Tail Dependence for Heavy-Tailed Scale Mixtures of Multivariate Distributions

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Abstract

The tail dependence of multivariate distributions is frequently studied via the tool of copulas. This paper develops a general method, which is based on multivariate regular variation, to evaluate the tail dependence of heavy-tailed scale mixtures of multivariate distributions, whose copulas are not explicitly accessible. Tractable formulas for tail dependence parameters are derived, and a sufficient condition under which the parameters are monotone with respect to the heavy-tail index is obtained. The multivariate elliptical distributions are discussed to illustrate the results.

Key words and phrases: Tail dependence, heavy tail, multivariate regular variation, copula, hazard rate ordering, multivariate elliptical distribution.


1 Introduction

Tail dependence parameters describe the amount of dependence in the upper tail or lower tail of a multivariate distribution and can be used to analyze the dependence among extreme values. This paper focuses on the tail dependence of multivariate distributions of random vectors in the form:

\[(X_1, \ldots, X_d) := (RT_1, \ldots, RT_d),\] (1.1)

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where \((T_1, \ldots, T_d)\) has the joint distribution \(G(t_1, \ldots, t_d)\) with some finite moments, and the scale variable \(R\), independent of \((T_1, \ldots, T_d)\), has a regularly varying right tail at infinity with survival function (see, e.g., [3] for detail),

\[
\bar{F}(r) := 1 - F(r) = \frac{L(r)}{r^\alpha}, \quad r > 0, \quad \alpha > 0,
\]

where \(L\) is a slowly varying function; that is, \(L\) is a positive function on \((0, \infty)\) with property

\[
\lim_{r \to \infty} \frac{L(cr)}{L(r)} = 1, \quad \text{for every } c > 0.
\]

The class of distributions of \((1.1)\) has a variety of interpretations in different applications, including, for example, multivariate elliptical distributions and various multivariate Pareto distributions as special cases.

It is possible to define and analyze tail dependence via copulas; however, the copula method’s effectiveness diminishes in higher dimensions and in the case of \((1.1)\), where the copulas are often not explicitly accessible. A copula \(C\) is a distribution function, defined on the unit cube \([0, 1]^d\), with uniform one-dimensional margins. Given a copula \(C\), if one defines

\[
F(t_1, \ldots, t_d) := C(F_1(t_1), \ldots, F_d(t_d)), \quad (t_1, \ldots, t_d) \in \mathbb{R}^d,
\]

then \(F\) is a multivariate distribution with univariate margins \(F_1, \ldots, F_d\). Given a distribution \(F\) with margins \(F_1, \ldots, F_d\), there exists a copula \(C\) such that \((1.4)\) holds. If \(F_1, \ldots, F_d\) are all continuous, then the corresponding copula \(C\) is unique and can be written as

\[
C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in [0, 1]^d.
\]

Thus, for multivariate distributions with continuous margins, the univariate margins and the multivariate dependence structure (as described by their copulas) can be separated [25, 18].

The survival copula is defined similarly. Consider a random vector \((X_1, \ldots, X_d)\) with continuous margins \(F_1, \ldots, F_d\) and copula \(C\). Observe that \(\bar{F}_i(X_i) = 1 - F_i(X_i), 1 \leq i \leq d\), is also uniformly distributed over \([0, 1]\), thus

\[
\hat{C}(u_1, \ldots, u_d) := \Pr\{\bar{F}_1(X_1) \leq u_1, \ldots, \bar{F}_d(X_d) \leq u_d\}
\]

is a copula. \(\hat{C}\) is called the survival copula of \((X_1, \ldots, X_d)\). The joint survival function of random vector \((X_1, \ldots, X_d)\) is expressed as

\[
\bar{F}(t_1, \ldots, t_d) := \Pr\{X_1 > t_1, \ldots, X_d > t_d\} = \hat{C}(\bar{F}_1(t_1), \ldots, \bar{F}_d(t_d)), \quad (t_1, \ldots, t_d) \in \mathbb{R}^d.
\]
It also follows that for any \((u_1, \ldots, u_d) \in [0, 1]^d\),
\[
C(u_1, \ldots, u_d) := \Pr\{F_1(X_1) > u_1, \ldots, F_d(X_d) > u_d\} = \hat{C}(1 - u_1, \ldots, 1 - u_d),
\] (1.5)
where \(\hat{C}\) is the joint survival function of copula \(C\).

The bivariate tail dependence parameter has been discussed extensively in [11]. Moreover, various multivariate tail dependence parameters can be introduced. Definition 1.1 details two notions of tail dependence which have been discussed in [22, 14].

**Definition 1.1.** Let \(X = (X_1, \ldots, X_d)\) be a random vector with continuous margins \(F_1, \ldots, F_d\) and copula \(C\). Let \(N\) denote the index set \(\{1, \ldots, d\}\).

1. \(X\) is said to be **upper-orthant tail dependent** if for some subset \(\emptyset \neq J \subset N\), the following limit exists and is positive.

\[
\tau_J = \lim_{u \to 1^-} \Pr\{F_j(X_j) > u, \forall j \in N \setminus J \mid F_i(X_i) > u, \forall i \in J\} > 0.
\] (1.6)

If for all \(\emptyset \neq J \subset N\), \(\tau_J = 0\), then we say \(X\) is upper-orthant tail independent.

