Tail Densities of Copulas and Extremal Dependence

Haijun Li

(Joint work with Peiling Wu)
Department of Mathematics
Washington State University

MMR2011, Beijing
Multivariate Heavy Tails

For notational convenience, let $X = (X_1, \ldots, X_d)$ be a non-negative random vector with distribution (df) $F$ and continuous margins $F_1, \ldots, F_d$.

- Assume that the margins are tail equivalent; that is, as $t \to \infty$,
  \[
  \frac{1 - F_i(t)}{1 - F_1(t)} = \frac{\overline{F}_i(t)}{\overline{F}_1(t)} \to 1, \ 1 \leq i \leq d.
  \]

- $X$ (or $F$) is said to be multivariate regularly varying (MRV) if there exists a Radon measure $\mu$, called the intensity measure, such that
  \[
  \lim_{t \to \infty} \frac{\mathbb{P}(X \in tB)}{\mathbb{P}(X_1 > t)} = \mu(B), \ \forall \text{ relatively compact sets } B \subset \mathbb{R}_+^d \setminus \{0\},
  \]
satisfying that $\mu(\partial B) = 0$ (Resnick, 2007).
The intensity measure $\mu$ has the following homogeneous property:

$$\mu(tB) = t^{-\alpha}\mu(B), \ \forall \ B, \text{ bounded away from 0},$$

where $\alpha > 0$ is known as the (heavy-) tail index.

The marginal dfs have regularly varying (heavy) right tails:

$$\overline{F}_i(x) = \mathbb{P}(X_i > x) = x^{-\alpha}L(x), \ x \in \mathbb{R}_+,$$

where $L : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is slowly varying; that is,

$$L(tx)/L(x) \to 1 \text{ for } x > 0 \text{ as } t \to \infty.$$

**Example:** Multivariate t or truncated t distribution ($\alpha = \text{degree of freedom}$), multivariate Pareto distribution ($\alpha = (\theta^{-1} + 1)/2$, where $\theta = \text{Gini index}$), .....
Heavy-Tail Phenomena

Heavy-tail phenomena have been widely observed in complex systems (e.g., engineering, biological, and social systems). The distribution tails are said to be

1. **Heavy**: $\alpha > 2$,
2. **Really heavy**: $1 < \alpha \leq 2$,
3. **Super heavy**: $0 < \alpha \leq 1$. 

Data networks: file sizes with $0.66 \leq \alpha \leq 1.05$ (Boston Study, 1995), and $0.4 \leq \alpha \leq 0.6$ (Calgary Study, 1996).

Micro/nano electronics: defect sizes with $\alpha > 2$ (see, e.g., Hwang, 2004).

Finance: daily returns with $\alpha \geq 1$ and monthly returns with $\alpha > 2$ (a lot of papers).
Heavy-Tail Phenomena

Heavy-tail phenomena have been widely observed in complex systems (e.g., engineering, biological, and social systems). The distribution tails are said to be

1. **heavy**: $\alpha > 2$,
2. **really heavy**: $1 < \alpha \leq 2$,
3. **super heavy**: $0 < \alpha \leq 1$.

- **Data networks**: file sizes with $0.66 \leq \alpha \leq 1.05$ (Boston Study, 1995), and $0.4 \leq \alpha \leq 0.6$ (Calgary Study, 1996).
- **Micro/nano electronics**: defect sizes with $\alpha > 2$ (see, e.g., Hwang, 2004).
- **Finance**: daily returns with $\alpha \geq 1$ and monthly returns with $\alpha > 2$ (a lot of papers).
Copula Method

In practice, one rarely observe components that are tail equivalent. To overcome this difficulty, one can equalize the margins via monotone transforms, such as the copula method.
Copula Method

In practice, one rarely observe components that are tail equivalent. To overcome this difficulty, one can equalize the margins via monotone transforms, such as the copula method.

- A copula $C$ is a $d$-dimensional distribution on support $[0, 1]^d$ with standard uniform margins (Nelsen, 2006).
- Every multivariate distribution $F$ with margins $F_1, \ldots, F_d$ can be written as $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ for some $d$-dimensional copula $C$ (Sklar, 1959).
- In the case of continuous margins, $C$ is unique and
  \[
  C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))
  \]
  where $F_i^{-1}(u_i)$ denotes the quantile functions of the $i$-th margin, $1 \leq i \leq d$. 
Set-Up in Terms of Copulas

Let $X = (X_1, \ldots, X_d)$ be a non-negative random vector with MRV df $F$ and continuous margins $F_1, \ldots, F_d$ that may not be tail equivalent.

