is connected and the interface is static, friction between different parts of the solid phase is impossible, and the energy dissipation is due exclusively to fluid viscosity. Assuming further that the fluid viscosity is not negligible, one can average the linearized microscopic acoustic equations to obtain viscoelastic equations of motion. These equations are similar to Buckingham's equations, but the significance of this similarity is not clear, because history dependence in Buckingham's equations is due to a different physical phenomenon. The memory kernels in the averaged acoustic equations seem to decay exponentially in time, whereas the kernels in Buckingham's theory decay at a much slower power rate. This discrepancy, together with the experimental results from Isakson *at al.* (2003), shows that the central issue, namely the role played by inter-granular friction in shaping up the effective constitutive equations, is not well understood at present.

To separate this issue from the problems of flow analysis, we study an idealized granular composite which consists of a consolidated visco-elastic matrix with a large number of periodically spaced holes containing rigid particles. The matrix is chosen to be viscoelastic, rather than perfectly elastic, because in geophysical materials the pores of the elastic skeleton are typically filled with fluid. The particles, representing the unconsolidated part of the solid phase, are in frictional contact with the matrix. The frictional forces are modelled initially by the pointwise Coulomb-type law with normal compliance (Kikuchi & Oden (1988)). The contact conditions of Coulomb type are formulated as inequalities involving tangential forces on the contact surface. The corresponding microscale problem has the form of a variational inequality. The most likely outcome of averaging such a model would be another variational inequality, or more generally, an inclusion. To guarantee that the averaged model has the form of an equation, rather than inequality, one needs to work with a microscale model of the same type. To obtain this simplified model, we approximate Coulomb contact conditions by nonlinear equations. This idea is borrowed from the papers on analysis of variational inequalities, in particular Kuttler & Shillor (1999).

Expecting small deviations from the effective displacements near the particle-matrix interface, we linearize the contact conditions about the effective displacement and velocity. These modified contact conditions still contain nonlinear functions of the effective variables. The nonlinear functions are further approximated by their Taylor polynomials. This procedure allows one to construct a family of models of increasing complexity, as needed. In particular, the simplest model corresponds to completely linearized contact conditions.

The effective equations are obtained using the method of two-scale asymptotic expansions and homogenization. (Sanchez-Palencia (1980), Bakhvalov & Panasenko (1989)). The approach adopted in this paper is similar to the one in Sanchez-Palencia (1980), where the focus is on derivation of the effective equations, rather than on proving convergence of the asymptotic expansions. Due to nonlinearity of the problem and complicated nature of the interface conditions, convergence issues merit a separate publication and will be investigated elsewhere. For the same reasons, constructing a two-scale asymptotics is a nontrivial problem, which we solve

† In the referential description, the interface is static. Linearized equations are derived assuming that spatial and referential descriptions are identical.

using a modification of the techniques developed by Gilbert & Panchenko (2003)for acoustics of porous media. Another reason for writing this paper is that averaged equations differ from the phenomenological models in Buckingham (2000), Harris (2001), Goodman & Cowin (1972), Kolymbas (1987), as well as from the acoustic equations of porous media obtained by homogenization in Sanchez-Palencia (1980), Burridge & Keller (1981), Gilbert & Mikelic (2000) and Gilbert & Panchenko (2003).

For the general non-linear contact conditions, the effective equations of motion are history dependent and nonlinear. The constitutive equations depend not only on the strain and strain rate, but also on the velocity and displacement. This is an essential feature reflecting the nature of the contact conditions. Moreover, the effective equation contains a term which may be interpreted as a macroscopic body force, similar in nature to the well known Stokes drag force. The effective model provides two constitutive equations: one for the effective stress, and another for the drag force. Simplified periodic geometry allows us to obtain explicit formulas for the constitutive equations. The effective stress contains the same nonlinear functions of the effective velocity as the contact conditions, and nonlinearities in the drag force are similar, but more complicated.

The paper is organized as follows. Section 2 contains description of geometry and microscopic governing equations. All simplification of the microscopic model are described there as well. In section 3 we outline the formal two-scale homogenization procedure applied to the problem at hand. Section 4 is devoted to the preliminary analysis of the nature of the effective equations. Finally, in Section 5, 6 and 7 we detail the averaging procedure for three different types of contact conditions and obtain explicit formulas for the constitutive equations for all three models.

2. Formulation of the problem

(a) *Geometry*

We consider a fixed cube $\Omega \in \mathbf{R}^3$ with periodic microstructure. The latter is obtained as follows. First, we define a periodicity cell (see Fig. 1).

Let $\mathcal{C}$ be a cube of side length one. We assume that $\mathcal{C}$ is a union of two disjoint domains Y and P. The *particle cell* P is simply connected and the distance between $\partial\mathcal{C}$ and P is positive. The boundary of P is the *particle-matrix interface* denoted by Γ_Y. For simplicity, we assume that Γ_Y is smooth. The complement Y of P in $\mathcal{C}$ is called the *matrix cell.*

The cell $\mathcal{C}$ is used as a building block. First, it is shrunk by a factor of $\varepsilon \in (0,1]$ to obtain the scaled cell $\mathcal{C}^\varepsilon$. Then Ω is filled with copies of $\mathcal{C}^\varepsilon$ (see Fig. 2).

The resulting periodic geometry consists of the perforated, connected matrix domain Ω^ε_M and the particulate domain Ω^ε_P, which is a union of many disjoint simply connected components (particles). The boundary of Ω^ε_M is a union of the external boundary $\partial\Omega$ and the particle-matrix interface Γ^ε.

(b) *Equations of motion and contact conditions.*

1. Particles. We assume that the particles are rigid bodies, moving with the same

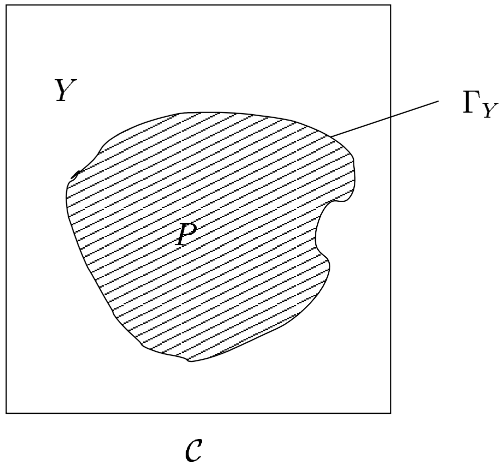

Figure 1. Periodicity cell.

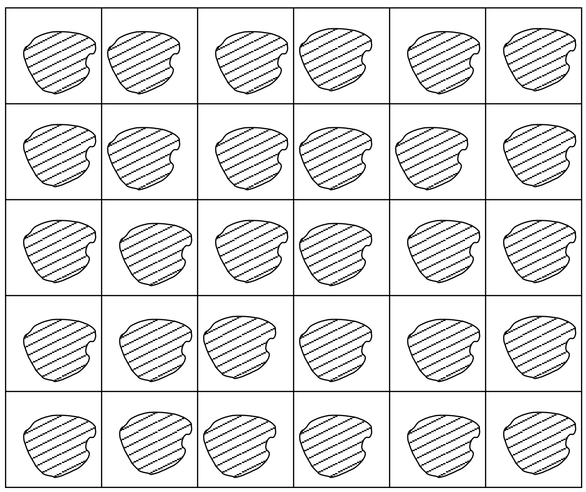

Figure 2. Perforated domain with particles.

prescribed velocity $\dot{\mathbf{U}}(t, \mathbf{x})$. The (rigid) displacement of each particle is denoted by $\mathbf{U}$.

2. Matrix. Since the particulate motion is prescribed, the only unknown in the model is the deformation state of the matrix. The matrix domain Ω_M^ε is occupied by a viscoelastic material with the constitutive equation given by a Kelvin-Voight law

$$\boldsymbol{\sigma} = Ae(\mathbf{u}) + Be(\mathbf{v}), \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, A, B are constant constitutive tensors, $e = \frac{1}{2}(\nabla + \nabla^T)$ is the symmetric part of the gradient matrix, and $\mathbf{u}, \mathbf{v}$ are displacement and velocity, respectively.