2. \(X\) is said to be **upper extremal dependent** if the following limit exists and is positive.

\[
\gamma = \lim_{u \to 1^-} \Pr\{F_j(X_j) > u, \forall j \in N \mid F_i(X_i) > u, \exists i \in N\} > 0.
\] (1.7)

If \(\gamma = 0\), then we say \(X\) is upper extremal independent.

The lower tail dependence of a copula is defined as the upper tail dependence of its survival copula in (1.5). The limits \(\tau_J\) (respectively, \(\gamma\)) are called the upper tail (extremal) dependence parameters. Obviously, tail dependence is a copula property; these parameters do not depend on the marginal distributions. If \(X_1, \ldots, X_n\) are independent, then the corresponding upper tail (extremal) dependence parameters are all zeros. Clearly, \(\tau_J \geq \gamma\), for all nonempty \(J \subset N\). Thus, the extremal dependence parameter provides a lower bound for orthant tail dependence parameters. The multivariate tail dependence parameters \(\tau_J\)'s have been used in [9] to analyze the contagion risk in banking systems, and the extremal dependence parameter \(\gamma\) in the bivariate case has been used in [5] to analyze the extremal dependence in financial return data.

Definition 1.1 implies that the tail dependence parameters of a distribution can be derived directly from its copula, and this has been done for the bivariate case [11, 4]. In [14, 15], using the copula method, Li derived explicit expressions of the orthant tail dependence for Marshall-Olkin distributions as well as multivariate extreme value distributions and their scaled mixtures. It is already evident in [14] that even if the copula is explicitly available,
the direct copula method has to equalize the distribution margins by taking component-wise marginal transforms (see (1.4)); this method becomes really cumbersome in higher dimensions. The copula method is obviously ineffective for (1.1) whose copulas are generally not explicitly accessible. This paper develops an alternative method to derive tractable tail dependence formulas for (1.1), which belong to the family of multivariate regularly varying distributions [1, 2]. Our method is rooted in the theory of multivariate regular variation [20, 21], and thus, unlike the copula method, our method avoids taking marginal transforms on the entire distribution. It is well-known that the bivariate normal distribution is asymptotically tail independent if its correlation coefficient $\rho$ is less than 1. Bivariate elliptical distributions, a special case of (1.1), possess the upper- (and lower-) orthant tail dependence property if their generating random variable is regularly varying [10, 22]. The formula for the extremal dependence $\gamma$ of bivariate elliptical distributions is also derived in [8]. Our method not only yields similar results for multivariate elliptical distributions, but also captures a more detailed relationship between various tail/extremal dependence parameters and theory of multivariate regular variation.

The paper is organized as follows. In Section 2, we first discuss the tail dependence functions, which can be used to express various tail dependence parameters. Furthermore, it is shown that for a multivariate regularly varying distribution, the upper tail dependence function and the intensity measure are equivalent in describing its extremal dependence structure. In Section 3, we derive explicit expressions for the tail dependence of (1.1). Multivariate elliptical distributions are also discussed as examples. Finally, some comments in Section 4 conclude the paper. Throughout this paper, the terms ‘increasing’ and ‘decreasing’ are used in the weak sense, and the measurability of functions is assumed. The notations $a \lor b$ and $a \land b$ mean the maximum and the minimum of $a$ and $b$ respectively.

2 Tail Dependence Function of Multivariate Regular Variation

Let $F$ be the distribution function (d.f.) of a $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ with continuous margins $F_1, \ldots, F_d$ and copula $C$. The lower and upper tail dependence functions, denoted by $b(\cdot; C)$ and $b^*(\cdot; C)$ respectively, are introduced in [13, 19, 12] as follows,

$$b(w; C) := \lim_{u \rightarrow 0^+} \frac{C(uw_j, \forall j \in N)}{u}, \quad \forall w = (w_1, \ldots, w_d) \in \mathbb{R}^d_+;$$

$$b^*(w; C) := \lim_{u \rightarrow 0^+} \frac{C(1 - uw_j, \forall j \in N)}{u}, \quad \forall w = (w_1, \ldots, w_d) \in \mathbb{R}^d_+, \quad \text{(2.1)}$$
provided that the limits exist. Since \( b(w; \hat{C}) = b^*(w; C) \) where \( \hat{C}(u_1, \ldots, u_d) = C(1 - u_1, \ldots, 1 - u_d) \) is the survival copula in (1.5), this paper focuses only on upper tail dependence. The explicit expression of \( b^* \) for elliptical distributions was obtained in [13]. A theory of tail dependence functions was developed in [19, 12] based on Euler’s homogeneous representation:

\[
b^*(w; C) = \sum_{j=1}^{d} w_j t_j(w_i, i \neq j | w_j), \quad \forall w = (w_1, \ldots, w_d) \in \mathbb{R}^d_+,
\]

(2.2)

where

\[
t_j(w_i, i \neq j | w_j) := \lim_{u_+ \rightarrow 0} \Pr\{F_i(X_i) > 1 - w_i u, \forall i \neq j | F_j(X_j) = 1 - w_j u\}, \quad j \in N.
\]

The \( t_j \)'s are called **upper conditional tail dependence functions**. For copulas with explicit expressions, the tail dependence functions are obtained directly from the copulas with relative ease. For copulas without explicit expressions, the tail dependence functions can be obtained from (2.2) by exploring closure properties of the related conditional distributions. In [19], for example, the tail dependence function of the multivariate \( t \) distribution is obtained by (2.2).