- Equalize the margins: $(U_1, \ldots, U_d) = (F_1(X_1), \ldots, F_d(X_d))$ has a copula $C$.
- The MRV property implies that

$$b^U(w) := \lim_{u \to 0} \frac{C(1 - uw_i, 1 \leq i \leq d)}{u}, \quad \forall w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d$$

$$= \lim_{u \to 0} \frac{\mathbb{P}(U_i > 1 - uw_i, 1 \leq i \leq d)}{\mathbb{P}(U_1 > 1 - u)}$$

known as the upper tail dependence function, exists (Klüppelberg et al, 2008, Joe et al, 2010).
Separating margins from rank-invariance dependence:

- $F_i$ is regularly varying with tail index $\alpha_i > 0$, $1 \leq i \leq d$.
- $b^U(w) = \mu(\prod_{i=1}^{d} (w_i^{-1/\alpha_i}, \infty])$, for all $w \in \mathbb{R}_+^d$.

Conversely, given a copula with upper tail dependence function $b^U$ and regularly varying margins $F_i$, $1 \leq i \leq d$,

$$F(x_1, \ldots, x_d) := C(F_1(x_1), \ldots, F_d(x_d))$$

is (multivariate) regularly varying (Li and Sun, 2009).

**Remark:** If continuous marginal dfs are strictly increasing, then $\overline{\mathbb{R}}_+^d$ and $(0, 1]^d$ are topologically equivalent. Since limiting properties are topological, the methods of tail dependence functions and intensity measure are equivalent as far as (upper) extreme value analysis is concerned.
Tail Dependence Functions and Intensity Measure

- One can also define the lower tail dependence function

\[
b^L(w) := \lim_{u \to 0} \frac{C(uw_i, \ 1 \leq i \leq d)}{u}, \ \forall w = (w_1, \ldots, w_d) \in \mathbb{R}^d_+,
\]

- The lower tail dependence function of a copula is the same as the upper tail dependence function of its survival copula.

- But both intensity measure and tail dependence function are cumulative in nature, and we want something that is more local. Such an extremal dependence analysis is especially useful for distributions that are specified by densities.
Theorem (de Haan and Resnick, 1987)

Assume the density $f$ of $F$ exists. If $\frac{f(tx)}{t^{-d}F_1(t)} \to \lambda(x) > 0$ on $\mathbb{R}^d_+ \setminus \{0\}$ and uniformly on $\{x > 0 \mid ||x|| = 1\}$, as $t \to \infty$, then

$$\lim_{t \to \infty} \frac{1 - F(tx)}{F_1(t)} = \mu([0, x]^c) = \int_{[0,x]^c} \lambda(y)dy.$$ 

Our Goal: Introduce the tail density $\lambda(\cdot)$ for copulas.
Theorem (de Haan and Resnick, 1987)

Assume the density $f$ of $F$ exists. If
\[
\frac{f(tx)}{t^{-d}F_1(t)} \to \lambda(x) > 0 \quad \text{on} \quad \mathbb{R}^d_+ \setminus \{0\}
\]
and uniformly on \( \{x > 0 \mid ||x|| = 1\} \), as \( t \to \infty \), then
\[
\lim_{t \to \infty} \frac{1 - F(tx)}{F_1(t)} = \mu([0, x]^c) = \int_{[0,x]^c} \lambda(y)dy.
\]

Our Goal: Introduce the tail density $\lambda(\cdot)$ for copulas.

- The notion of tail density is local and geometric (Balkema and Embrechts, 2007).
- Asymptotic analysis of tail risk/reliability measures often involves directly functionals of tail densities (Joe and Li, 2010).
Consider a bivariate MTCJ copula \( C(x, y) = (x^{-\theta} + y^{-\theta} - 1)^{-1/\theta} \) with density \( c(x, y) = (1 + \theta)(xy)^{-\theta-1}(x^{-\theta} + y^{-\theta} - 1)^{-2-1/\theta}, \theta > 0 \).