The momentum balance equation for the matrix is

$$\dot{\mathbf{v}} = div\ \boldsymbol{\sigma} + \mathbf{f}, \quad \text{in } \Omega_M^\varepsilon. \tag{2.2}$$

3. Boundary conditions. Boundary conditions on the contact surface Γ^ε are prescribed for the traction $\boldsymbol{\sigma}\mathbf{n}$ where $\mathbf{n}$ is the unit normal to Γ^ε exterior with respect to Ω^ε_M. To keep track of normal and tangential forces, we write

$$\boldsymbol{\sigma}\mathbf{n} = \sigma_n \mathbf{n} + \boldsymbol{\sigma}_T,$$

where $\sigma_n = \boldsymbol{\sigma}_{ij} n_i n_j$. The magnitude of the normal force is given by

$$\sigma_n = -p(u_n - \varepsilon g) \quad \text{on } \Gamma^\varepsilon, \tag{2.3}$$

where εg is a given function (gap) between the matrix and a particle, measured in normal direction. The function p is the pressure function. The tangential traction on Γ^ε is prescribed by the following conditions.

$$|\boldsymbol{\sigma}_T| \le F(u_n - \varepsilon g), \tag{2.4}$$

$$|\boldsymbol{\sigma}_T| < F(u_n - \varepsilon g) \Rightarrow \mathbf{v}_T - \dot{\mathbf{U}}_T = 0, \tag{2.5}$$

$$|\boldsymbol{\sigma}_T| = F(u_n - \varepsilon g) \Rightarrow \mathbf{v}_T - \dot{\mathbf{U}}_T = -\lambda \boldsymbol{\sigma}_T. \tag{2.6}$$

In (2.5), (2.6), $\mathbf{v}_T = \mathbf{v} - v_n\mathbf{n}$. The inequality (2.4) means that $|\boldsymbol{\sigma}_T|$ does not exceed the friction bound F, so the maximal magnitude of the tangential force is prescribed in terms of the normal deformation. Condition (2.5) means that there is no sliding of particles relative to the matrix when the tangential force is smaller than the friction bound. According to (2.6), sliding occurs when the tangential force reaches the friction bound. The relative velocity of sliding is proportional to the tangential force with the friction coefficient $\lambda > 0$.

The set of contact conditions (2.3)-(2.6) is the so-called Coulomb friction law with normal compliance (Kikuchi & Oden (1988)). The functions p and F are non-linear. Based on experimental evidence, it is reasonable to choose (Kikuchi & Oden (1988))

$$p(z) = \begin{cases} z^m, & \text{when } z \ge 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where m is positive. The function F is prescribed similarly, with a larger value of m. *Remark.* At present, it is not known whether the Coulomb law of friction actually holds pointwise on the contact surface (Kikuchi & Oden (1988)). It is clear that the condition (2.6) is not frame-indifferent. Frame-indifference of traction is one of the fundamental axioms of rational continuum mechanics (see, e.g. Truesdell (1997)). Perhaps, the pointwise version of the Coulomb law should be eventually replaced by a more realistic non-local, history dependent, and frame indifferent contact conditions.

On the external boundary $\partial\Omega$ we prescribe periodic boundary conditions.

4. Initial conditions. The initial conditions are

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \qquad \mathbf{v}(0, x) = \mathbf{v}_0(x), \quad x \in \Omega^\varepsilon_M. \tag{2.8}$$

Because particulate motion is prescribed, the formulation of the problem may be simplified by introducing relative displacement

$$\boldsymbol{\xi} = \mathbf{u} - \mathbf{U} \tag{2.9}$$

and the relative velocity

$$\boldsymbol{\zeta} = \mathbf{v} - \dot{\mathbf{U}} \tag{2.10}$$

of the matrix. In terms of $\boldsymbol{\xi}, \boldsymbol{\zeta}$ the contact conditions on Γ^ε are written as follows.

$$\sigma_n = -p(\xi_n - \varepsilon\tilde{g}), \tag{2.11}$$

$$|\boldsymbol{\sigma}_T| \leq F(\xi_n - \varepsilon\tilde{g}), \tag{2.12}$$

$$|\boldsymbol{\sigma}_T| < F(\xi_n - \varepsilon\tilde{g}) \Rightarrow \boldsymbol{\zeta}_T = 0, \tag{2.13}$$

$$|\boldsymbol{\sigma}_T| = F(\xi_n - \varepsilon\tilde{g}) \Rightarrow \boldsymbol{\zeta}_T = -\lambda\boldsymbol{\sigma}_T. \tag{2.14}$$

where $\tilde{g}$ is a given relative gap function.

(c) *Approximation of the contact conditions*

Since dealing with the inequality-type contact conditions (2.12)-(2.14) is still difficult, one can approximate them by an equation. In this paper we use the approximation

$$\boldsymbol{\sigma}_T = -F(\xi_n - \varepsilon\tilde{g})\frac{\boldsymbol{\zeta}_T}{(|\boldsymbol{\zeta}_T|^2 + \delta)^{1/2}}, \qquad \text{where } \delta > 0, \tag{2.15}$$

from Kuttler & Shillor (1999) . Convergence of solutions of the approximate problem to the solution of the exact problem as $\delta \to 0$ was proved in Kuttler & Shillor (1999) . In this paper we do not pass to the limit $\delta \to 0$. Instead, we fix a small $\delta > 0$ (which is a "good approximate model" in the above sense), and homogenize the problem with the simplified contact conditions (2.15). The dependence on δ is therefore suppressed in the remainder of this paper.

Next, we regularize the functions p, F. These functions, given by formulas similar to (2.7), are smoothed out near $z = 0$ (see Figure 3).

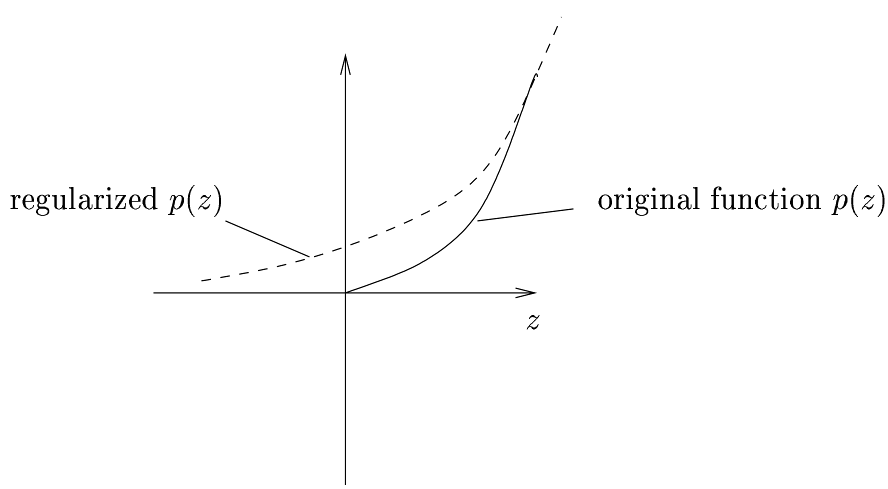

Figure 3. Original p is shown by a solid line, regularized p by a dashed line.

The regularized p and F are further approximated by polynomials. This leads to the simplified contact conditions

$$\boldsymbol{\sigma}_T = -F(\xi_n - \varepsilon\tilde{g})G(\boldsymbol{\zeta}_T)\boldsymbol{\zeta}_T, \tag{2.16}$$

in which p, F are polynomial functions, and $(|\boldsymbol{\zeta}_T|^2+\delta)^{-1/2}$ is also approximated by a polynomial function G of $\boldsymbol{\zeta}_T$. Since ξ_n (respectively, $\boldsymbol{\zeta}_T$) are linear functions of components of the vectors $\boldsymbol{\xi}$ ($\boldsymbol{\zeta}$), polynomial functions of $\xi_n - \varepsilon\tilde{g}$ are also polynomial functions of components of $\boldsymbol{\xi}$.

The vectors $\boldsymbol{\xi}, \boldsymbol{\zeta}$ depend on the small micro-structural parameter ε. In what follows we are going to use two-scale asymptotic approximations

$$\begin{aligned} \boldsymbol{\xi}(t,\mathbf{x},\varepsilon) &= \boldsymbol{\xi}^0(t,\mathbf{x}) + \varepsilon\boldsymbol{\xi}^1(t,\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon}),\\ \boldsymbol{\zeta}(t,\mathbf{x},\varepsilon) &= \boldsymbol{\zeta}^0(t,\mathbf{x}) + \varepsilon\boldsymbol{\zeta}^1(t,\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon}), \end{aligned} \tag{2.17}$$

where the leading terms $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$ satisfy an ε-independent effective equation, and $\boldsymbol{\xi}^1, \boldsymbol{\zeta}^1$ are correctors. This makes it natural to linearize (2.16) about effective $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$ to be specified later, and keep only terms of orders $\varepsilon^0, \varepsilon$. The chief purpose of linearization is to separate the first two terms in the asymptotics of $\boldsymbol{\sigma}\mathbf{n}$.