If follows from (1.6) and (2.1) that the upper tail dependence parameters can be expressed as

\[
\tau_J = \frac{b^*(1, \ldots, 1; C)}{b^*(1, \ldots, 1; C_J)}, \quad \text{for all } \emptyset \neq J \subset N,
\]

(2.3)

where \( C_J \) is the multivariate margin of \( C \) with component indexes in \( J \). It is shown in [12] that \( b^*(w; C) > 0 \) for all \( w \in \mathbb{R}^d_+ \) if and only if \( b^*(1, \ldots, 1; C) > 0 \). Unlike the \( \tau_J \), however, the tail dependence function provides all the extremal dependence information of the copula \( C \) as specified by its extreme value copula (EV copula). The upper EV copula of \( C \), denoted by \( C^{UEV} \), is defined as \( C^{UEV}(u_1, \ldots, u_d) := \lim_{n \rightarrow \infty} C^n(u_1^{1/n}, \ldots, u_d^{1/n}) \) for any \( (u_1, \ldots, u_d) \in [0,1]^d \) if the limit exists [11].

Define the upper **exponent function** \( a^*(\cdot; C) \) of copula \( C \) as

\[
a^*(w; C) := \sum_{S \subseteq N, S \neq \emptyset} (-1)^{|S|-1} b^*_S(w_i, i \in S; C_S),
\]

(2.4)

where \( b^*_S(w_i, i \in S; C_S) \) denotes the upper tail dependence function of the margin \( C_S \) of \( C \) with component indexes in \( S \). It follows from (1.7) and (2.4) that the extremal dependence parameter can be expressed as

\[
\gamma = \frac{b^*(1, \ldots, 1; C)}{a^*(1, \ldots, 1; C)}.
\]

(2.5)
Similar to tail dependence functions, the exponent function has the following homogeneous representation:

\[ a^*(\mathbf{w}; C) = \sum_{j=1}^{d} w_j \tilde{t}_j(w_i, i \neq j \mid w_j), \quad \forall \mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d_+, \]  

(2.6)

where

\[ \tilde{t}_j(w_i, i \neq j \mid w_j) = \lim_{u \to 0^+} \Pr\{F_i(X_i) \leq 1 - w_i u, \forall i \neq j \mid F_j(X_j) = 1 - w_j u\}, \quad j \in N. \]

It is shown in [12] that tail dependence functions \( \{b^*_S(w_i, i \in S; C_S)\} \) and the exponent function \( a^*(\mathbf{w}; C) \) are uniquely determined from each other.

**Theorem 2.1.** ([19], [12]) Let \( C \) be a \( d \)-dimensional copula. Then the upper EV copula

\[ C_{UEV}(u_1, \ldots, u_d) = \exp\{-a^*(-\log u_1, \ldots, -\log u_d)\}, \quad \forall (u_1, \ldots, u_d) \in [0, 1]^d \]

where \( a^* \) is given by either (2.4) or (2.6).

Using the language of extreme value theory (see, e.g., Ch. 7 of [17]), if the upper tail dependence function \( b^*(\cdot; C) \) exists, then the copula \( C \) is in the copula domain of attraction of the EV copula \( C_{UEV} \) and \( C_{UEV} \) is uniquely determined by \( b^*(\cdot; C) \). Next, we will show that the upper tail dependence function and the intensity measure for multivariate regular variation are also uniquely determined from each other. For notational convenience, we denote hereafter by \([a, b]\) the Cartesian product \( \prod_{i=1}^{d} [a_i, b_i] \) where \( a, b \in \mathbb{R}_{d}^d \) and \( a_i \leq b_i \) for each \( i \). The following definition can be found in [20, 7, 21].

**Definition 2.2.** A random vector \( X \) is said to be multivariate regularly varying (MRV) if there exists a Radon measure \( \mu \) (i.e., finite on compact sets), called the intensity measure, on \( \mathbb{R}^d \setminus \{0\} \)\(^1\) such that

\[ \lim_{t \to \infty} \frac{\Pr\{X \in tB\}}{\Pr\{|X| > t\}} = \mu(B), \]

for any relatively compact set \( B \subset \mathbb{R}^d \setminus \{0\} \) that satisfies \( \mu(\partial B) = 0 \), where \( |\cdot| \) denotes a norm on \( \mathbb{R}^d \).

If \( X \) is MRV with \( \mu([0, 1]^c) > 0 \), where \( 1 \) denotes the vector of ones, then for every relatively compact set \( B \subset \mathbb{R}^d \setminus \{0\} \) that satisfies \( \mu(\partial B) = 0 \),

\[ \lim_{t \to \infty} \frac{\Pr\{X \in tB\}}{\Pr\{X \in t[0, 1]^c\}} = \tilde{\mu}(B), \]

(2.8)

\(^1\)Here \( \mathbb{R}^d = [-\infty, \infty]^d \) is compact and the punctured version \( \mathbb{R}^d \setminus \{0\} \) is modified via one-point uncompactification (see, e.g., [21]).
where $\tilde{\mu}(B) = \mu(B)/\mu([0,1]^c)$. In particular, any MRV d.f. with support in $\mathbb{R}_+^d = [0, \infty]^d$ admits the following spectral representation.

