Dependence Comparison of Multivariate Extremes

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Let \((X_n, n \geq 1)\) denote a sequence of iid non-negative, \(d\)-dimensional random vectors with common joint df \(F\). Write

- \(X_n = (X_{n,1}, \ldots, X_{n,d})\) with marginal dfs \(F_1, \ldots, F_d\), and
- \(M_{n,j} := \max\{X_{i,j}, 1 \leq i \leq n\}, 1 \leq j \leq d\).

We are interested in the limiting distribution of properly normalized \(M_n := (M_{n,1}, \ldots, M_{n,d})\) as \(n \to \infty\).
Multivariate Extremes

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We are interested in the limiting distribution of properly normalized \(M_n := (M_{n,1}, \ldots, M_{n,d})\) as \(n \to \infty\).

- For simplicity, we assume the margins are regularly varying and tail equivalent:

\[
\bar{F}_j(s) := \mathbb{P}(X_{n,j} > s) = s^{-\alpha}L_j(s), 1 \leq j \leq d,
\]

where \(L_i(s)/L_j(s) \to 1\) as \(s \to \infty\), \(i \neq j\), and \(L_j(\cdot), 1 \leq j \leq d\), are slowly varying.

- Then the properly rescaled \(M_{n,j}\) converges weakly to the Fréchet distribution (i.e., \(G(s) = \exp\{-s^{-\alpha}\}, s \geq 0\) with parameter \(\alpha, 1 \leq j \leq d\).
Multivariate Regular Variation

- Assume that $F$ is a multivariate regularly varying distribution (MRV) with intensity measure $\mu$, i.e.,

$$
\lim_{t \to \infty} \frac{\mathbb{P}(X_n \in tB)}{\mathbb{P}(X_{n,1} > t)} = \mu(B), \ \forall \text{ relatively compact sets } B \subset \mathbb{R}_+^d \setminus \{0\},
$$

satisfying that $\mu(\partial B) = 0$ (Resnick, 1987 and 2007).

- The intensity measure $\mu$ is a Radon measure with homogeneous property

$$
\mu(tB) = t^{-\alpha} \mu(B), \ \forall \ B, \text{ bounded away from } 0,
$$

where $\alpha > 0$ is known as the tail index.

- $F_j, 1 \leq j \leq d$, is regularly varying with tail index $\alpha > 0$.

- Examples: Multivariate t distribution, multivariate Pareto distributions, certain elliptical distributions, ...
Max Domain of Attraction

Notations: For any $x, y \in \mathbb{R}^d$, the sum $x + y$, quotient $x/y$, and the vector inequalities are all operated component-wise.
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**Theorem (de Haan & Resnick, 1977; Marshall & Olkin, 1983)**

There exist normalization vectors $a_n$ and $b_n > 0$ such that

$$\mathbb{P}
\left(
\frac{M_n - a_n}{b_n} \leq x
\right)
\to G(x), \text{ as } n \to \infty, \forall x$$

where $G$ is a $d$-dimensional distribution with Fréchet margins $G_j(s) = \exp\{-s^{-\alpha}\}, 1 \leq j \leq d$, if and only if $F$ is MRV with intensity measure $\mu([0, x]^c) := -\log G(x)$. 

One-dimensional result is due to Gnedenko (1943). The parametric feature enjoyed by univariate extremes is lost in the multivariate context.
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- The parametric feature enjoyed by univariate extremes is lost in the multivariate context.
Pickands (1980) Representation

Let $S_{1+}^d = \{ a : a = (a_1, \ldots, a_d) \in \mathbb{R}_+^d, \sum_{i=1}^d a_i = 1 \}$. 

$$G(x) = \exp \left\{ -c \int_{S_{1+}^d} \max_{1 \leq i \leq d} \{(a_i/x_i)^\alpha\} \mathbb{Q}(da) \right\},$$

where $c > 0$ and $\mathbb{Q}$ is a probability measure defined on $S_{1+}^d$. 
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\]

where \( c > 0 \) and \( Q \) is a probability measure defined on \( S_{1+}^d \).