Approaching the origin along the ray passing though \((w_1, w_2)\), \(w_1 > 0\) and \(w_2 > 0\), we have, as \(u \to 0\),

\[
c(uw_1, uw_2) \approx u^{-1}[(1 + \theta)(w_1w_2)^{-\theta-1}(w_1^{-\theta} + w_2^{-\theta})^{-2-1/\theta}]
\]
**Idea:** Let \((U_1, \ldots, U_d) \overset{d}{\sim} C\). When \(u\) is sufficiently small,

\[
\text{tail density} \approx \frac{\mathbb{P}(1 - uw_i \leq U_i \leq 1 - u(w_i - dw_i), 1 \leq i \leq d) / u^\kappa}{dw_1 \cdots dw_d} \\
\approx u^{d-\kappa} \frac{\mathbb{P}(1 - uw_i \leq U_i \leq 1 - uw_i + d(uw_i), 1 \leq i \leq d)}{d(uw_1) \cdots d(uw_d)}
\]

In what follows, \(\kappa = 1\).
Tail Densities of a Copula $C$

Assume that all the necessary regularity conditions hold.

**Definition**

The upper and lower tail densities of $C$ are defined as

$$
\lambda^U(w) := \lim_{u \to 0} \frac{D_w \bar{C}(1 - uw_i, 1 \leq i \leq d)}{u}, \\
\lambda^L(w) := \lim_{u \to 0} \frac{D_w C(uw_i, 1 \leq i \leq d)}{u}, \quad \forall \ w = (w_1, \ldots, w_d) \in \mathbb{R}^d_+
$$

where $D_w$ denotes the $d$-order partial differentiation operator.

- The lower tail density of the survival copula $\hat{C}$ is the upper tail density of $C$.
- $C$ is upper (lower) tail independent if and only if $\lambda^U = 0$ ($\lambda^L = 0$).
Properties

- Let $c$ denote the density of $C$, then

$$\lambda^U(w) = \lim_{u \to 0} u^{d-1} c(1 - uw_i, 1 \leq i \leq d),$$

$$\lambda^L(w) = \lim_{u \to 0} u^{d-1} c(uw_i, 1 \leq i \leq d), \quad \forall w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d$$

- For a $d$-dimensional copula $C$, the tail densities are homogeneous of order $1 - d$. That is, $\lambda^U(tw) = t^{1-d} \lambda^U(w)$ for any $w \in \mathbb{R}_+^d$ and $t > 0$.

- The Euler’s homogeneous theorem implies that the tail densities are directionally decreasing and convex, and go down to zero at infinity.
Relation with Tail Dependence Functions

Recall that $b^U$ and $b^L$ are, respectively, the upper and lower tail dependence functions of a copula $C$. Define the upper exponent function:

$$a^U(w) := \lim_{u \to 0} \frac{\mathbb{P}(U_i > 1 - uw_i, \text{ for some } i)}{u}, \quad \forall w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d.$$ 

The lower exponent function $a^L$ can be defined similarly or via duality.

Theorem

Recall that $D_w$ is the $d$-order partial differentiation operator.

1. $\lambda^U(w) = D_w b^U(w) = (-1)^{d-1} D_w a^U(w)$ for all $w \in \mathbb{R}_+^d$.
2. $\lambda^L(w) = D_w b^L(w) = (-1)^{d-1} D_w a^L(w)$ for all $w \in \mathbb{R}_+^d$. 
Relation with Tail Density of MRV

Recall that $\lambda(\cdot)$ denotes the tail density of an MRV df $F$ such that
\[
\lim_{t \to \infty} \frac{1 - F(tx)}{F_1(t)} = \int_{[0,x]^c} \lambda(y)dy, \quad x \in \mathbb{R}_+^d.
\]

Once again, $\lambda^U$ denotes the upper tail density of the copula $C$ of $F$.

**Theorem**

Let $\alpha$ denote the tail index of $F$, then
\[
\lambda(w_1, \ldots, w_d) = \alpha^d (w_1 \cdots w_d)^{-\alpha - 1} \lambda^U(w_1^{-\alpha}, \ldots, w_d^{-\alpha})
\]
\[
= |J(w_1^{-\alpha}, \ldots, w_d^{-\alpha})| \lambda^U(w_1^{-\alpha}, \ldots, w_d^{-\alpha}),
\]

where $|J(w_1^{-\alpha}, \ldots, w_d^{-\alpha})|$ is the Jacobian determinant of the homeomorphism $y_i = w_i^{-\alpha}, \ 1 \leq i \leq d$. 
Archimedean Tail Densities

Let $C(u; \phi) = \phi(\sum_{i=1}^{d} \phi^{-1}(u_i))$ be an Archimedean copula where the Laplace transform $\phi$.