**Theorem 2.3.** ([20, 7, 21]) If $X$ is non-negative with MRV distribution $F$, then there exists a Radon measure $\tilde{\mu}$ on $\mathbb{R}_+^d$ such that
\[
\lim_{t \to \infty} \frac{1 - F(t \mathbf{x})}{1 - F(t \mathbf{1})} = \lim_{t \to \infty} \frac{\Pr\{X/t \in [0, \mathbf{x}]^c\}}{\Pr\{X/t \in [0, \mathbf{1}]^c\}} = \tilde{\mu}([0, \mathbf{x}]^c),
\]
for all continuous points $\mathbf{x}$ of $\tilde{\mu}$, where $\tilde{\mu}([0, \mathbf{x}]^c) = c \int_{S_+^{d-1}} \max_{1 \leq j \leq d} (u_j/x_j)^\alpha S(d\mathbf{u})$ for some constant $c$ and a probability measure $S$ on $S_+^{d-1} := \{x \in \mathbb{R}_+^d : \|x\| = 1\}$.

There is a well-known relation (see Proposition 5.15 in [20] for detail) between MRV distributions and multivariate extreme value (MEV) distributions with identical Fréchet margins $H(x; 1) := \exp\{-x^{-1}\}$ for $x > 0$. In general, the margins of an MEV distribution are expressed in terms of the generalized extreme value family,
\[
H(x; \gamma) := \exp\{-(\max\{1 + \gamma x, 0\})^{-1/\gamma}\}, \ x \in \mathbb{R}, \ \gamma \in \mathbb{R}.
\]
Note, however, that the parametric feature enjoyed by the univariate EV distributions vanishes in the multivariate context.

**Theorem 2.4.** Consider a non-negative random vector $X = (X_1, \ldots, X_d)$ with MRV d.f. $F$ and continuous margins $F_1, \ldots, F_d$. Let $C^F$ and $\mu$ denote, respectively, the copula and the intensity measure of $F$. If the margins are tail equivalent (i.e. $F_i(x)/F_j(x) \to 1$ as $x \to \infty$ for any $i \neq j$), then the upper tail dependence function $b^*(\cdot; C^F)$ exists and
\[
1. \ b^*(\mathbf{w}; C^F) = \frac{\mu([\prod_{i=1}^d [w_i^{-1/\alpha}, \infty))]}{\mu([1, \infty) \times \mathbb{R}_+^{d-1})} \text{ and } a^*(\mathbf{w}; C^F) = \frac{\mu([\prod_{i=1}^d [0, w_i^{-1/\alpha})^c])}{\mu([0,1] \times \mathbb{R}_+^{d-1})^c};
2. \ \frac{\mu([w, \infty])}{\mu([0,1]^c)} = \frac{b^*([w_1^{-\alpha}, \ldots, w_d^{-\alpha})]C^F)}{a^*([1,1]C^F)}, \text{ and } \frac{\mu([0, w]^c)}{\mu([0,1]^c)} = \frac{a^*([\prod_{i=1}^d w_i^{-\alpha}]C^F)}{a^*([1,1]C^F)}.
\]

**Proof.** Since each $F_i$, $i \in N$, is regularly varying, from (1.2) and (1.3) we have that $\bar{F}_i(x) = L_i(x)/x^\alpha$ for $x \geq 0$. To estimate $\bar{F}_i^{-1}(w_i u)$ when $u \to 0^+$ for fixed $w_i > 0$, consider
\[
\bar{F}_i(w_i^{1/\alpha} x) = \frac{L_i(w_i^{1/\alpha} x)}{w_i x^\alpha} = w_i^{-1} \bar{F}_i(x) g_i(w_i, x),
\]
where $g_i(w_i, x) := L_i(w_i^{1/\alpha} x)/L_i(x) \to 1$ as $x \to \infty$. Substitute $\bar{F}_i(x) = w_i u$ into above expression and take $\bar{F}_i^{-1}$ on both sides, we obtain that $\bar{F}_i^{-1}(w_i u) = w_i^{-1/\alpha} \bar{F}_i^{-1}(ug_i(w_i, \bar{F}_i^{-1}(w_i u)))$. Asymptotically, $\bar{F}_i^{-1}(w_i u) \approx w_i^{-1/\alpha} \bar{F}_i^{-1}(u)$ as $u \to 0^+$. 7
For any fixed $w = (w_1, \ldots, w_d)$ with $w_i > 0$, $i \in N$, consider

$$
\nu^*(w; C^F) = \lim_{u \to 0^+} \frac{\Pr\{F_i(X) > 1 - w_i u, \forall i \in N\}}{\Pr\{F_i(X) > 1 - u\}} = \lim_{u \to 0^+} \frac{\Pr\{X_i > \bar{F}_i^{-1}(w_i u), \forall i \in N\}}{\Pr\{X_1 > \bar{F}_1^{-1}(u)\}}
$$

$$
= \lim_{u \to 0^+} \frac{\Pr\{X_i > w_i^{-1/\alpha} \bar{F}_i^{-1}(u)\}}{\Pr\{X_1 > \bar{F}_1^{-1}(u)\}}, \quad (2.10)
$$

Since $\bar{F}_i(x) / \bar{F}_1(x) \to 1$ as $x \to \infty$, we have from Proposition 2.6 of [21] that $\bar{F}_i^{-1}(u) / \bar{F}_1^{-1}(u) \to 1$ as $u \to 0^+$. For any small $\epsilon > 0$, when $u$ is sufficiently small,

$$
1 - \epsilon < g_i(w_i, \bar{F}_i^{-1}(w_i u)) < 1 + \epsilon, \text{ for all } i \in N.
$$

Thus, when $u$ is sufficiently small,

$$
\frac{\bar{F}_i^{-1}(u(1 - \epsilon))}{\bar{F}_1^{-1}(u)} \geq \frac{\bar{F}_i^{-1}(u g_i(w_i, \bar{F}_i^{-1}(w_i u)))}{\bar{F}_1^{-1}(u)} \geq \frac{\bar{F}_i^{-1}(u(1 + \epsilon))}{\bar{F}_1^{-1}(u)} . \quad (2.11)
$$