- The non-parametric nature is reflected by the fact that \( cQ \), called the spectral or angular measure, is not specified.
- The multivariate extreme value distribution (MEV) \( G \) is positively associated (Marshall and Olkin, 1983); that is, as \( n \) is sufficiently large, we have, “loosely speaking”,

\[
\mathbb{E}\left( f(M_n)g(M_n) \right) \geq \mathbb{E}\left( f(M_n) \right) \mathbb{E}\left( g(M_n) \right)
\]

for all non-decreasing functions \( f, g : \mathbb{R}^d \rightarrow \mathbb{R} \).
Orthant Dependence Order

Let $X$ and $Y$ be two $\mathbb{R}^d_+$-valued random vectors. $X$ is said to be smaller than $Y$ in the sense of orthant dependence order, denoted as $X \leq_{od} Y$, if $X$ and $Y$ have identical corresponding marginal dfs and $\mathbb{P}(X \leq x) \leq \mathbb{P}(Y \leq x)$, $\forall x \in \mathbb{R}^d_+$. 
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One can establish directly dependence comparisons for MEV df $G$ (Joe, 1997), but it would be better that the orthant order of $G$s can be expressed in terms of the underlying sample $X_n$s.
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One can establish directly dependence comparisons for MEV df $G$ (Joe, 1997), but it would be better that the orthant order of $G$s can be expressed in terms of the underlying sample $X_n$s.

Theorem

1. Let $(X_n, n \geq 1)$ and $(X'_n, n \geq 1)$ be two iid samples with dfs that are in the domains of attraction of $G$ and $G'$.
2. Let $(M_n, n \geq 1)$ and $(M'_n, n \geq 1)$ denote the vectors of component-wise maxima of $(X_n, n \geq 1)$ and $(X'_n, n \geq 1)$.

$$X_n \leq_{od} X'_n \Rightarrow M_n \leq_{od} M'_n \Rightarrow G \leq_{od} G'.$$
Let $X$ and $Y$ be two $\mathbb{R}_+^*$-valued random variables. $X$ is said to be smaller than $Y$ in the sense of tail order, denoted as $X \leq_{stn} Y$, if there exists a constant $t_0 > 0$ (usually large) such that $P(X > t) \leq P(Y > t)$, $\forall \ t > t_0$. 

$\leq_{stn}$ is reflexive, and transitive. $\leq_{stn}$ is antisymmetric if tail identically distributed random variables are considered to be equivalent. If $X$ is stochastically smaller than $Y$, then $X \leq_{stn} Y$. 

$X \leq_{stn} Y \iff$ There exists a small open neighborhood of $\infty$ within which $X$ is stochastically smaller than $Y$. 

$X \leq_{stn} Y$ implies that $\limsup_{t \to \infty} P(X > t) \leq P(Y > t)$.
Stochastic Tail Order

Let $X$ and $Y$ be two $\mathbb{R}_+^*$-valued random variables. $X$ is said to be smaller than $Y$ in the sense of tail order, denoted as $X \leq_{sto} Y$, if there exists a constant $t_0 > 0$ (usually large) such that

$$\mathbb{P}(X > t) \leq \mathbb{P}(Y > t), \quad \forall \ t > t_0.$$

- $\leq_{sto}$ is reflexive, and transitive. $\leq_{sto}$ is antisymmetric if tail identically distributed random variables are considered to be equivalent.

- If $X$ is stochastically smaller than $Y$, then $X \leq_{sto} Y$.

- $X \leq_{sto} Y$ $\iff$ There exists a small open neighborhood of $\infty$ within which $X$ is stochastically smaller than $Y$.

- $X \leq_{sto} Y$ implies that

$$\limsup_{t \to \infty} \frac{\mathbb{P}(X > t)}{\mathbb{P}(Y > t)} \leq 1.$$
Asymptotic Portfolio Loss (apl) Order

Mainik & Rüschendorf (2012) introduce an asymptotic order and define $X \leq_{\text{apl}} Y$ if

$$\limsup_{t \to \infty} \frac{\mathbb{P}(X > t)}{\mathbb{P}(Y > t)} = \lim_{t \to \infty} \left[ \sup_{s > t} \frac{\mathbb{P}(X > s)}{\mathbb{P}(Y > s)} \right] \leq 1.$$
Mainik & Rüschendorf (2012) introduce an asymptotic order and define $X \leq_{\text{apl}} Y$ if

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- Notice that $\sup_{s > t} \frac{P(X > s)}{P(Y > s)}$ is decreasing in $t$. If $X \leq_{\text{apl}} Y$ with $\limsup_{t \to \infty} \frac{P(X > t)}{P(Y > t)} = 1$, then in any open neighborhood $(c, \infty]$ of $\infty$, one can find that $P(X > s) \geq P(Y > s)$ for some $s > c$.