(*d*) *Microscale equations*

Writing the equation of motion (2.2) and the initial conditions (2.8) in terms of $\boldsymbol{\xi}, \boldsymbol{\zeta}$, and linearizing contact conditions (2.16) we obtain the microscale problem to be averaged. It consists of

1. Momentum balance equation for the matrix.

$$\dot{\boldsymbol{\zeta}} = div\ \boldsymbol{\sigma} + \mathbf{F} \quad \text{in } \Omega_M^\varepsilon. \tag{2.18}$$

2. Contact conditions on Γ^ε*.*

$$\begin{aligned} \sigma_n &= -p(\xi_n^0) - \varepsilon p'(\xi_n^0)(\xi_n^1 - \tilde{g}),\\ \boldsymbol{\sigma}_T &= -F(\xi_n^0)G(\boldsymbol{\zeta}_T^0)\boldsymbol{\zeta}_T^0 - \\ &\quad \varepsilon\left[F'(\xi_n^0)G(\boldsymbol{\zeta}_T^0)\boldsymbol{\zeta}_T^0 + F(\xi_n^0)\nabla G(\boldsymbol{\zeta}_T^0)\cdot\boldsymbol{\zeta}_T^1\boldsymbol{\zeta}_T^0 + F(\xi_n^0)G(\boldsymbol{\zeta}_T^0)\boldsymbol{\zeta}_T^1\right], \end{aligned} \tag{2.19}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}^0 + \varepsilon\boldsymbol{\xi}^1$, $\boldsymbol{\zeta} = \boldsymbol{\zeta}^0 + \varepsilon\boldsymbol{\zeta}^1$, (see (2.17).
3. Initial conditions.

$$\boldsymbol{\xi}(0,\mathbf{x}) = \mathbf{u}_0 - \mathbf{U}(0,\cdot), \qquad \boldsymbol{\zeta}(0,\mathbf{x}) = v_0 - \dot{\mathbf{U}}(0,\cdot). \tag{2.20}$$

In the equation (2.18),

$$\boldsymbol{\sigma} = Ae(\boldsymbol{\xi}) + Be(\boldsymbol{\zeta}), \tag{2.21}$$

and the non-homogeneous term is

$$\mathbf{F} = \mathbf{f} - \ddot{\mathbf{U}}. \tag{2.22}$$

To obtain (2.18), we used the fact that the gradients of $\mathbf{U}, \dot{\mathbf{U}}$ are constant.

On the outer boundary $\partial\Omega$ we prescribe periodic boundary conditions.

3. Two-scale expansions and formal homogenization.

The starting point of the formal homogenization procedure (Sanchez-Palencia (1980)) is to postulate two-scale asymptotic expansions

$$\begin{aligned}\boldsymbol{\xi}(t,\mathbf{x},\varepsilon) &= \boldsymbol{\xi}^0(t,\mathbf{x}) + \varepsilon\boldsymbol{\xi}^1(t,\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon}) + \varepsilon^2\boldsymbol{\xi}^2(t,\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon}) + \cdots,\\ \boldsymbol{\zeta}(t,\mathbf{x},\varepsilon) &= \boldsymbol{\zeta}^0(t,\mathbf{x}) + \varepsilon\boldsymbol{\zeta}^1(t,\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon}) + \varepsilon^2\boldsymbol{\zeta}^2(t,\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon}) + \cdots,\end{aligned} \tag{3.1}$$

where $\boldsymbol{\xi}^k, \boldsymbol{\zeta}^k$ are periodic functions of the *fast variable* $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$. Since

$$\frac{\partial f}{\partial x_k}(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}) = (\partial_{x_k} + \frac{1}{\varepsilon}\partial_{y_k})f(t,\mathbf{x},\mathbf{y})_{|\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}},$$

we can write

$$e(\boldsymbol{\zeta}) = e_x(\boldsymbol{\zeta}^0) + e_y(\boldsymbol{\zeta}^1) + \varepsilon[e_x(\boldsymbol{\zeta}^1) + e_y(\boldsymbol{\zeta}_2)] + \cdots \tag{3.2}$$

$$e(\boldsymbol{\xi}) = e_x(\boldsymbol{\xi}^0) + e_y(\boldsymbol{\xi}^1) + \varepsilon[e_x(\boldsymbol{\xi}^1) + e_y(\boldsymbol{\xi}_2)] + \cdots \tag{3.3}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \varepsilon\boldsymbol{\sigma}^1 + \varepsilon^2\boldsymbol{\sigma}^2 + \cdots, \tag{3.4}$$

$$\boldsymbol{\sigma}^0 = Ae_x(\boldsymbol{\xi}^0) + Ae_y(\boldsymbol{\xi}^1) + Be_x(\boldsymbol{\zeta}^0) + Be_y(\boldsymbol{\zeta}^1) \tag{3.5}$$

$$\boldsymbol{\sigma}^1 = Ae_x(\boldsymbol{\xi}^1) + Ae_y(\boldsymbol{\xi}_2) + Be_x(\boldsymbol{\zeta}^1) + Be_y(\boldsymbol{\zeta}_2). \tag{3.6}$$

Both sides in (3.2)-(3.6) are functions of $t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}$. Substituting (3.2)-(3.6) into the momentum equation (2.18) and keeping only the terms of orders $\varepsilon^{-1}, \varepsilon^0$ we obtain the equations

$$div_y\boldsymbol{\sigma}^0 = 0, \tag{3.7}$$

(term of oder ε^{-1}), and

$$\dot{\boldsymbol{\zeta}}^0 = div_x\boldsymbol{\sigma}^0 + div_y\boldsymbol{\sigma}^1 + \mathbf{F}, \tag{3.8}$$

(term of order ε^0). The contact conditions for $\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1$ are obtained from (2.19):

$$\boldsymbol{\sigma}^0\mathbf{n} = -p(\boldsymbol{\xi}^0\cdot\mathbf{n})\mathbf{n} - F(\boldsymbol{\xi}^0\cdot\mathbf{n})G(\zeta^0_T)\zeta^0_T, \tag{3.9}$$

$$\begin{aligned}\boldsymbol{\sigma}^1\mathbf{n} \;=\; & -p'(\boldsymbol{\xi}^0\cdot\mathbf{n})(\xi^1_n - \tilde{g})\mathbf{n} - F'(\boldsymbol{\xi}^0\cdot\mathbf{n})G(\zeta^0_T)\zeta^0_T -\\ & F(\boldsymbol{\xi}^0\cdot\mathbf{n})\nabla G(\zeta^0_T)\cdot\zeta^1\zeta^0_T - F(\boldsymbol{\xi}^0\cdot\mathbf{n})G(\zeta^0_T)\zeta^1_T.\end{aligned} \tag{3.10}$$

The next step is to "lift" (3.7)-(3.10) to a space of double dimensions so that $\frac{\mathbf{x}}{\varepsilon}$ is replaced by the independent of $\mathbf{x}$ fast variable $\mathbf{y}$. This is useful for two reasons. One is that $\mathbf{x}$ and $\mathbf{y}$ separate, and the second is that geometry of the problem becomes independent of ε. In the equations (3.7) and (3.8), $\mathbf{x}$ is a point in Ω^ε_M, a perforated domain with perforations dependent on ε. In the lifted equations, $\mathbf{x} \in \Omega$ (domain with no holes), and $\mathbf{y} \in Y$ (unit matrix cell with a hole, see Fig. 1).

Using formulas (3.5), (3.6), from (3.7) and (3.9) we obtain

$$div_y\left[Ae_y(\boldsymbol{\xi}^1) + Be_y(\boldsymbol{\zeta}^1)\right] = 0, \quad \text{in } \Omega\times Y\times[0,T], \tag{3.11}$$

$$\begin{aligned}(Ae_y(\boldsymbol{\xi}^1) + Be_y(\boldsymbol{\zeta}^1))\mathbf{n}(\mathbf{y}) \;=\; & -\left[Ae_x(\boldsymbol{\xi}^0) + Be_x(\boldsymbol{\zeta}^0)\right](t,\mathbf{x})\mathbf{n}(\mathbf{y}) -\\ & p(\boldsymbol{\xi}^0(t,\mathbf{x})\cdot\mathbf{n}(\mathbf{y}))\mathbf{n}(\mathbf{y}) - F(\xi^0_n)G(\zeta^0_T)\zeta^0_T,\\ & \mathbf{y}\in\Gamma_Y,\ \mathbf{x}\in\Omega,\ t\in[0,T].\end{aligned} \tag{3.12}$$

For future reference, we also state the weak formulation of (3.11), (3.12):

$$\int_0^T \int_Y (Ae_y(\boldsymbol{\xi}^1) + Be_y(\boldsymbol{\zeta}^1)) \cdot e(\mathbf{w}) d\mathbf{y} dt + \int_0^T \int_{\Gamma_Y} (Ae_x(\boldsymbol{\xi}^0) + Be_x(\boldsymbol{\zeta}^0))\mathbf{n} \cdot \mathbf{w} d\alpha dt + \\ \int_0^T \int_{\Gamma_Y} [p(\boldsymbol{\xi}^0 \cdot \mathbf{n})\mathbf{n} + F(\boldsymbol{\xi}^0 \cdot \mathbf{n}) G(\boldsymbol{\zeta}^0_T)\boldsymbol{\zeta}^0_T] \cdot \mathbf{w} d\alpha dt = 0, \tag{3.13}$$

for each smooth function $\mathbf{w}(t, \mathbf{y})$, vanishing at $t = T$ ands satisfying periodic boundary conditions on ∂Y.

The unknowns in (3.11), (3.12) and (3.13) are $\boldsymbol{\xi}^1, \boldsymbol{\zeta}^1$, viewed as functions of $\mathbf{y}$ which also depend on the parameters $t, \mathbf{x}$.

