Higher Order Tail Densities of Copulas and Hidden Regular Variation

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Abstract

A notion of higher order tail densities for copulas is introduced using multivariate regular variation of copula densities, and densities of multivariate extremes with various margins can then be studied in a unified fashion. We show that the tail of a multivariate density can be decomposed into the tail density of the underlying copula, coupled with marginal tail transforms of the three types: Fréchet, Gumbel, and Weibull types. We also derive the relation between the tail density and tail order functions of a copula in the context of hidden regular variation. Some illustrative examples are given.

Key words and phrases: Multivariate regular variation, tail dependence, upper exponent function, tail order function.

1 Introduction

To facilitate tail inference with various multivariate distributions, a better understanding of the strength of dependence in joint lower or joint upper distribution tails is often needed. In particular, in this paper we are interested in analyzing scale-invariant tail dependence strength of a multivariate distribution, separated from its univariate margins; that is, we are interested in analyzing tail dependence via the copula approach.

The main purpose of this paper is to develop a general copula tail density approach, so that tail properties can be derived directly from joint tails of multivariate densities. Most

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multivariate distributions are specified by densities and the tail density approach is especially tractable when a multivariate density has a simple, explicit expression, whereas its joint cumulative distribution function does not have a closed form. This research is motivated by the need to analyze the tail risk measures that are often expressed in terms of tail densities of the multivariate copulas of underlying loss distributions [12, 25].

Let \( X = (X_1, \ldots, X_d) \) be a random vector with distribution \( F \) and continuous marginal distributions \( F_1, \ldots, F_d \). Let \( \overline{F} \), and \( \overline{F}_1, \ldots, \overline{F}_d \) denote the corresponding survival functions. Assume throughout this paper that \( F \) has a density function \( f \). The tail behavior of \( F \) or \( f \) is often described using the notion of multivariate regular variation [22, 24], and without loss of generality, we assume that \( F \) concentrates on \( \mathbb{R}^d_+ = [0, \infty)^d \). A univariate Borel-measurable function \( V : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be regularly varying at \( \infty \) with tail index \( \rho \in \mathbb{R} \), denoted by \( V \in \text{RV}_\rho \), if

\[
\lim_{t \to \infty} \frac{V(tx)}{V(t)} = x^\rho, \quad x > 0.
\]

for some univariate regularly varying function \( V \in \text{RV}_{-\alpha} \) where \( \alpha > 0 \). The tail density \( \lambda(\cdot) \) in (1.1) was introduced in [3], and de Haan and Resnick proved in [4] that if furthermore (1.1) converges uniformly on the unit sphere of \( \mathbb{R}^d_+ \), then the regular variation (1.1) of a density implies multivariate regular variation of its cumulative distribution function \( F \); i.e.,

\[
\lim_{t \to \infty} \frac{1 - F(tx)}{V(t)} = \int_{[0,x]^c} \lambda(v) dv, \quad x > 0,
\]

where \([0, x]^c = (\prod_{i=1}^d [0, x_i])^c\) denotes the complement of \( \prod_{i=1}^d [0, x_i] \) for \( x = (x_1, \ldots, x_d) \). Here and in the sequel, vector addition/product and vector inequalities are operated component-wise. The fact that (1.1) implies (1.2) can be viewed as a multivariate extension of Karamata’s theorem for univariate regular variation. In contrast to the univariate case, however, the uniform convergence on the unit sphere of \( \mathbb{R}^d_+ \) is needed in the multivariate case to control the function’s variation moving from ray to ray originated from \( 0 \). There are multivariate distributions that satisfy (1.1) but not (1.2) (see [4]). The regular variation property (1.2) is crucial in deriving multivariate extreme value distributions for random samples drawn from distribution \( F \) [24].

The scale-invariant tail behavior of a multivariate distribution \( F \) can be studied using its marginally transformed distribution \( F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)) \), known as the copula of \( F \) [11]. A copula is invariant under marginal increasing transforms and thus preserves the scale-invariant dependence structure of the distribution. The strongest form of scale-invariant
dependence in the distribution tails is the first-order tail dependence when as $u \to 0^+$,

$$F(F^{-1}_1(u), \ldots, F^{-1}_d(u)) \sim u$$
in the lower tail and

$$\mathcal{F}(F^{-1}_1(u), \ldots, F^{-1}_d(u)) \sim u$$
in the upper tail, which are also studied under the notion of tail comonotonicity in [6, 7]; see [11, 14, 15] for more details about the first-order tail dependence that is simply referred to as tail dependence in the literature. Here and in the sequel, for any invertible function $h(\cdot)$, $h^{-1}(\cdot)$ denotes its inverse, and the tail equivalence of two functions $f(x) \sim g(x)$ as $x \to a$, $a \in \mathbb{R}$, means that