- Coupling method? Closure properties? $\leq_{\text{apl}}$ is not closed under weak convergence.

- Mainik & Rüschendorf (2012) show that if $R_1 \leq_{\text{apl}} R_2$, then $R_1 V \leq_{\text{apl}} R_2 V$ for any bounded random variable $V \geq 0$ that is independent of $R_1, R_2$. 
Theorem (Tail Coupling)

Two positive random variables $X \leq_{sto} Y$ iff there exists a random variable $Z$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and non-decreasing functions $\psi_1$ and $\psi_2$ such that $X \overset{d}{=} \psi_1(Z)$ and $Y \overset{d}{=} \psi_2(Z)$ and $\mathbb{P}(\psi_1(Z) \leq \psi_2(Z) | Z > z_0) = 1$ for some $z_0 > 0$.

Let $X$ and $Y$ have continuous distributions $F$ and $G$ with support $[0, \infty)$ respectively, and let $F^{-1}(\cdot)$ and $G^{-1}(\cdot)$ denote the corresponding right continuous inverses.
**Theorem (Tail Coupling)**

Two positive random variables $X \leq_{sto} Y$ iff there exists a random variable $Z$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and non-decreasing functions $\psi_1$ and $\psi_2$ such that $X \overset{d}{=} \psi_1(Z)$ and $Y \overset{d}{=} \psi_2(Z)$ and $\mathbb{P}(\psi_1(Z) \leq \psi_2(Z) \mid Z > z_0) = 1$ for some $z_0 > 0$.

Let $X$ and $Y$ have continuous distributions $F$ and $G$ with support $[0, \infty)$ respectively, and let $F^{-1}(\cdot)$ and $G^{-1}(\cdot)$ denote the corresponding right continuous inverses.

- $\Rightarrow$: $X \leq_{sto} Y$ implies that $F^{-1}(u) \leq G^{-1}(u)$, $\forall u > u_0$ for some $0 < u_0 < 1$. Let $U$ be a random variable with standard uniform distribution, and thus $\mathbb{P}(F^{-1}(U) \leq G^{-1}(U) \mid U > u_0) = 1$.

- $\Leftarrow$: $\mathbb{P}(X > t) = \mathbb{P}(Z > z_0)\mathbb{P}(\psi_1(Z) > t \mid Z > z_0) \leq \mathbb{P}(Z > z_0)\mathbb{P}(\psi_2(Z) > t \mid Z > z_0) \leq \mathbb{P}(\psi_2(Z) > t) = \mathbb{P}(Y > t)$, $\forall t \geq \psi_1(z_0)$. 
Closure Properties

- $X \leq_{sto} Y$ implies that $g(X) \leq_{sto} g(Y)$ for any continuous function $g$ that is eventually increasing to $\infty$.

- If $X_1, X_2$ are independent, and $Y_1, Y_2$ are independent, then $X_1 \leq_{sto} Y_1$ and $X_2 \leq_{sto} Y_2$ imply that

$$
\psi(X_1, X_2) \leq_{sto} \psi(Y_1, Y_2)
$$

for any continuous function $\psi$ that is eventually (component-wise) increasing to $\infty$. 

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  for any continuous function $\psi$ that is eventually (component-wise) increasing to $\infty$.

- $R_1 \leq_{sto} R_2$ implies that $R_1 V \leq_{sto} R_2 V$ for any non-negative random variable $V$ that is independent of $R_1, R_2$.

- If $[X \mid \Theta = \theta] \leq_{sto} [Y \mid \Theta = \theta]$ for all $\theta$ in the bounded support of $\Theta$, then $X \leq_{sto} Y$. 
Examples

Let \( X \sim \text{Weibull}(1,k) \), and \( Y \sim \text{Exp}(1) \). If the shape parameter \( k > 1 \) (IFR), then \( X \leq_{\text{sto}} Y \). If the shape parameter \( k < 1 \) (DFR), then \( X \geq_{\text{sto}} Y \).
Examples

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If $X \sim \text{Exp}(1)$, and $Y \sim \text{Pareto Type II}(2)$, then $X \leq_{\text{sto}} Y$. Note that $X$ and $Y$ are regularly varying with respective tail indexes 2 and 1.
Examples

- Let $X \sim \text{Weibull}(1,k)$, and $Y \sim \text{Exp}(1)$. If the shape parameter $k > 1$ (IFR), then $X \leq_{\text{sto}} Y$. If the shape parameter $k < 1$ (DFR), then $X \geq_{\text{sto}} Y$.