We use (3.11), (3.12) to express $\boldsymbol{\xi}^1, \boldsymbol{\zeta}^1$ in terms of $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$. This is done by first postulating a representation (ansatz)

$$\boldsymbol{\xi}^1(t, \mathbf{x}, \mathbf{y}) = \mathbf{A}(t, \mathbf{y})\boldsymbol{\xi}^0(t, \mathbf{x}), \tag{3.14}$$

where $\mathbf{A}$ is an operator involving several unknown *fast-variable functions* of $t, \mathbf{y}$. If the ansatz is chosen correctly, then after substitution of (3.14), $\mathbf{x}$ and $\mathbf{y}$ in (3.11), (3.12) separate. Next, one requires that (3.11),(3.12) hold for any choice of $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$. Then the *slow-variable functions* containing components of $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$ can be "factored out", and (3.11), (3.12) reduces to several cell problems for fast-variable functions.

Despite the fact that the homogenization procedure outlined above is well known (Sanchez-Palencia (1980)), its implementation for the problem at hand is not trivial. The choice of the ansatz is the heart of the matter. It should reflect the nature of the problem, in particular the contact conditions (3.12). There are no general methods for selecting the structure of $\mathbf{A}$ in (3.14). Direct computation shows that the standard homogenization substitution

$$\boldsymbol{\xi}^1 = A_j(\mathbf{y})\partial_{x_j}\boldsymbol{\xi}^0(\mathbf{x}, t) \tag{3.15}$$

does not work. The difficulty arises because of nonlinear (in $\boldsymbol{\xi}^0$) and dissipative nature of (3.12).

From the mathematical point of view, a good ansatz should lead to separation of fast and slow variables in (3.11), (3.12), so that solving these equations is reduced to solving well-posed cell problems for the functions which form the kernel of $\mathbf{A}$.

From the point of view of continuum mechanics, the choice is motivated by the following principle:
Instantaneous dissipation on the microscale leads to history-dependent dissipation on the macroscale.
This is intuitively clear, since for small ε, the particles become more numerous, and the distance between them goes to zero, so that frictional dissipation on the surface of one particle is likely to influence the energy dissipation on the surface of several neighboring particles.

With this in mind, consider the right hand side of (3.12). It can be written as

$$\left[-Ae(\boldsymbol{\xi}^0) - Be(\boldsymbol{\zeta}^0)\right](t, \mathbf{x})\mathbf{n}(\mathbf{y}) + \sum_j g_j(\mathbf{y}) f_j(t, \mathbf{x}), \tag{3.16}$$

where the slow-variable functions f_j are products of powers of components of $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$. Evidently, the substitution should contain the same slow-variable functions as (3.16), otherwise the variables will not separate. Moreover, the history dependence should be incorporated. Therefore, we introduce the following ansatz:

$$\boldsymbol{\xi}^1(t,\mathbf{x},\mathbf{y}) = K_1^{pq}(\mathbf{y})e(\boldsymbol{\xi}^0)_{pq} + \int_0^t K_2^{pq}(t-\tau,\mathbf{y})e(\boldsymbol{\zeta}^0)(\tau,\mathbf{x})d\tau + \sum_j \int_0^t M_j(t-\tau,\mathbf{y})f_j(\tau,\mathbf{x})d\tau, \tag{3.17}$$

where the fast-variable functions K_1^{pq}, K_2^{pq} and M_j should be determined by solving cell problems.

4. Analysis of the effective equation.

After the cell problems are solved, the effective equation for $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$ can be obtained from (3.8) by substituting (3.17) and applying the averaging operator $\langle\cdot\rangle$ defined by

$$\langle f\rangle(t,\mathbf{x}) = \frac{1}{|Y|}\int_Y f(t,\mathbf{x},\mathbf{y})d\mathbf{y}. \tag{4.1}$$

Averaging (3.8), integrating by parts in the term $div_y\boldsymbol{\sigma}^1$ and using periodicity on the external boundary of the unit cell, we obtain the effective equation

$$\dot{\boldsymbol{\zeta}}^0 = div_x\widehat{\mathbf{T}} + \widehat{\mathbf{S}} + \mathbf{F}, \tag{4.2}$$

where

$$\begin{aligned}\widehat{\mathbf{T}} &= \langle\boldsymbol{\sigma}^0\rangle,\\ \widehat{\mathbf{S}} &= \langle\boldsymbol{\sigma}^1\mathbf{n}\rangle_\Gamma.\end{aligned} \tag{4.3}$$

In the equation for $\widehat{\mathbf{S}}$, $\langle\cdot\rangle_\Gamma$ denotes the interface averaging operator

$$\langle f\rangle_\Gamma(t,\mathbf{x}) = \frac{1}{|Y|}\int_{\Gamma_Y} f(t,\mathbf{x},\mathbf{y})d\alpha(\mathbf{y}). \tag{4.4}$$

While $\widehat{\mathbf{T}}$ can be identified with the effective stress tensor, the term $\widehat{\mathbf{S}}$ represents the density of a body force, which we call generalized drag force. The constitutive equations for $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{S}}$ are given by (4.3).

Combining (3.17) with (3.5) and (3.10), one can make two conclusions.

i) The effective equations are nonlinear and history-dependent. The effective stress contains the same nonlinear functions of $\boldsymbol{\xi}^0$, $\boldsymbol{\zeta}^0$ as the contact conditions (3.9). The nonlinearities in the drag force are similar, but more complicated.

ii) The effective medium is not a simple material (see e.g. Truesdell (1997)), that is, both $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{S}}$ depend not only on $e(\boldsymbol{\xi}^0), e(\boldsymbol{\zeta}^0)$, but also on $\boldsymbol{\xi}^0$ and $\boldsymbol{\zeta}^0$.

5. Model case I: Linear contact conditions.

First we consider the case

$$p(\boldsymbol{\xi}^0) = 0, \quad F = 1, \quad G = const, \; G > 0, \tag{5.1}$$

so the contact conditions on Γ_Y are

$$\boldsymbol{\sigma}^0\mathbf{n} = -G\boldsymbol{\zeta}_T^0, \qquad \boldsymbol{\sigma}^1\mathbf{n} = -G\boldsymbol{\zeta}_T^1. \tag{5.2}$$

The suggested ansatz is as follows.

$$\boldsymbol{\xi}^1(t,\mathbf{x},\mathbf{y}) = K_1^{pq}(\mathbf{y})e(\boldsymbol{\xi}^0)_{pq}(t,\mathbf{x}) + \int_0^t K_2^{pq}(t-\tau,\mathbf{y})e(\partial_t\boldsymbol{\xi}^0)_{pq}(\tau,\mathbf{x})d\tau + \int_0^t M^p(t-\tau,\mathbf{y})\partial_t\xi_p^0(\tau,\mathbf{x})d\tau. \quad (5.3)$$

where the *scalars* $\xi_p^0, e(\boldsymbol{\xi}^0)_{pq}$ are the components of the vector $\boldsymbol{\xi}^0$ and tensor $e(\boldsymbol{\xi}^0)$ respectively, and K_1^{pq}, K_2^{pq}, M^p are *vectors* to be determined. We substitute (5.3) and (5.2) into (3.13) and group the terms containing the same slow-variable functions. Factoring out the slow-variable terms, we obtain the cell problems for determination of the fast-variable functions.

(*a*) *Cell Problems.*

The problem for K_1^{pq} is obtained collecting the terms containing $e(\boldsymbol{\xi}^0)(t,\mathbf{x})_{pq}$.

$$\int_Y A(e(K_1^{pq}) + I^{pq})(\mathbf{y}) \cdot e(\mathbf{w})(\mathbf{y})d\mathbf{y} = 0, \quad (5.4)$$

where $I_{kl}^{pq} = \delta_{kp}\delta_{lq}$, so that $A_{ijkl}I_{kl}^{pq} = A_{ijpq}$.
The problem for $K_2^{pq}(0,\mathbf{y})$**.** Collect terms containing $e(\partial_t\boldsymbol{\xi}^0)_{pq}(t,\mathbf{x})$.

$$\int_Y B(e(K_1^{pq} + I^{pq})(\mathbf{y}) \cdot e(\mathbf{w})(\mathbf{y})d\mathbf{y} + \int_Y Be(K_2^{pq}(0,\mathbf{y})) \cdot e(\mathbf{w})d\mathbf{y} = 0. \quad (5.5)$$

The problem for K_2^{pq}**.** Collect terms containing $e(\partial_t\boldsymbol{\xi}_0)_{pq}(\tau,\mathbf{x})$.

$$-\int_Y Be(K_2^{pq}(0,\mathbf{y}) \cdot e(\mathbf{w})d\mathbf{y}d - \int_0^T \int_Y Be(K_2^{pq}) \cdot \partial_t e(\mathbf{w}(t,\mathbf{y}))d\mathbf{y}dt + \int_0^T \int_Y Ae(K_2^{pq}) \cdot e(\mathbf{w}(t,\mathbf{y}))d\mathbf{y}dt = 0, \quad (5.6)$$

The problem for $M^p(0,\mathbf{y})$**.** Collect terms containing $\partial_t\xi_p^0(t,\mathbf{x})$.

$$\int_Y Be(M^p)(0,\mathbf{y}) \cdot e(\mathbf{w})d\mathbf{y} + \int_{\Gamma_Y} Gw_{p,T}d\alpha = 0. \quad (5.7)$$

The problem for $M^p(t,\mathbf{y})$**.** Collect terms containing $\partial_t\xi_p^0(\tau,\mathbf{x})$.

$$-\int_Y Be(M^p(0,\mathbf{y}) \cdot e(\mathbf{w})d\mathbf{y} - \int_0^T \int_Y Be(M^p) \cdot \partial_t e(\mathbf{w}(t,\mathbf{y}))d\mathbf{y}dt + \int_0^T \int_Y Ae(M^p) \cdot e(\mathbf{w})d\mathbf{y}dt = 0. \quad (5.8)$$

The details on the derivation of the cell problems are presented in Appendix A.