$$\lim_{x \to a} [f(x)/g(x)] = 1.$$ The strength of higher order scale-invariant tail dependence can be characterized via the lower and upper tail orders $\kappa_L, \kappa_U \geq 1$ (see [10, 5]) when

$$F(F^{-1}_1(u), \ldots, F^{-1}_d(u)) \sim u^{\kappa_L} \ell_L(u), \quad u \to 0^+, \quad (1.3)$$
in the lower tail and

$$\mathcal{F}(F^{-1}_1(u), \ldots, F^{-1}_d(u)) \sim u^{\kappa_U} \ell_U(u), \quad u \to 0^+, \quad (1.4)$$
in the upper tail, where $\ell_L(\cdot)$, and $\ell_U(\cdot)$ are slowly varying functions at 0 (i.e., $\ell_L(1/t), \ell_U(1/t) \in RV_0$). Smaller values of $\kappa_L$ ($\kappa_U$) indicate stronger dependence in the joint lower (upper) tail, and in contrast to the first-order tail dependence, there can be intermediate tail dependence when the tail order is between 1 and $d$. The tail orders are easy to compute when the distributions and quantile functions have closed forms [5], and it becomes difficult to compute for the distributions that are only specified by their densities. A copula tail density approach was developed in [18] to study the first-order scale-invariant tail dependence of multivariate distributions with tractable densities. Furthermore, it was shown in [18] (also see [16]) that the tail density $\lambda(\cdot)$ in the case of multivariate regular variation (1.2) can be decomposed into the copula tail density and marginal power transforms. In this paper, we extend the tail density approach to analyze the scale-invariant tail dependence and tail order for copulas that are specified only by their densities. Specifically, we show that the tails of various multivariate densities can be written in terms of higher order copula tail densities and marginal tail transforms of the three types (Fréchet, Gumbel, and Weibull). We also show that under mild regularity conditions, regular variation of tail densities of copulas, together with regularly varying margins, imply hidden regular variation (HRV); that is, multivariate regular variation resided within the interior of $\mathbb{R}^d_+$ where the joint tail probability decays to zero faster than marginal univariate regular variation.

We introduce in Section 2 the higher order tail density of a copula and apply it to analyze the tails of multivariate densities. We prove in Section 3 a multivariate copula version of Karamata’s theorem for the distributions with hidden regular variation. Some remarks in Section 4 conclude the paper.
2 Tail densities of copulas

A copula $C$ is a multivariate distribution with uniformly distributed univariate margins on $[0,1]$. Sklar’s theorem (see, e.g., Section 1.6 in [11]) states that every multivariate distribution $F$ with margins $F_1, \ldots, F_d$ can be written as $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ for some $d$-dimensional copula $C$. In fact, in the case of continuous univariate margins, $C$ is unique and

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)).$$

Let $(U_1, \ldots, U_d)$ denote a random vector with distribution $C$ and $U_i, 1 \le i \le d$, being uniformly distributed on $[0, 1]$. The survival copula $\hat{C}$ is defined as follows:

$$\hat{C}(u_1, \ldots, u_d) = \mathbb{P}(1 - U_1 \le u_1, \ldots, 1 - U_d \le u_d) = \mathbb{C}(1 - u_1, \ldots, 1 - u_d) \quad (2.1)$$

where $\mathbb{C}$ is the joint survival function of $C$. The survival copula $\hat{C}$ can be used to transform lower tail properties of $(U_1, \ldots, U_d)$ into the corresponding upper tail properties of $(1 - U_1, \ldots, 1 - U_d)$. Assume throughout this paper that the density $c(\cdot)$ of copula $C$ exists, and that $c(\cdot)$ is continuous in some small open neighborhoods of $0$ and $1 = (1, \ldots, 1)$ (i.e., ultimately continuous at $0$ and $1$).