Since $\bar{F}_1^{-1}(u)$ is regularly varying at 0 (see, e.g., Proposition 2.6 of [21]), or more precisely, $\bar{F}_1^{-1}(uc) / \bar{F}_1^{-1}(u) \to c^{-1/\alpha}$ as $u \to 0^+$ for any $c > 0$, then by taking the limits in (2.11) we have

$$(1 - \epsilon)^{-1/\alpha} \geq \frac{\bar{F}_1^{-1}(u g_i(w_i, \bar{F}_i^{-1}(w_i u)))}{\bar{F}_1^{-1}(u)} \geq (1 + \epsilon)^{-1/\alpha}, \text{ as } u \to 0^+ ,$$

for any small $\epsilon > 0$. That is, when $u$ is sufficiently small,

$$(1 - \epsilon)^{-1/\alpha} \bar{F}_1^{-1}(u) \geq \bar{F}_1^{-1}(u g_i(w_i, \bar{F}_i^{-1}(w_i u))) \geq (1 + \epsilon)^{-1/\alpha} \bar{F}_1^{-1}(u), \forall \epsilon > 0, i \in N.$$

Combining these inequalities with (2.10), then for any $\epsilon > 0$ we have

$$
\lim_{u \to 0^+} \frac{\Pr\{X_i > w_i^{-1/\alpha}(1 - \epsilon)^{-1/\alpha} \bar{F}_1^{-1}(u), \forall i \in N\}}{\Pr\{X_1 > \bar{F}_1^{-1}(u)\}} \leq \nu^*(w; C^F) \leq \lim_{u \to 0^+} \frac{\Pr\{X_i > w_i^{-1/\alpha}(1 + \epsilon)^{-1/\alpha} \bar{F}_1^{-1}(u), \forall i \in N\}}{\Pr\{X_1 > \bar{F}_1^{-1}(u)\}},
$$

which implies, after substituting $t = \bar{F}_1^{-1}(u)$,

$$
\nu^*(w; C^F) = \lim_{t \to \infty} \frac{\Pr\{X_i > w_i^{-1/\alpha} t, i \in N\}}{\Pr\{X_1 > t\}}, \quad (2.12)
$$

---

2This result is stated in [21] in terms of increasing regularly varying functions, but is easily verified to be true for decreasing regularly varying functions.
Set $A = \prod_{i=1}^{d}[w_i^{-1/\alpha}, \infty]$ and $B = [1, \infty] \times \mathbb{R}_+^{d-1}$, then (2.7) implies
\[
b^*(w; C^F) = \frac{\mu(A)}{\mu(B)} = \frac{\mu(\prod_{i=1}^{d}[w_i^{-1/\alpha}, \infty])}{\mu([1, \infty] \times \mathbb{R}_+^{d-1})}.
\]

Similar to (2.10), it follows from (2.4) that
\[
a^*(w; C^F) = \lim_{u \to 0^+} \frac{\Pr\{X_i > 1 - w_i u, \exists i \in N\}}{\Pr\{F_i(X_i) > 1 - u\}} = \lim_{u \to 0^+} \frac{\Pr\{X_i > w_i^{-1/\alpha} \bar{F}_i^{-1}(ug_i(w_i, \bar{F}_i^{-1}(w_i u))), \exists i \in N\}}{\Pr\{X_i > \bar{F}_i^{-1}(u)\}}.
\]

Borrowing from the derivation of (2.12) then using (2.7), we have
\[
a^*(w; C^F) = \lim_{t \to \infty} \frac{\Pr\{X_i > w_i^{-1/\alpha} t, \exists i \in N\}}{\Pr\{X_i > t\}} = \frac{\mu((\prod_{i=1}^{d}(0, w_i^{-1/\alpha}])^c)}{\mu([1, \infty] \times \mathbb{R}_+^{d-1})}.
\]

The expressions in part 2 can be easily obtained from part 1. □

Observe that the rescaled intensity measure $\tilde{\mu}(B) = \mu(B)/\mu([0, 1]^c)$ in part 2 of Theorem 2.4 (also see (2.8) and (2.9)) satisfies that $\tilde{\mu}((0, 1]^c) = 1$. The intensity measures $\mu$ and $\tilde{\mu}$ are uniquely determined from each other.

In general, $\bar{F}_i(x)/\bar{F}_j(x) \to r_{ij}$ as $x \to \infty$ for any $i \neq j$ [21]. If $0 < r_{ij} < \infty$, then Theorem 2.4 still holds by properly adjusting the marginal scaling constants. If $r_{ij} = 0$ or $r_{ij} = \infty$, then some margins have heavier tails than others and more subtle separate marginal scalings are needed to derive the limiting results.

### 3 Tail Dependence of Heavy-Tailed Scale Mixtures of Multivariate Distributions

As we showed in Theorem 2.4, the upper tail dependence function represents a re-scaled version of the intensity measure. The link presented in Theorem 2.4 allows people on one hand to develop, via tail dependence functions, tractable parametric models for the intensity measure, and on the other hand to derive, via multivariate regular variation, tail dependence functions in the situations where copulas have neither explicit analytic expressions nor closure properties for their conditional distributions. We illustrate the latter part in this section for the random vectors of the form (1.1).