(b) Averaged Equations for the Model I

To obtain the averaged equations, we integrate (3.8) over the periodicity cell and divide by $|Y|$, (see equation (4.2)).

Using the contact condition (5.2) for $\boldsymbol{\sigma}_1$ and the formula (5.3) we obtain

$$\widehat{\mathbf{S}}(\boldsymbol{\zeta}^0) = -G\big(S_1^{pq}e(\boldsymbol{\zeta}^0)_{pq} + \int_0^t S_2^{pq}(t-\tau)e(\boldsymbol{\zeta}^0)_{pq}(\tau,x)d\tau + S_3^p\zeta_p^0 + \int_0^t S_4^p(t-\tau)\zeta_p^0(\tau,x)d\tau\big), \quad (5.9)$$

where

$$S_1^{pq} = \langle K_{1,T}^{pq} + K_{2,T}^{pq}(0,\cdot)\rangle_\Gamma \qquad S_2^{pq}(t) = \langle K_{2,T}^{pq}(t,\cdot)\rangle_\Gamma$$
$$S_3^p = \langle M_T^p(0,\cdot)\rangle_\Gamma \qquad S_4^p(t) = \langle \partial_t M_T^p(t,\cdot)\rangle_\Gamma. \quad (5.10)$$

The equation (5.9) shows that the drag force is history-dependent, and depends linearly on the averaged velocity $\boldsymbol{\zeta}^0$ and the averaged strain rate $e(\boldsymbol{\zeta}^0)$. The latter dependence shows that the drag force at point $\mathbf{x}$ is determined by the history of averaged velocities in a small neighborhood of $\mathbf{x}$, that is the drag force is non-local in space as well as in time.

Next we use (3.5) and (5.3) to calculate the effective stress:

$$\widehat{\mathbf{T}} = A(I+T_1)e(\boldsymbol{\xi}^0) + B(I+T_1)e(\boldsymbol{\zeta}^0) + (A+B\partial_t)\int_0^t T_2(t-\tau)e(\boldsymbol{\zeta}^0)(\tau,\cdot)d\tau + (A+B\partial_t)\int_0^t T_3(t-\tau)\boldsymbol{\zeta}^0(\tau,\cdot)d\tau, \quad (5.11)$$

where the components of $T_i, i=1,2,3$ are related to $K_{1,2}^{pq}, M^p$ as follows.

$$T_1^{klpq} = \langle e(K_1^{pq})_{kl}\rangle, \quad (5.12)$$

$$T_2^{pqkl} = \langle e(K_2^{pq})_{kl}(t-\tau,\cdot)\rangle, \quad (5.13)$$

$$T_3^{klp} = \langle e(M^p)_{kl}(t-\tau,\cdot)\rangle, \quad (5.14)$$

The constitutive equation (5.11) shows that the effective material is not simple, since the last two terms in (5.11) are velocity-dependent.

6. Model II: Quadratic contact conditions

In this case, we assume $p(\xi_n^0) = p\xi_n^0,\;\; F(\xi_n^0) = \xi_n^0,\;\; G(\boldsymbol{\zeta}_T^0) = G$, where $p>0, G>0$ are constants.

The contact conditions (3.9), (3.10) become

$$\boldsymbol{\sigma}^0\mathbf{n} = -p\xi_n^0\mathbf{n} - G\xi_n^0\boldsymbol{\zeta}_T^0, \quad (6.1)$$

and

$$\boldsymbol{\sigma}^1\mathbf{n} = -p(\xi_n^1 - g_0)\mathbf{n} - G(\xi_n^1-g_0)\boldsymbol{\zeta}_T^0 - G\xi_n^0\boldsymbol{\zeta}_T^1, \quad (6.2)$$

respectively. The fast-variable problem (3.13) becomes:

$$\int_0^T \int_{\Gamma_Y} p\xi_n^0 \mathbf{n} \cdot \mathbf{w} d\alpha dt + \int_0^T \int_{\Gamma_Y} G\xi_n^0 \boldsymbol{\zeta}_T^0 \cdot \mathbf{w} d\alpha dt$$
$$+ \int_0^T \int_Y [Ae_y(\boldsymbol{\xi}^1) + Be_y(\boldsymbol{\zeta}^1)] e(\mathbf{w}) d\mathbf{y} dt + \int_0^T \int_Y [Ae_x(\boldsymbol{\xi}^0) + Be_x(\boldsymbol{\zeta}^0)] e(\mathbf{w}) d\mathbf{y} dt = 0 \tag{6.3}$$

To specify the ansatz (3.17), we note that the contact condition (6.1) contains terms linear in $\boldsymbol{\xi}^0$ and quadratic terms of the form $\xi_p^0 \zeta_q^0$ (components of the dyadic $\boldsymbol{\xi}^0 \otimes \boldsymbol{\zeta}^0$). Thus the slow-variable terms in the ansatz for $\boldsymbol{\xi}^1$ should contain similar terms. Because the constitutive equations for the matrix contain $e(\boldsymbol{\xi})$ and $e(\boldsymbol{\zeta})$, the corresponding slow-variable terms must appear in the ansatz as well. Moreover, due to the dissipative nature of the contact conditions and the constitutive equations we expect the effective medium to be history-dependent. Therefore we choose the following substitution.

$$\boldsymbol{\xi}^1 = K_1^{pq}(\mathbf{y}) e(\boldsymbol{\xi}^0)_{pq} + \int_0^t K_2^{pq}(t-\tau, \mathbf{y}) e(\boldsymbol{\zeta}^0)_{pq}(\tau, \mathbf{x}) d\tau +$$
$$\int_0^t \xi_p^0(\tau, \mathbf{x}) M^{pq}(t-\tau, \mathbf{y}) \partial_t \xi_q^0(\tau, \mathbf{x}) d\tau + \int_0^t L^p(t-\tau, \mathbf{y}) \xi_p^0(\tau, \mathbf{x}) d\tau. \tag{6.4}$$

Substituting (6.4) into (6.3), and collecting slow-variable terms we get following cell problems.
The problem for K_1^{pq}. Collect the terms containing $e(\boldsymbol{\xi}^0)_{pq}(t, \mathbf{x})$.

$$\int_Y A(I^{pq} + e_y(K_1^{pq})) \cdot e(\mathbf{w}) d\mathbf{y} = 0. \tag{6.5}$$

The problem for $K_2^{pq}(0, \mathbf{y})$. Collect the terms containing $e(\partial_t \boldsymbol{\xi}^0)_{pq}(t, \mathbf{x})$.

$$\int_Y B(e(K_2^{pq}(0, \cdot)) + e(K_1^{pq}) + I^{pq}) \cdot e(\mathbf{w}) d\mathbf{y} = 0. \tag{6.6}$$

The problem for $K_2^{pq}(t, \mathbf{y})$. Collect the terms containing $e(\partial_t \boldsymbol{\xi}^0)_{pq}(\tau, \mathbf{x})$.

$$\int_0^T \int_Y (Ae_y(K_2^{pq}) + Be_y(\partial_t K_2^{pq})) \cdot e(\mathbf{w}) d\mathbf{y} dt = 0. \tag{6.7}$$

The initial condition is given by the solution of the cell problem (6.6).
The problem for $M^{pq}(0, \mathbf{y})$. Collect the terms containing $\boldsymbol{\xi}_p^0 \partial_t \xi_q^0(t, \mathbf{x})$.

$$\int_Y Be_y(M^{pq}(0, \cdot)) \cdot e\mathbf{w}) + \int_{\Gamma_Y} Gn_p w_{T,q} d\alpha = 0. \tag{6.8}$$

The problem for $M^{pq}(t, \mathbf{y})$. Collect the terms containing $\xi_p^0 \partial_t \xi_q^0(\tau, \mathbf{x})$.

$$\int_0^T \int_Y (Ae_y(M^{pq}) + Be_y(\partial_t M^{pq})) \cdot e(\mathbf{w}) d\mathbf{y} = 0 \tag{6.9}$$

The initial condition is given by the solution of (6.8).
The problem for $L^p(0,\mathbf{y})$. Collect the terms containing $\xi_p^0(t,\mathbf{x})$.

$$\int_Y Be_y(L^p(0,\cdot))\cdot e(\mathbf{w})d\mathbf{y} + \int_{\Gamma_Y} pn_p\mathbf{n}\cdot\mathbf{w}d\alpha = 0. \tag{6.10}$$

The problem for $L^p(t,\mathbf{y})$. Collect the terms containing $\xi_p^0(\tau,\mathbf{x})$.

$$\int_0^T\int_Y (Ae_y(L^p) + Be_y(\partial_t L^p))\cdot e(\mathbf{w})d\mathbf{y} = 0 \tag{6.11}$$

The initial condition is given by the solution of (6.10).