Lower Tail Density

If $\phi$ is regularly varying at $\infty$ with tail index $\theta > 0$, then

$$
\lambda^L(w) = \prod_{i=1}^{d} \left(1 + \frac{i - 1}{\theta}\right) \left(\prod_{i=1}^{d} w_i\right)^{-1-1/\theta} \left(\sum_{i=1}^{d} w_i^{-1/\theta}\right)^{-\theta-d}.
$$

Upper Tail Density

If $\phi^{-1}$ is regularly varying at 1 with tail index $\beta > 1$, then

$$
\lambda^U(w) = \prod_{i=1}^{d} ((i - 1)\beta - 1) \left(\prod_{i=1}^{d} w_i\right)^{\beta-1} \left(\sum_{i=1}^{d} w_i^\beta\right)^{-d-1/\beta}.
$$
Lower Archimedean Tail Density

Theorem 2.5. Let $X = (X_1,\ldots,X_d)$ be a non-negative MRV random vector with intensity measure $\mu$, copula $C$ and continuous margins $F_1,\ldots,F_d$. If the marigins are tail equivalent (i.e. $\frac{\bar{F}_i(t)}{\bar{F}_i(t)} \to 1$ as $t \to \infty$ for any $i \neq j$) with heavy-tail index $\beta_0$, then the upper tail dependence function $\lambda^*(\cdot)$ exists and

$$
\lambda^*(w) = \partial_d \partial_w \mu([w-1^{\beta},\infty)) \mu([1,\infty) \times \bar{R}_d-1^+);
$$

$$
\partial_d \partial_w 1^{\cdots} \partial_w \mu([w,\infty)) \mu([0,1] \times \bar{R}_d-1^+) = (-1)^d a^*(1) \cdot \beta_d \cdot \prod_{i=1}^d w-\beta-1^i \lambda^*(w-\beta);
$$

Tail approximation via tail density

In this section, we derive the tail asymptotics for $\text{VaR}_p(||X||)$, as $p \to 1$, for any fixed norm $||\cdot||$ on $\mathbb{R}^d$. The results discussed in [6] can be obtained by taking the $l_1$-norm and the tail dependence function of Archimedean copulas.
Upper Archimedean Tail Density

Example 2.4. Consider a bivariate Clayton copula $C(u, v; \theta) = (u - \theta + v - \theta - 1)^{-1} \theta$, $\theta > 0$.

We know that Clayton copula only has lower tail dependence, no upper tail dependence. Its lower tail density function is given by the following steps:

The bivariate copula density function with parameter $\theta$ is

$$c(u, w_i, 1 \leq i \leq 2) = (1 + \theta) u^{-1} (w - \theta_1 + w - \theta_2 - u \theta)^{-1} \theta^{-2} (w_1 w_2)^{-\theta - 1}, \theta > 0.$$ 

Therefore the lower tail density function is $\lambda(w) = \lim_{u \to 0} u \cdot c(u, w, 1 \leq i \leq 2) = \lim_{u \to 0} u \cdot (1 + \theta) (w_1 w_2)^{-\theta - 1} (w - \theta_1 + w - \theta_2)^{-1} \theta^{-2} = (1 + \theta) (w_1 w_2)^{-\theta - 1} (w - \theta_1 + w - \theta_2)^{-1} \theta^{-2}, \theta > 0$.

The expression of $\lambda(w)$ implies some properties:

1. $\lambda(0, w_2) = d(w_1, 0) = \infty$
2. $\lambda(0, 0) = \infty$
3. $\lambda(\infty, w_2) = d(w_1, \infty) = 0$
4. $\lambda(\infty, \infty) = 0$. 

The graph of this bivariate Gumbel copula is as follows ($\delta = 2$).
Consider a $d$-dimensional symmetric $t$ distribution with cdf

$$f(x; \nu, \Sigma) = \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})(\nu\pi)^{d/2}} |\Sigma|^{-\frac{1}{2}} \left[ 1 + \frac{1}{\nu} (x^T \Sigma^{-1} x) \right]^{-\frac{\nu+d}{2}}, \quad x \in \mathbb{R}^d$$

where $\nu > 0$ is the degree of freedom, and $\Sigma$ is a $d \times d$ dispersion matrix.