- If $X \sim \text{Exp}(1)$, and $Y \sim \text{Pareto Type II}(2)$, then $X \leq_{\text{sto}} Y$.

- If $X \sim \text{Fréchet}(2)$, and $Y \sim \text{Pareto type II}(1)$, and then $X \leq_{\text{sto}} Y$. Note that $X$ and $Y$ are regularly varying with respective tail indexes 2 and 1.
Examples

- If $X \sim \text{Pareto Type I} (0.5, 1)$, $Y \sim \text{Pareto Type II} (1)$, then $X \leq_{\text{sto}} Y$. Note that $X$ and $Y$ are regularly varying with same tail index 1.
Examples

- If $X \sim$ Pareto Type I $(0.5, 1)$, $Y \sim$ Pareto Type II $(1)$, then $X \leq_{sto} Y$. Note that $X$ and $Y$ are regularly varying with same tail index 1.

- Let $R_1$ and $R_2$ have regularly varying distributions with tail indexes $\alpha_1$ and $\alpha_2$ respectively. If $\alpha_1 > \alpha_2$, then $R_1 \leq_{sto} R_2$. 
Examples

- If $X \sim$ Pareto Type I $(0.5, 1)$, $Y \sim$ Pareto Type II $(1)$, then $X \leq_{sto} Y$. Note that $X$ and $Y$ are regularly varying with same tail index 1.

- Let $R_1$ and $R_2$ have regularly varying distributions with tail indexes $\alpha_1$ and $\alpha_2$ respectively. If $\alpha_1 > \alpha_2$, then $R_1 \leq_{sto} R_2$.

- Let $R$ be regularly varying with tail indexes $\alpha$. If $V_1$ and $V_2$ are random variables, independent of $R$, such that $\mathbb{E}(V_1^{\alpha}) < \mathbb{E}(V_2^{\alpha})$, then $RV_1 \leq_{sto} RV_2$ (via Breiman’s theorem).
A random vector $X \in \mathbb{R}^d$ is called elliptically distributed if $X$ has the representation:

$$X \overset{d}{=} \mu + RAU$$

where $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ and $U$ is uniformly distributed on $\mathbb{S}^d_2 := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ and $R \geq 0$ is independent of $U$.

We denote this by $X \sim \mathcal{E}(\mu, \Sigma, R)$ where $\Sigma = AA^\top$.

Recall: $\mathbb{S}^d_1 := \{x \in \mathbb{R}^d : \|x\|_1 = 1\}$
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We denote this by $X \sim \mathcal{E}(\mu, \Sigma, R)$ where $\Sigma = AA^\top$.

Recall: $S_1^d := \{x \in \mathbb{R}^d : \|x\|_1 = 1\}$

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**Tail Comparison of Elliptical Distributions**

Let $X \sim \mathcal{E}(\mu_1, \Sigma_1, R_1)$ and $Y \sim \mathcal{E}(\mu_2, \Sigma_2, R_2)$. If

$$\mu_1 \leq \mu_2, \ R_1 \leq_{sto} R_2, \ \text{and} \ \xi^\top \Sigma_1 \xi \leq \xi^\top \Sigma_2 \xi, \ \text{for fixed} \ \xi \in S_1^d$$

then $|\xi^\top X| \leq_{sto} |\xi^\top Y|$.
Remarks

- This is our $\leq_{\text{sto}}$-version of a similar result that is obtained in Mainik & Rüschendorf (2012) using the $\leq_{\text{apl}}$ order.
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- Anderson (1955), Fefferman, Jodeit & Perlman (1972) show that if $\mu_1 = \mu_2$, $R_1 \overset{d}{=} R_2$, and
  \[ \xi^T \Sigma_1 \xi \leq \xi^T \Sigma_2 \xi, \forall \xi \in \mathbb{R}^d, \]
  then $E(\psi(X)) \leq E(\psi(Y))$ for all symmetric and convex functions $\psi : \mathbb{R}^d \mapsto \mathbb{R}$, such that the expectations exist.
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- This is our $\leq_{st0}$-version of a similar result that is obtained in Mainik & Rüschendorf (2012) using the $\leq_{apl}$ order.