(a) Averaged equation for the model II

Constitutive equation for the effective stress. We use (3.5) and (6.4) to express $\boldsymbol{\sigma}^0$ in terms of $\boldsymbol{\xi}^0, \boldsymbol{\zeta}^0$. Averaging out the fast variable we obtain

$$\widehat{\mathbf{T}} = A(T_1+I)e(\boldsymbol{\xi}^0) + B(T_1+I)e(\boldsymbol{\zeta}^0) + (A+B\partial_t)\int_0^t T_2(t-\tau)e(\boldsymbol{\zeta}^0)(\tau,\cdot)d\tau +$$
$$(A+B\partial_t)\int_0^t T_3(t-\tau)(\boldsymbol{\xi}^0\otimes\boldsymbol{\zeta}^0)(\tau,\cdot)d\tau + (A+B\partial_t)\int_0^t T_4(t-\tau)\boldsymbol{\zeta}^0(\tau,\cdot)d\tau, \tag{6.12}$$

where

$$T^1_{pqkl} = \langle e(K_1^{pq})_{kl}\rangle, \qquad T^2_{pqkl} = \langle e(K_2^{pq})_{kl}\rangle, \tag{6.13}$$

$$T^3_{pqkl} = \langle e(M^{pq})_{kl}\rangle, \qquad T^4_{pkl} = \langle\int_Y e(L^p)_{kl}\rangle, \tag{6.14}$$

The effective drag force. Using the contact conditions (6.2) we write

$$\widehat{\mathbf{S}} = \langle -p(\xi_n^1 - g^0)\mathbf{n} + G(\xi_n^1 - g^0)\boldsymbol{\zeta}_T^0 + G\xi_n^0\boldsymbol{\zeta}_T^1\rangle_\Gamma.$$

We make use of the formula $\boldsymbol{\zeta}_T = (\mathbf{n}\times\boldsymbol{\zeta})\times\mathbf{n}$ for the "tangential projection" of a vector $\boldsymbol{\zeta}$. Componentwise,

$$\zeta_{T,l} = \epsilon_{jkl}\epsilon_{jus}n_k n_u\zeta_s = \zeta_s\mathcal{N}_{ls}. \tag{6.15}$$

Next, we make use of the formula (6.4) together with (6.15) to obtain

$$\langle -G(\xi_n^1\zeta_{T,l}^0 + \xi_n^0\zeta_{T,l}^1\rangle_\Gamma = \partial_t\big[\xi_s^0 e(\boldsymbol{\xi}^0)_{pq}W^1_{pqls} +$$
$$\int_0^t \xi_s^0(t,\cdot)e(\boldsymbol{\zeta}^0)_{pq}(\tau,\cdot)W^2_{pqls}(t-\tau)d\tau + \int_0^t \xi_s^0(t,\cdot)\xi_p^0\zeta_q^0(\tau,\cdot)W^3_{pqls}(t-\tau)d\tau +$$
$$\int_0^t \xi_s^0(t,\cdot)\xi_p^0(\tau,\cdot)W^4_{pls}(t-\tau)d\tau\big], \tag{6.16}$$

where

$$W^1_{pqls} = \langle -G\mathcal{N}_{ls}n_m K^{pq}_{1,m}\rangle_\Gamma, \qquad W^2_{pqls} = \langle -G\mathcal{N}_{ls}n_m K^{pq}_{2,m}\rangle_\Gamma, \tag{6.17}$$

$$W^3_{pqls} = \langle -G\mathcal{N}_{ls} n_m M^{pq}_m \rangle_\Gamma, \qquad W^4_{pls} = \langle -G\mathcal{N}_{ls} n_m L^p_m \rangle_\Gamma, \tag{6.18}$$

Similarly,

$$\langle -p(\xi^1_n - g^0)n_l - Gg^0\zeta^0_{T,l}\rangle_\Gamma = V^1_l + \zeta^0_s V^2_{ls} + e(\boldsymbol{\xi}^0)_{pq} V^3_{pqls} + \int_0^t e(\boldsymbol{\zeta}^0)_{pq}(\tau,\cdot) V^4_{pqls}(t-\tau)d\tau + \int_0^t \xi^0_p(\tau,\cdot)\zeta^0_q(\tau,\cdot) V^5_{pqls}(t-\tau)d\tau + \int_0^t \xi^0_p(\tau,\cdot) V^6_{pls}(t-\tau)d\tau, \tag{6.19}$$

where

$$V^1_l = \langle pg^0 n_l\rangle_\Gamma, \qquad V^2_{ls} = \langle Gg^0\mathcal{N}_{ls}\rangle_\Gamma,$$
$$V^3_{pql} = \langle -pK^{pq}_{1,m} n_m n_l\rangle_\Gamma, \qquad V^4_{pql} = \langle -pK^{pq}_{2,m} n_m n_l\rangle,$$
$$V^5_{pql} = \langle -pM^{pq}_m n_m n_l\rangle, \qquad V^6_{pl} = \langle -pL^p_m n_m n_l\rangle_\Gamma, \tag{6.20}$$

Combining (6.16) and (6.19) we obtain

$$\widehat{S}_l = V^1_l + \zeta^0_s V^2_{ls} + e(\boldsymbol{\xi}^0)_{pq} V^3_{pql} + \int_0^t e(\boldsymbol{\zeta}^0)_{pq}(\tau,\cdot) V^4_{pql}(t-\tau)d\tau + \int_0^t \xi^0_p(\tau,\cdot)\zeta^0_q(\tau,\cdot) V^5_{pql}(t-\tau)d\tau + \int_0^t \xi^0_p(\tau,\cdot) V^6_{pl}(t-\tau)d\tau + \partial_t\big[\xi^0_s\big\{e(\boldsymbol{\xi}^0)_{pq} W^1_{pqls} + \int_0^t e(\boldsymbol{\zeta}^0)_{pq}(\tau,\cdot) W^2_{pqls}(t-\tau)d\tau + \int_0^t \xi^0_p\zeta^0_q(\tau,\cdot) W^3_{pqls}(t-\tau)d\tau + \int_0^t \xi^0_p(\tau,\cdot) W^4_{pls}(t-\tau)d\tau\big\}\big]. \tag{6.21}$$

7. Model III: Power type contact condition

(a) Contact conditions, ansatz and cell problems.

In this case we take $p(z) = z^\gamma, F(z) = z^\beta$, where $0 < \gamma < \beta$ are integers. We also linearize the δ-approximation:

$$\frac{\boldsymbol{\zeta}_T}{(|\zeta_T|^2 + \delta)^{1/2}} \approx R\boldsymbol{\zeta}_T + \mathbf{r},$$

where R and r are constant matrix and vector, respectively. The contact conditions are

$$\boldsymbol{\sigma}^0\mathbf{n} = -(\xi^0_n)^a\mathbf{n} - (\xi^0_n)^b(R\boldsymbol{\zeta}_T + \mathbf{r}), \tag{7.1}$$

$$\boldsymbol{\sigma}^1\mathbf{n} = -a(\xi^0_n)^{a-1}(\xi^1_n - g_0)\mathbf{n} - b(\xi^0_n)^{b-1}(\xi^1_n - g_0)(R\boldsymbol{\zeta}^0_T + \mathbf{r}) - (\xi^0_n)^b R\boldsymbol{\zeta}^1_T, \tag{7.2}$$

where $a > 1, b > a$ are integers.

The right hand side of (7.1) is a sum of terms of two types:

$$C_a {\xi^0}^{a_1}_{k_1} {\xi^0}^{a_2}_{k_2} {\xi^0}^{a_3}_{k_3}, \qquad C_{b,l} {\xi^0}^{b_1}_{k_1} {\xi^0}^{b_2}_{k_2} {\xi^0}^{b_3}_{k_3} {\zeta^0}_l, \tag{7.3}$$

where the non-negative integers a_1, a_2, a_3 and b_1, b_2, b_3 satisfy $a_1 + a_2 + a_3 = a$ and $b_1 + b_2 + b_3 = b$, respectively, and (k_1, k_2, k_3) is a permutation of $(1, 2, 3)$.

To keep track of numerous exponents and indices, it is convenient to use multi-index notation. In the multi-index notation, the expressions (7.3) are written as $C_a {\xi^0}^\alpha_{\mathcal{K}}$, $C_b {\xi^0}^\beta_{\mathcal{K}}$ with the multi-indices $\alpha = a_1a_2a_3$, $|\alpha| = a_1 + a_2 + a_3 = a$, $\beta = b_1b_2b_3$, $|\beta| = b_1 + b_2 + b_3 = b$, and $\mathcal{K} = k_1k_2k_3$. The whole contact condition (7.1) becomes

$$\boldsymbol{\sigma}^0 \mathbf{n} = \sum_{|\alpha|=0}^{a} C_\alpha {\xi^0}^\alpha \mathbf{n} + \sum_{|\beta|\le b} C_\beta {\xi^0}^\beta \boldsymbol{\zeta}^0 + \sum_{|\beta|\le b} D_\beta {\xi^0}^\beta \mathbf{r}, \tag{7.4}$$

where $C_\alpha, C_\beta, D_\beta$ depend on (powers of) components of $\mathbf{n}$, components of R and binomial coefficients.