Let $T_{i+} := T_i \vee 0, \ i \in N$, and assume in this section that $0 < E(T_i^{\alpha+\epsilon}) < \infty$ for some $\epsilon > 0$. Since $R$ is regularly varying with survival function (1.2), then, by Breiman’s theorem
function $\Pr$ the intensity measure affine transform of a random vector that is regularly varying. Applying this formula to (3.1), down to the determination of the intensity measure of $(X, Y)$ by Theorem 2.4, the expression of the upper tail dependence function of $(X, Y)$ boils down to the determination of the intensity measure of $(Y_1, \ldots, Y_d)$ in (3.1).

Proposition A.1 in [2] presents a general formula for the intensity measure of a random affine transform of a random vector that is regularly varying. Applying this formula to (3.1), the intensity measure $\mu$ of (3.1) is given by, for any Borel measurable subset $B \subseteq \mathbb{R}_+$,

$$\mu(B) = E(\nu(D[(E(T_{i+}^\alpha))^{-1/\alpha}T_{1+}, \ldots, (E(T_{d+}^\alpha))^{-1/\alpha}T_{d+}]](B)))$$

where $\nu$ is the intensity measure of $(X_1, \ldots, X_d)$. Using (2.9), we have for any non-negative $w = (w_1, \ldots, w_d)$,

$$\nu([0, w]^c) = \lim_{t \to \infty} \frac{\Pr\{R > t \land \bigwedge_{i=1}^d w_i\}}{\Pr\{R > t\}} = \bigwedge_{i=1}^d w_i^{-\alpha}.$$

Using the inclusion-exclusion relation and the fact that $\sum_{\emptyset \not= S \subseteq N} (-1)^{|S|-1} \bigwedge_{i \in S} w_i = \bigwedge_{i \in N} w_i$ for all non-negative $w_1, \ldots, w_d$, we also have

$$\nu([w, \infty]) = \bigwedge_{i=1}^d w_i^{-\alpha}.$$

Substitute these two expressions into (3.2), then

$$\mu([0, w]^c) = E\left(\bigwedge_{i=1}^d \frac{w_i^{-\alpha}T_{i+}^\alpha}{E(T_{i+}^\alpha)}\right), \text{ and } \mu([w, \infty]) = E\left(\bigwedge_{i=1}^d \frac{w_i^{-\alpha}T_{i+}^\alpha}{E(T_{i+}^\alpha)}\right),$$

which lead to the expression of the upper tail dependence function.

In summary, we give the following theorem.

**Theorem 3.1.** Consider a random vector $X = (RT_1, \ldots, RT_d)$ in the form of (1.1) with d.f. $F$ and continuous margins $F_1, \ldots, F_d$. Assume that $0 < E(T_{i+}^{\alpha+\epsilon}) < \infty$, $i \in N$, for some $\epsilon > 0$, then the upper tail dependence and exponent functions are given by

$$b^*(w; C^F) = E\left(\bigwedge_{i=1}^d \frac{w_i T_{i+}^\alpha}{E(T_{i+}^\alpha)}\right) \text{ and } a^*(w; C^F) = E\left(\bigvee_{i=1}^d \frac{w_i T_{i+}^\alpha}{E(T_{i+}^\alpha)}\right).$$
Applying Theorem 3.1 to (2.3) and (2.5), we obtain the following corollary.

**Corollary 3.2.** Making the same assumptions as Theorem 3.1, the upper tail and extremal dependence parameters are given by

\[
\tau_J = \frac{E \left( \bigwedge_{i=1}^d (E(T_{i+}^{\alpha}))^{-1} T_{i+}^{\alpha} \right)}{E \left( \bigwedge_{i \in J} (E(T_{i+}^{\alpha}))^{-1} T_{i+}^{\alpha} \right)} \text{ for } \emptyset \neq J \subseteq N \quad \text{and} \quad \gamma = \frac{E \left( \bigwedge_{i=1}^d (E(T_{i+}^{\alpha}))^{-1} T_{i+}^{\alpha} \right)}{E \left( \bigvee_{i=1}^d (E(T_{i+}^{\alpha}))^{-1} T_{i+}^{\alpha} \right)}.
\]

We illustrate our main results using multivariate elliptical distributions. Let \( \Sigma \) be a \( d \times d \) positive semi-definite matrix, and \( U = (U_1, \ldots, U_m) \) be uniformly distributed on the unit sphere (with respect to Euclidean distance) in \( \mathbb{R}^m \). Consider the following stochastic representation,

\[
(X_1, \ldots, X_d)^T = (\mu_1, \ldots, \mu_d)^T + RA(U_1, \ldots, U_m)^T,
\]

where \( A \) is an \( d \times m \) matrix with \( AA^T = \Sigma \) and \( R \) is a non-negative random variable independent of \( U \). The distribution of \((X_1, \ldots, X_d)\) is known as an elliptical contoured distribution and is one of most widely used radially symmetric multivariate distributions. The examples of elliptical distributions include the multivariate normal, \( t \)- and logistic distributions. We refer to [6] for details on properties of elliptical distributions.