- Anderson (1955), Fefferman, Jodeit & Perlman (1972) show that if $\mu_1 = \mu_2$, $R_1 \overset{d}{=} R_2$, and

$$\xi^\top \Sigma_1 \xi \leq \xi^\top \Sigma_2 \xi, \forall \xi \in \mathbb{R}^d,$$

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- This is known as “dilatation”, which can be defined on any locally convex topological linear space $\mathbb{V}$ (traced back to Karamata, 1932). Dilatation provides various versions of continuous majorization (Marshall & Olkin, 1979).
Tail Orthant Dependence Order

- Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be non-negative random vectors with dfs $F$ and $F'$ respectively.
- Assume that margins of $F$ and $F'$ are tail equivalent.
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- Assume that margins of $F$ and $F'$ are tail equivalent.

**Definition**

$X$ is said to be smaller than $Y$ in the sense of tail orthant dependence order, denoted as $X \leq_{tod} Y$, if for all $w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d$ with at least two components being finite,

$$\max_{1 \leq i \leq d} \{X_i/w_i\} \geq_{sto} \max_{1 \leq i \leq d} \{Y_i/w_i\}.$$  

That is, $X \leq_{tod} Y$ is equivalent to

$$\mathbb{P}(X_1 \leq tw_1, \ldots, X_d \leq tw_d) \leq \mathbb{P}(Y_1 \leq tw_1, \ldots, Y_d \leq tw_d)$$

for all $t > t_w$ for some $t_w > 0$.  

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If margins are not tail equivalent...

- If some corresponding margins of $F$ and $F'$ are not tail equivalent, one can still define the “tail orthant order” to compare their tail behaviors in upper orthants. But all corresponding margins of $F$ and $F'$ have to be tail equivalent in order to compare their tail dependence.
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- If some margins of $F$ (or $F'$) are not tail equivalent, then one can still define the tail orthant dependence order but scaling functions would be different among the components.
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- If some margins of $F$ (or $F'$) are not tail equivalent, then one can still define the tail orthant dependence order but scaling functions would be different among the components.

- Another alternative is to convert all the margins of $F$ and $F'$ to standard Pareto margins, resulting in Pareto copulas (Klüppelberg & Resnick, 2008), and then compare their Pareto copulas using the $\leq_{\text{tod}}$ order.
Properties

- $X \leq_{od} Y$ implies $X \leq_{tod} Y$.
- Let $G$ and $G'$ be two multivariate extreme value distributions with standard Fréchet margins. Then $G \leq_{od} G'$ is equivalent to $G \leq_{tod} G'$.
- $(X_1, \ldots, X_d) \leq_{tod} (Y_1, \ldots, Y_d)$ implies that $(g_1(X_1), \ldots, g_d(X_d)) \leq_{tod} (g_1(Y_1), \ldots, g_d(Y_d))$ for all continuous functions $g_1, \ldots, g_d: \mathbb{R}_+ \mapsto \mathbb{R}_+$ that are asymptotically increasing and homogeneous of the same order $\kappa > 0$.
- If $[X \mid \Theta = \theta] \leq_{tod} [Y \mid \Theta = \theta]$ for all $\theta$ in the bounded support of $\Theta$, then $X \leq_{tod} Y$.
- If $X \leq_{tod} Y$ and $Z \leq_{tod} W$ and $(X, Y)$ and $(Z, W)$ are independent, then $X \lor Z \leq_{tod} Y \lor W$. 

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- Assume that corresponding margins $F$ and $F'$ are tail equivalent.
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Assume that corresponding margins $F$ and $F'$ are tail equivalent.

**Theorem**

$X_n \leq_{tod} Y_n$ implies that $G \leq_{od} G'$
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**Theorem**

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**Theorem**

If $\mu((x, \infty]) < \mu'((x, \infty])$ for all $x \in S^d_{1+}$ such that at least two components are finite, then $X_n \leq_{tod} Y_n$