The contact condition (7.2) is even more complicated. We represent it schematically as

$$(\boldsymbol{\sigma}^1 \mathbf{n})_l = \sum_{\mu,\nu,q} {\xi^0}^\mu {\zeta^0}^\nu \mathcal{A}^{ql}_{\mu\nu}(\mathbf{y}) \xi^1_q + \sum_{\mu,\nu,q} {\xi^0}^\mu {\zeta^0}^\nu \mathcal{B}^{ql}_{\mu\nu}(\mathbf{y}) \zeta^1_q + \sum_{\mu,\nu,q} {\xi^0}^\mu {\zeta^0}^\nu \mathcal{C}_{\mu\nu}(\mathbf{y})_l, \quad l = 1,2,3, \tag{7.5}$$

where multi-indices μ, ν are such that $|\mu| \le (b-1)$, $|\nu| \le 1$ and $q = 1, 2, 3$. The factors $\mathcal{A}, \mathcal{B}, \mathcal{C}$ depend on components of $\mathbf{n}$ R, $\mathbf{r}$ and binomial coefficients.

The relation between $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^0$. Motivated by (7.4), we choose the relation between $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^0$ as follows.

$$\boldsymbol{\xi}^1 = K_1^{pq} e(\boldsymbol{\xi}^0)_{pq} + \int_0^t K_2^{pq}(t-\tau) e(\boldsymbol{\zeta}^0)_{pq}(\tau) d\tau +$$
$$\sum_{\substack{|\beta| \le b \\ \mathcal{K}}} \int_0^t M_\beta^{\mathcal{K},q}(t-\tau) {\xi^0}^\beta_{\mathcal{K}} \zeta^0_q(\tau) d\tau + \sum_{\substack{|\alpha| \le b, \\ \mathcal{K}}} \int_0^t L_\alpha^{\mathcal{K}}(t-\tau) {\xi^0}^\alpha_{\mathcal{K}}(\tau) d\tau, \tag{7.6}$$

where the summation is over p, q, multiindices α, β and $\mathcal{K}$. In particular, $\mathcal{K}$ runs over all permutations of $1, 2, 3$.

As before, the vectors $K_1^{pq}, K_{pq}^2, M_\alpha^{\mathcal{K},q}, L_\beta^{\mathcal{K}}$ are found by solving the cell problems. The latter are obtained on substituting (7.6) into the fast-variable problem (3.13) and collecting the terms containing the same slow-variable functions.

The problems for K_1^{pq}, K_2^{pq} are just (6.5)-(6.7). The other cell problems have the structure similar to see (6.9), (6.11). This structure can be described as follows. Let $X(t, \mathbf{y})$ denote any of the vectors $M_\alpha^{\mathcal{K},q}, L_\beta^{\mathcal{K}}$. The evolutionary cell problem for $e(X)$ is (the weak formulation of) a linear system of ordinary differential equations with respect to time, namely

$$Ae(X) + B\partial_t e(X) = 0, \tag{7.7}$$

where A, B are the constitutive tensors of the matrix. The initial condition $e(X(0, \mathbf{y}))$ for (7.7) is found by solving another cell problem. The latter is of the form (compare with (6.8), (6.10))

$$div(Be(X(0, \mathbf{y}))) = F_X, \tag{7.8}$$

where F_X is determined for each particular X by the contact conditions (7.1). Specifically, each X in (7.6) is multiplied by the corresponding (unique) slow-variable function $Sl_X(\boldsymbol{\xi}^0)(t, x)$. The right hand side of (7.8) is determined by the coefficient at Sl_X in (7.4).

(b) Effective stress.

Let us denote the solution operator to (7.7)by $e^{-B^{-1}At}$. Then

$$e(X(t,\mathbf{y})) = e^{-B^{-1}At}e(X(0,\mathbf{y})), \tag{7.9}$$

where $X(0,\mathbf{y})$ solves (7.8). The term corresponding to X in the expression (7.6) for $\boldsymbol{\xi}^1$ will be of the form

$$\int_0^t X(t-\tau,\mathbf{y})Sl_X(\boldsymbol{\xi}^0)(\tau,x)d\tau. \tag{7.10}$$

Next, we determine the term corresponding to X in the expression (3.5) for $\boldsymbol{\sigma}^0$. Applying the differential operator e_y to both sides of (7.10) and taking into account (7.9) we obtain

$$\int_0^t e^{B^{-1}A(t-\tau)}e_y(X(0,\mathbf{y}))Sl_X(\boldsymbol{\xi}^0)(\tau,x)d\tau. \tag{7.11}$$

The contribution of the terms containing X to the effective stress tensor will be written as

$$(A+B\partial_t)\int_0^t e^{B^{-1}A(t-\tau)}\langle e_y(X(0,\mathbf{y}))\rangle Sl_X(\boldsymbol{\xi}^0)(\tau,x)d\tau, \tag{7.12}$$

Using the formula (7.11) for each of the terms in the expression $\boldsymbol{\sigma}^0 = Ae_x(\boldsymbol{\xi}^0)+Be_x(\boldsymbol{\zeta}^0)+Ae_y(\boldsymbol{\xi}^1)+Be_y(\boldsymbol{\zeta}^1)$ we obtain the constitutive equation for the effective stress

$$\begin{aligned}\widehat{T} \;=\; & A(T^1+I)e(\boldsymbol{\xi}^0)+B(T^1+I)e(\boldsymbol{\zeta}^0)+(A+B\partial_t)\left[\textstyle\int_0^t T^2(t-\tau)e(\boldsymbol{\xi}^0)(\tau,\cdot)d\tau\right]+\\ & (A+B\partial_t)\left[\sum_{\mathcal{K},\beta,q}\int_0^t T_\beta^{\mathcal{K},q}(t-\tau){\xi^0}_{\mathcal{K}}^{\beta}{\zeta^0}_q(\tau,\cdot)d\tau+\sum_{\mathcal{K},\alpha}\int_0^t T_\alpha^{\mathcal{K}}(t-\tau){\xi^0}_{\mathcal{K}}^{\alpha}d\tau\right].\end{aligned} \tag{7.13}$$

where

$$\begin{aligned}&T^1_{pqkl} = \langle e(K_1^{pq})_{kl}\rangle, \qquad T^2_{pqkl} = \langle e(K_2^{pq})_{kl}\rangle,\\ &T_\beta^{\mathcal{K},q} = \langle e^{-B^{-1}A(t-\tau)}M_\beta^{\mathcal{K},q}(0,t-\tau))\rangle,\\ &T_\alpha^{\mathcal{K}} = \langle e^{-B^{-1}A(t-\tau)}M_\alpha^{\mathcal{K}}(0,t-\tau))\rangle.\end{aligned} \tag{7.14}$$

(c) Effective drag force

To obtain the effective drag force, we substitute $\boldsymbol{\xi}^1,\boldsymbol{\zeta}^1$ determined by (7.6) into the condition (7.5) and average in $\mathbf{y}$ over Γ_Y. The resulting, rather complicated, equation is of the form

$$\widehat{\mathbf{S}} = \widehat{S}^1+\widehat{S}^3+\widehat{S}^3 \tag{7.15}$$

where $\widehat{S}^m, m=1,2,3$ correspond to the three terms in (7.5), namely

$$\begin{aligned}\widehat{S}^1_l \;=\; & \sum_{\mu,\nu,s}{\xi^0}^{\mu}{\zeta^0}^{\nu}(t,\mathbf{x})\Bigg[V^{1,l}_{\mu\nu}e(\boldsymbol{\xi}^0)(t,\mathbf{x})+\int_0^t V^{2,l}_{\mu\nu}(t-\tau)e(\boldsymbol{\zeta}^0)(\tau,\mathbf{x})d\tau+\\ & \sum_{|\beta|\le b,\mathcal{K}}\int_0^t V^{3,l}_{\mu\nu,\beta,\mathcal{K}}(t-\tau){\xi^0}^{\beta}_{\mathcal{K}}\zeta^0_q(\tau,x)d\tau+\sum_{|\alpha|\le b,\mathcal{K}}\int_0^t V^{4,l}_{\mu\nu,\alpha,\mathcal{K}}(t-\tau){\xi^0}^{\alpha}_{\mathcal{K}}(\tau,x)d\tau,\Bigg]\end{aligned} \tag{7.16}$$

where

$$
\begin{array}{ll}
V^{1,lpq}_{\mu\nu} = \langle \mathcal{A}^{sl}_{\mu\nu} K^{pq}_{1,s}\rangle_\Gamma, & V^{2,lpq}_{\mu\nu} = \langle \mathcal{A}^{sl}_{\mu\nu} K^{pq}_{2,s}\rangle_\Gamma, \\
V^{3,lq}_{\mu\nu,\beta,\mathcal{K}} = \langle \mathcal{A}^{sl}_{\mu\nu} M^{\mathcal{K}}_{\beta,s}\rangle_\Gamma, & V^{4,l}_{\mu\nu,\beta,\mathcal{K}} = \langle \mathcal{A}^{sl}_{\mu\nu} L^{\mathcal{K}}_{\alpha,s}\rangle_\Gamma.
\end{array}
\tag{7.17}
$$