Let \((T_1, \ldots, T_d)^T = A(U_1, \ldots, U_m)^T\), then we have

\[
(X_1, \ldots, X_d) = (\mu_1, \ldots, \mu_d) + R(T_1, \ldots, T_d),
\]

which is a scale mixture of \((T_1, \ldots, T_d)\). The random vector \((X_1, \ldots, X_d)\) is regularly varying if and only if \( R \) has a regularly varying right tail [10, 22]. A formula for the bivariate tail dependence parameter has been also established in [10, 22], and the upper tail dependence function of a multivariate elliptical distribution was derived explicitly in [13]. The multivariate upper tail dependence and exponent functions, upper tail and extremal dependence parameters for multivariate elliptical distributions are now given by Theorem 3.1 and Corollary 3.2. We discuss the bivariate case in detail. Let

\[
\Sigma = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_{22}^2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \sigma_{11} & 0 \\ \frac{\sigma_{12}}{\sigma_{11}} & \sigma_{22} \sqrt{1 - \rho^2} \end{bmatrix},
\]

where \( \rho = \frac{\sigma_{12}}{\sigma_{11} \sigma_{22}} \). Thus,

\[
T_1 = \sigma_{11} U_1, \quad T_2 = \frac{\sigma_{12}}{\sigma_{11}} U_1 + \sigma_{22} \sqrt{1 - \rho^2} U_2.
\]

Observe that, marginally, \( RT_1 \) and \( RT_2 \) have one-dimensional elliptical distributions, and \( \sigma_{11}^{-1} RT_1 \) and \( \sigma_{22}^{-1} RT_2 \) have the same distribution [6]. Assume that \( R \) has a regularly varying
right tail with heavy-tail index $\alpha > 0$. Since $(X_1, X_2)$ and $(RT_1, RT_2)$ have the same tail dependence parameter, we have

$$
\tau_1 = \lim_{t \to \infty} \Pr\{\sigma_{22}^{-1}RT_2 > t \mid \sigma_{11}^{-1}RT_1 > t\}.
$$

Obviously, $(T_1, T_2)$ has a bounded support, thus it follows from Corollary 3.2 that

$$
\tau_1 = \frac{E(\sigma_{11}^{-1}T_1^+ \wedge \sigma_{22}^{-1}T_2^+)^\alpha}{E(\sigma_{11}^{-1}T_1^+)^\alpha} = \frac{E(U_1^+ \wedge (\rho U_1 + \sqrt{1-\rho^2}U_2^+))^\alpha}{E(U_1^+)^\alpha},
$$

(3.3)

where $U_1^+ := U_1 \vee 0$ and $(\rho U_1 + \sqrt{1-\rho^2}U_2^+)_+ := (\rho U_1 + \sqrt{1-\rho^2}U_2^+) \vee 0$.

If $\rho = 1$, then trivially $\tau_1 = 1$. Suppose that $\rho < 1$. Transferring to polar coordinates, we have

$$
U_1 = \cos \Theta, \quad U_2 = \sin \Theta, \quad \text{and} \quad \rho U_1 + \sqrt{1-\rho^2}U_2 = \sin(\Theta + \theta_0),
$$

where $\Theta$ is uniformly distributed on $[0, 2\pi]$ and $\theta_0 = \tan^{-1} \frac{\rho}{\sqrt{1-\rho^2}}$. Clearly,

$$
E(U_1^+)^\alpha = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^\alpha \theta d\theta = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \cos^\alpha \theta d\theta = \frac{1}{\pi} \int_{0}^{1} \frac{u^\alpha}{\sqrt{1-u^2}} du.
$$

To calculate the quantity in the numerator of (3.3), consider the boundary case when

$$
u_1 = \rho u_1 + \sqrt{1-\rho^2}u_2,
$$

which leads to the solution $\theta_1 = \tan^{-1} \frac{\nu_2}{\nu_1} = \tan^{-1} \frac{1-\rho}{\sqrt{1-\rho^2}}$. That is,

$$
\cos \theta \geq \sin(\theta + \theta_0) \geq 0 \quad \text{if} \quad -\theta_0 \leq \theta \leq \theta_1
$$

$$
0 \leq \cos \theta \leq \sin(\theta + \theta_0) \quad \text{if} \quad \theta_1 \leq \theta \leq \frac{\pi}{2}.
$$

Simple trigonometric arguments show that $\theta_0 + 2\theta_1 = \frac{\pi}{2}$. Thus,

$$
E(U_1^+ \wedge (\rho U_1 + \sqrt{1-\rho^2}U_2^+))^\alpha = \frac{1}{2\pi} \int_{-\theta_0}^{\theta_1} \sin^\alpha(\theta + \theta_0) d\theta + \frac{1}{2\pi} \int_{\theta_1}^{\frac{\pi}{2}} \cos^\alpha \theta d\theta
$$

$$
= \frac{1}{2\pi} \int_{0}^{\frac{\pi}{2}-\theta_1} \sin^\alpha \theta d\theta + \frac{1}{2\pi} \int_{\theta_1}^{\frac{\pi}{2}} \cos^\alpha \theta d\theta.
$$

Since $\cos \theta_1 = \frac{1-\rho}{\sqrt{2-2\rho}} = \left(\frac{1+\rho}{2}\right)^{\frac{1}{2}}$, we obtain that

$$
E(U_1^+ \wedge (\rho U_1 + \sqrt{1-\rho^2}U_2^+))^\alpha = \frac{1}{\pi} \int_{0}^{\left(\frac{1+\rho}{2}\right)^{\frac{1}{2}}} \frac{u^\alpha}{\sqrt{1-u^2}} du.
$$
Hence,
\[
\tau_1 = \frac{\int_0^{(1+\rho^2)^{1/2}} \frac{\alpha^\rho u^\alpha}{\sqrt{1-u^2}} du}{\int_0^1 \frac{\alpha^\rho u^\alpha}{\sqrt{1-u^2}} du},
\]
which is identical to the one obtained in [10, 22].