The Tail Density of t Copula

$$\lambda^U(w) = \lambda^L(w) = \pi^{-\frac{d-1}{2}} |\Sigma|^{-\frac{1}{2}} \nu^{-d} \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu+1}{2})} \left[ (w - \frac{1}{\nu})^T \Sigma^{-1} w - \frac{1}{\nu} \right]^{-\frac{\nu+d}{2}} \prod_{i=1}^{d} \frac{\nu+1}{\nu} \zeta_{i}^{-\frac{\nu+1}{\nu}}$$

for any $w \in \mathbb{R}^d_+$. 
Norm-Based Tail Densities

- Define a norm $\|w\| := (w^T \Sigma^{-1} w)^{1/2}$, $w \in \mathbb{R}^d$.
- Let $|J(w_1^{-1/\nu}, \ldots, w_d^{-1/\nu})|$ be the Jacobian determinant of the homeomorphism $y_i = w_i^{-1/\nu}$, $1 \leq i \leq d$.
- Rewrite the upper tail density of a t copula:

\[
\lambda^U(w_1, \ldots, w_d) = \zeta \prod_{i=1}^d w_i^{-\frac{\nu+1}{\nu}} ((w^{-1/\nu})^T \Sigma^{-1} w^{-1/\nu})^{-\frac{\nu+d}{2}}
\]

\[
= \zeta \nu^d |J(w_1^{-1/\nu}, \ldots, w_d^{-1/\nu})| \times \|w^{-1/\nu}\|^{-\nu-d}
\]

- For the symmetric t distribution itself,

\[
\lambda(w_1, \ldots, w_d) = \zeta \nu^d \|w\|^{-\nu-d} \text{ (de Haan and Resnick, 1987).}
\]

- In fact, the tail densities of Archimedean and t copulas depend only on some norm on $\mathbb{R}_+^d$ and the Jacobian determinant of some (topologically invariant) homeomorphism.
A vine copula is a copula constructed from a set of $d(d - 1)/2$ bivariate copulas by using successive mixing according to a tree structure on finite indexes 1, ..., $d$ (Kurowicka and Cooke, 2006; Kurowicka and Joe, 2011).
Vine Copula $C$ of $(U_1, \ldots, U_d)$

- $\{c_{i,j}, 1 \leq i < j \leq d\} = \text{A set of the densities of bivariate linking copulas.}$
- For any $S \subseteq \{1, \ldots, d\}$, define the $S$-marginal density $c_S := c_S(u_S)$.
- Define the conditional distribution of $U_k$ given $U_S = u_S$ $C_{k|S} := C_{k|S}(u_k|u_S)$, $k \notin S$.

The density $c_{\{1,\ldots,d\}}$ of $C$ is constructed recursively as follows.

**D-Vine Construction (Bedford and Cooke, 2001 and 2002)**

**Baseline:** For any $1 \leq i \leq d - 1$, the density of the $\{i, i + 1\}$ margin is $c_{i,i+1}$.

**Recursion:**

\[
\frac{c_{\{1,\ldots,d\}}}{c_{\{2,\ldots,d-1\}}} = c_{1,d} \left( C_{1|2,\ldots,d-1}, C_{d|2,\ldots,d-1} \right) \frac{c_{\{1,\ldots,d-1\}}}{c_{\{2,\ldots,d-1\}}} \frac{c_{\{2,\ldots,d\}}}{c_{\{2,\ldots,d-1\}}}. 
\]
The Recursion of D-Vine Lower Tail Densities

- For any $S \subseteq \{1, \ldots, d\}$, $\lambda^L_S(w_S) = \text{lower tail density of the } S\text{-margin } C_S$.
- For any $S \subseteq \{1, \ldots, d\}$ and $k \notin S$, define the lower tail conditional distribution of $U_k$ given $U_S = u_S$:

$$
t_{k|S}(w_k|w_S) := \lim_{u \to 0} C_{k|S}(uw_k|uw_S).
$$

**Theorem**

Fix $S = \{2, \ldots, d - 1\}$. If all the bivariate baseline linking copulas have lower tail dependence, then

$$
\frac{\lambda^L(w)}{\lambda^L_S(w_S)} = c_{1,d} \left( t_{1|S}(w_1|w_S), t_{d|S}(w_d|w_S) \right) \frac{\lambda^L_{\{1\} \cup S}(w)}{\lambda^L_S(w_S)} \frac{\lambda^L_{\{d\} \cup S}(w)}{\lambda^L_S(w_S)}.
$$
Examples of D-Vine Tail Densities