Similarly, $\widehat{S}^2$ corresponds to the second term in the right hand side of (7.5).

$$
\begin{aligned}
\widehat{S}^2_l \;=\; & \sum_{\mu,\nu,s} \xi^{0\,\mu}\zeta^{0\,\nu}(t,\mathbf{x})\partial_t \Bigg[W^{1,l}_{\mu\nu} e(\boldsymbol{\xi}^0)(t,\mathbf{x}) + \int_0^t W^{2,l}_{\mu\nu}(t-\tau)e(\boldsymbol{\zeta}^0)(\tau,\mathbf{x})d\tau + \\
& \sum_{|\beta|\le b,\mathcal{K}} \int_0^t W^{3,l}_{\mu\nu,\beta,\mathcal{K}}(t-\tau)\xi^{0\,\beta}_{\ \mathcal{K}}\zeta^0_q(\tau,\mathbf{x})d\tau + \sum_{|\alpha|\le b,\mathcal{K}} \int_0^t W^{4,l}_{\mu\nu,\alpha,\mathcal{K}}(t-\tau)\xi^{0\,\alpha}_{\ \mathcal{K}}(\tau,\mathbf{x})d\tau, \Bigg]
\end{aligned}
\tag{7.18}
$$

where

$$
\begin{array}{ll}
W^{1,lpq}_{\mu\nu} = \langle \mathcal{B}^{sl}_{\mu\nu} K^{pq}_{1,s}\rangle_\Gamma, & W^{2,lpq}_{\mu\nu} = \langle \mathcal{B}^{sl}_{\mu\nu} K^{pq}_{2,s}\rangle_\Gamma, \\
W^{3,lq}_{\mu\nu,\beta,\mathcal{K}} = \langle \mathcal{B}^{sl}_{\mu\nu} M^{\mathcal{K}}_{\beta,s}\rangle_\Gamma, & W^{4,l}_{\mu\nu,\beta,\mathcal{K}} = \langle \mathcal{B}^{sl}_{\mu\nu} L^{\mathcal{K}}_{\alpha,s}\rangle_\Gamma.
\end{array}
\tag{7.19}
$$

Finally, $\widehat{S}^3$ is simply

$$
\widehat{S}^3_l = \sum_{\mu,\nu,q} \xi^{0\,\mu}\zeta^{0\,\nu}(t,\mathbf{x})\langle \mathcal{C}_{\mu\nu,l}\rangle_\Gamma. \tag{7.20}
$$

8. Appendix

Here we derive the cell problem for K^{pq}_2 assuming that the contact conditions are given by (5.2) (Model I). Substituting the ansatz (5.3) into the weak formulation (3.13) of the fast- variable problem, we try to choose K^{pq}_2 so that the sum of all terms containing $e(\boldsymbol{\zeta}^0)_{pq}(\tau,\mathbf{x})$ is zero, that is

$$
\int_Y \int_0^T \int_0^t \left[Ae(K^{pq}_2) + Be(\partial_t K^{pq}_2)\right](t-\tau,\mathbf{y})e(\boldsymbol{\zeta}^0)_{pq}(\tau,\mathbf{x}) \cdot e(\mathbf{w})(t,\mathbf{y})d\tau dt d\mathbf{y} = 0. \tag{8.1}
$$

Interchange the time-integrations in (8.1):

$$
\int_Y \int_0^T \int_\tau^T \left[Ae(K^{pq}_2) + Be(\partial_t K^{pq}_2)\right](t-\tau,\mathbf{y})e(\boldsymbol{\zeta}^0)_{pq}(\tau,\mathbf{x}) \cdot e(\mathbf{w})(t,\mathbf{y})dt d\tau d\mathbf{y} = 0. \tag{8.2}
$$

Next, replace (8.2) with

$$
\int_Y \int_\tau^T \left[Ae(K^{pq}_2) + Be(\partial_t K^{pq}_2)\right](t-\tau,\mathbf{y})\cdot e(\mathbf{w})(t,\mathbf{y})dt d\mathbf{y} = 0, \quad \text{for almost all } \tau \in [0,T). \tag{8.3}
$$

Clearly, if K^{pq}_2 satisfies (8.3), it also satisfies (8.2). For a fixed τ, replace t in (8.3) with $s = t-\tau$. Then $K^{pq}_2(s,\mathbf{y})$ must satisfy

$$
\int_Y \int_0^{T-\tau} \left[Ae(K^{pq}_2) + Be(\partial_s K^{pq}_2)\right](s,\mathbf{y})\cdot e(\mathbf{w})(s+\tau,\mathbf{y})ds d\mathbf{y} = 0, \quad \text{for almost all } \tau \in [0,T). \tag{8.4}
$$

Instead of (8.4) consider

$$
\int_Y \int_0^T \left[Ae(K^{pq}_2) + Be(\partial_s K^{pq}_2)\right](s,\mathbf{y}) \cdot e(\mathbf{w})(s,\mathbf{y})ds d\mathbf{y} = 0. \tag{8.5}
$$

Fix τ and pick an arbitrary test function $\hat{\mathbf{w}}(s, \mathbf{y})$ in (8.5). It must be zero for $s \geq T$. The shifted function $\hat{\mathbf{w}}(s+\tau, \mathbf{y})$ is still Y-periodic, smooth , and is zero for $s \geq T - \tau$. Hence using $\hat{\mathbf{w}}(s+\tau, \mathbf{y})$ in (8.5) produces (8.4). This shows that K_2^{pq} satisfying (8.5) must satisfy (8.4) and therefore (8.1). Finally, integrating (8.5) by parts in s, using the initial condition for K_2^{pq} and replacing s with t yields (5.6).

References

Bakhvalov, N. and Panasenko, G. 1989 *Homogenization: averaging processes in periodic media.* Kluwer, Boston.

Buckingham, M. J. 1997 Theory of acoustic attenuation, dispersion, and pulse propagation in unconsolidated granular materials, including marine sediments. *Journ. Acoust. Soc. Am.* **102**, 2579-2596.

Buckingham, M. J. 1998 Theory of compressional and shear waves in fluid-like marine sediments. *Journ. Acoust. Soc. Am.* **103**, 288-299.

Buckingham, M. J. 2000 Wave propagation, stress relaxation, and grain-to-grain shearing in saturated, unconsolidated marine sediments. *Journ. Acoust. Soc. Am.* **108** (6), 2796-2815.

Burridge, R. and Keller, J.B. 1981 Poroelasticity equations derived from microstructure. *Journ. Acoust. Soc. Am.* **70**, 1140-1146.

Gilbert, R. P. and Mikelic, A. 2000 Homogenizing the acoustic properties of the seabed: Part I. *Nonlinear Analysis*, **40**, 185-212.

Gilbert, R. P. and Panchenko, A. 1999 Acoustics of a stratified poroelastic composite. *Zeitschrift für Analysis und ihre Anwendungen.* **18**(4), 977-1001.

Gilbert, R. P. and Panchenko, A. 2003 Effective acoustic equations for a two-phase medium with microstructure. *Mathematical and Computer Modeling*, to appear.

Goodman, M. A. and Cowin, S. C 1971 A continuum theory for granular materials. *Arch. Rat. Mech. Analysis.* **44**, 249-266.

Hill, J. M. 2000 Simlarity "hot-spot" solutions for a hypoplastic granular material. *Proceedings of Royal Society A.* **456**, 2653-2671.

Harris, D. 2001 Characteristic relations for a model for the flow of granular materials. *Proceedings of Royal Society A* **457**, 349-370.

Isakson, M., McJunkin, S. and Chotiros, N. P. 2003 A comparison of inversions for elastic and poro-elastic models for high-frequency reflection from a smooth water-sediment interface. Paper presented at the 16th ASCE Egineering Mechanics Conference, July 2003, Seatlle.

Jenkins, J. T. and Savage, S. B. 1983 A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles. *Journal of Fluid Mechanics* **130**, 187-202.

Kikuchi, N. and Oden, J. T. 1988 *Contact problems in elasticity*, SIAM, Philadelphia.

Kirchner, N. P. 2002 Thermodynamically consistent modelling of abrasive granular materials. I. Non-equilibrium theory. *Proceedings of Royal Society A.* **458**, 2153-2176.

Kolymbas, D. 1987 A novel constitutive law for soils. in *Proc. 2nd Int. Conf. Constitutive laws for Engineering Materials* (ed. C. S. Desai), 319-326.

Kuttler, K. L. and Shillor, M 1999 Set-valued pseudomonotone maps and degenerate evolution inclusions. *Comm. Contemp. Math.* **1** 87-123.

Sanchez-Palencia, E. 1980 *Non-Homogenious Media and Vibration Theory*, Springer, Berlin.

Schaeffer, D. G. 1992 A mathematical model for localization in granular flow. *Proceedings of Royal Society A* **436**, 217-250.

Truesdell, C. A. 1997 *A first course in rational continuum mechanics*, Academic Press, NY.