An intriguing issue is whether or not these tail dependence parameters are monotone in response to changes in the heavy-tail index \(\alpha\). It was demonstrated by the numerical results in [22] that the tail dependence parameter \(\tau_1\) is decreasing in \(\alpha\). We show that this is indeed the case using the ratio-of-moments expressions obtained in Corollary 3.2.

A non-negative random variable \(X\) is said to be smaller than a non-negative random variable \(Y\) in the hazard rate ordering, denoted by \(X \leq_{hr} Y\), if the hazard rate of \(X\) is larger than that of \(Y\). A detailed discussion on the hazard rate ordering can be found, for example, in Chapter 1 of [24], from which, the following result also holds.

**Lemma 3.3.** Let \(X\) and \(Y\) be non-negative random variables. If \(X \leq_{hr} Y\), then \(\frac{E X^\alpha}{E Y^\alpha}\) is decreasing in \(\alpha\).

**Proof.** Theorem 1.B.12 in [24] states that if \(X \leq_{hr} Y\), then
\[
E g_2(X)E g_1(Y) \leq E g_1(X)E g_2(Y),
\]
for all non-negative real functions \(g_1\) and \(g_2\) satisfying that \(g_1(\cdot)\) is increasing and \(g_2(\cdot) = \frac{g_1(\cdot)}{g_1(\cdot)}\) is increasing. Consider the situation that \(\alpha_1 \leq \alpha_2\), \(g_1(x) = x^{\alpha_1}\) and \(g_2(x) = x^{\alpha_2}\) for \(x \geq 0\), and (3.4) is reduced to
\[
E X^{\alpha_2} E Y^{\alpha_1} \leq E X^{\alpha_1} E Y^{\alpha_2}.
\]
The monotonicity of \(\frac{E X^\alpha}{E Y^\alpha}\) follows. \(\square\)

**Proposition 3.4.** Let \(RT_i\) be the \(i\)th component of (1.1), where \(T_i\) and \(R\) satisfy the regularity conditions specified in Theorem 3.2. Suppose that \(E(T^\alpha_{i+}) = E(T^\alpha_{j+}), i \neq j\).

1. If \(\wedge_{i=1}^d T_{i+} \leq_{hr} \wedge_{i \in J} T_{i+}\), then \(\tau_J\) is decreasing in \(\alpha\).

2. If \(\wedge_{i=1}^d T_{i+} \leq_{hr} \vee_{i=1}^d T_{i+}\), then \(\gamma\) is decreasing in \(\alpha\).

**Proof.** If \(E(T^\alpha_{i+}) = E(T^\alpha_{j+}), i \neq j\), then from Corollary 3.2,
\[
\tau_J = \frac{E(\wedge_{i=1}^d T_{i+})^\alpha}{E(\wedge_{i \in J} T_{i+})^\alpha}, \text{ for all } \emptyset \neq J \subset N, \text{ and } \gamma = \frac{E(\wedge_{i=1}^d T_{i+})^\alpha}{E(\vee_{i=1}^d T_{i+})^\alpha}.
\]
The monotone properties follow from Lemma 3.3. \(\square\)
Notice that the inequality (3.4) resembles the property of total positivity of order 2, and in fact, Proposition 3.4 can be established directly by using the theory of total positivity.

To show that $\tau_1$ of a bivariate elliptical distribution is decreasing in $\alpha$, we need, by virtue of Lemma 3.3, to establish that $U_{1+} \geq hr U_{1+} \wedge (\rho U_1 + \sqrt{1 - \rho^2} U_2)_+$. From [24] (condition 1.B.3 in page 16), it is sufficient to show that

$$\frac{\Pr\{U_{1+} > t\}}{\Pr\{U_{1+} \wedge (\rho U_1 + \sqrt{1 - \rho^2} U_2)_+ > t\}}$$

is increasing in $t \in [0, s]$, (3.5)

where $s \leq 1$ is the right endpoint of the support of $U_{1+} \wedge (\rho U_1 + \sqrt{1 - \rho^2} U_2)_+$. Again, using the polar coordinate system, we have for any $t \in [0, s]$,

$$\Pr\{U_{1+} > t\} = \frac{\cos^{-1} t}{\pi},$$

$$\Pr\{U_{1+} \wedge (\rho U_1 + \sqrt{1 - \rho^2} U_2)_+ > t\} = \frac{(\cos^{-1} t - \sin^{-1} t)_+}{2\pi}.$$

It is easy to verify that $\frac{\frac{1}{2} \cos^{-1} t - \sin^{-1} t}{\pi}$ is increasing in $t \in [0, s]$, and then (3.5) follows. Therefore, $\tau_1$ is decreasing in $\alpha$.

4 Concluding Remarks

In this paper, we have developed a general method based on multivariate regular variation to derive tail dependence parameters for heavy-tailed scale mixtures of multivariate distributions. Our method properly re-scales the marginal distributions so that, asymptotically, the scaled random variables have the same tail marginal distributions, thus the calculation of their tail dependence avoids taking the marginal transforms on the entire distribution. Our tail analysis leads to tractable expressions of tail dependence parameters which depend on joint moments of the random variables being mixed and the heavy-tail index of the mixing random variable. Our method also establishes the link between multivariate tail dependence parameters and theory of multivariate regular variation. The structural properties of multivariate tail dependence, such as monotonicity of tail dependence with respect to structural parameters of the distribution, have also been discussed for bivariate elliptical distributions. Yet, a general theory remains to be developed.

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References