The 3-dimensional D-vine:

\[
\lambda^L(w_1, w_2, w_3) = \lambda^L_{12}(w_1, w_2) \cdot \lambda^L_{23}(w_2, w_3) \cdot c_{13}(t_{1|2}(w_1|w_2), t_{3|2}(w_3|w_2)).
\]

The 4-dimensional D-vine:

\[
\lambda^L(w_1, w_2, w_3, w_4) = \lambda^L_{12}(w_1, w_2) \cdot \lambda^L_{23}(w_2, w_3) \cdot \lambda^L_{34}(w_3, w_4) \\
\cdot c_{13}(t_{1|2}(w_1|w_2), t_{3|2}(w_3|w_2)) \cdot c_{24}(t_{2|3}(w_2|w_3), t_{4|3}(w_4|w_3)) \\
\cdot c_{14}(t_{1|23}(w_1|w_2, w_3), t_{4|23}(w_4|w_2, w_3)).
\]
What happens if some or all bivariate baseline linking copulas of a D-vine $C$ are tail independent? Joe et al (2010) has a partial answer:

- $C$ must be tail independent, e.g., as $u \to 0$, $C(uw_i, 1 \leq i \leq d) \sim u^\kappa h(w)$, $\kappa > 1$.
- Some margins of $C$ can still be tail dependent.
What happens if some or all bivariate baseline linking copulas of a D-vine $C$ are tail independent? Joe et al (2010) has a partial answer:

- $C$ must be tail independent, e.g., as $u \to 0$, $C(uw_i, 1 \leq i \leq d) \sim u^\kappa h(w)$, $\kappa > 1$.
- Some margins of $C$ can still be tail dependent.

But can we quantify the order of scaling, e.g., $\kappa$, in the case of tail independence of a vine copula?

- Expansion of Pickands densities, see Frick and Reiss (2008).
- Tail expansions of copulas, see Jaworski (2010).
- Intermediate tail dependence, see Hua and Joe (2010).
Seeking Recursions of Regular Variation Near Tails

Consider the 3-dimensional D-vine

\[ c(uw_1, uw_2, uw_3) = c_{12}(uw_1, uw_2) \cdot c_{23}(uw_2, uw_3) \]
\[ \cdot c_{13}(C_{1|2}(uw_1|uw_2), C_{3|2}(uw_3|uw_2)), \text{ when } u \text{ is small.} \]

As functions of \( u \), the regular variations of constructs \( c_{12}, c_{23}, C_{1|2}, C_{3|2} \), and \( c_{13} \) should yield the regular varying property of \( c \).
Seeking Recursions of Regular Variation Near Tails

Consider the 3-dimensional D-vine

\[ c(uw_1, uw_2, uw_3) = c_{12}(uw_1, uw_2) \cdot c_{23}(uw_2, uw_3) \]
\[ \cdot c_{13}(C_{1|2}(uw_1|uw_2), C_{3|2}(uw_3|uw_2)), \text{ when } u \text{ is small.} \]

As functions of \( u \), the regular variations of constructs \( c_{12}, c_{23}, C_{1|2}, C_{3|2}, \) and \( c_{13} \) should yield the regular varying property of \( c \).

**Example:** If the baseline linking copulas \( C_{12} \) and \( C_{23} \) are Morgenstern copulas, and the linking copula \( C_{13} \) is a t copula, then \( c_{12}(uw_1, uw_2) \) and \( c_{23}(uw_2, uw_3) \) are asymptotically constant as \( u \to 0 \) and

\[ C_{1|2}(uw_1|uw_2) \sim uh_{1|2}(w_1, w_2), C_{3|2}(uw_3|uw_2) \sim uh_{3|2}(w_2, w_3), \text{ as } u \to 0. \]

Thus, \( c(uw_1, uw_2, uw_3) \sim u^{-1}h(w_1, w_2, w_3), \text{ as } u \to 0. \) That is, \( \kappa = 2. \)