The upper tail density of $C$ with tail order $\kappa_U$, denoted by $\lambda_U(\cdot; \kappa_U)$, is defined as follows:

$$\lambda_U(w; \kappa_U) := \lim_{u \to 0^+} \frac{c(1 - uw_i, 1 \le i \le d)}{u^{\kappa_U - d}\ell(u)} > 0, \; w = (w_1, \ldots, w_d) \in [0, \infty)^d \setminus \{0\}, \quad (2.2)$$

provided that the non-zero limit exists for some $\kappa_U \ge 1$ and some function $\ell(\cdot)$ that is slowly varying at $0$. Similarly, the lower tail density of $C$ with tail order $\kappa_L$, denoted by $\lambda_L(\cdot; \kappa_L)$, is defined as follows:

$$\lambda_L(w; \kappa_L) := \lim_{u \to 0^+} \frac{c(uw_i, 1 \le i \le d)}{u^{\kappa_L - d}\ell(u)} > 0, \; w = (w_1, \ldots, w_d) \in [0, \infty)^d \setminus \{0\}, \quad (2.3)$$

provided that the limit exists for some $\kappa_L \ge 1$ and some slowly varying function $\ell(\cdot)$. Clearly, the tail density functions are homogeneous; that is,

$$\lambda_U(tw; \kappa_U) = t^{\kappa_U - d}\lambda_U(w; \kappa_U), \quad \lambda_L(tw; \kappa_L) = t^{\kappa_L - d}\lambda_L(w; \kappa_U), \quad (2.4)$$

for any $t > 0$ and $w \in [0, \infty)^d \setminus \{0\}$. It follows from (2.4) that tail densities $\lambda_U(\cdot; \kappa_U)$ and $\lambda_L(\cdot; \kappa_L)$ that are non-zero at some points are positive everywhere. Since the lower tail density of a copula $C$ is the upper tail density of the survival copula $\hat{C}$ (see (2.1)), we focus only on the upper tail density.

To incorporate univariate margins, we assume, without loss of generality, that marginal distributions $F_i$’s belong to the max-domain of attraction of one of the three extreme value
distribution families; that is, Fréchet, Gumbel or Weibull distributions (see, e.g., Chapter 6 of [11]). Let $F$ be a $d$-dimensional distribution with continuous margins $F_1, \ldots, F_d$ that are right-tail equivalent in the sense that

$$f_i(t) \sim f_1(t), \quad \text{as } t \to x_{F_i} = x_{F_1}, \quad i = 1, \ldots, d,$$

(2.5)

where $x_{F_i}$ is the right-end point of the support of $F_i$, $f_i$ is the density of the $i$th margin $F_i$, $i = 1, \ldots, d$. Note that (2.5) implies that the usual tail equivalence $F_i(t) \sim F_1(t)$, $1 \leq i \leq d$, as $t \to x_{F_1}$, but the reverse may not be true because some densities may have dampened oscillations as $t \to x_{F_1}$.

**Proposition 2.1** (the Fréchet case where $x_{F_1} = \infty$). Assume that marginal densities are right-tail equivalent in the sense of (2.5). If the marginal density $f_i \in \text{RV}_{-\alpha-1}$, $i = 1, \ldots, d$, $\alpha > 0$, and the limit (2.2) holds locally uniformly for all $w \in [0, \infty)^d \backslash \{0\}$, then $F$ has a tail density $\lambda(\cdot)$ as defined in (1.1) and $\lambda(\cdot)$ is related to the upper tail density $\lambda_U(\cdot)$ of $C$ as follows: for any $w = (w_1, \ldots, w_d) > 0$,

$$\lambda(w) := \lim_{t \to \infty} \frac{f(tw)}{t^{-d V_{\alpha}(t)}} = \alpha^d(w_1^{-\alpha}, \ldots, w_d^{-\alpha})^{-\alpha-1} \lambda_U(w_1^{-\alpha}, \ldots, w_d^{-\alpha}; \kappa_U)$$

$$= \lambda_U(w_1^{-\alpha}, \ldots, w_d^{-\alpha}) J(w_1^{-\alpha}, \ldots, w_d^{-\alpha}) |J(w_1^{-\alpha}, \ldots, w_d^{-\alpha})|,$$

(2.6)

where $V \in \text{RV}_{-\alpha}$ and $J(w_1^{-\alpha}, \ldots, w_d^{-\alpha})$ is the Jacobian determinant of the homeomorphic transform $y_i = w_i^{-\alpha}$, $1 \leq i \leq d$.

**Proof.** Let $c(\cdot)$ denote the density of copula $C$, and then the density $f$ of $F$ is given by

$$f(x) = c(F_1(x_1), \ldots, F_d(x_d)) \prod_{i=1}^d f_i(x_i), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d,$$

(2.7)

and thus

$$f(tx) = c(F_1(tx_1), \ldots, F_d(tx_d)) \prod_{i=1}^d f_i(tx_i), \quad t > 0, \quad x = (x_1, \ldots, x_d) > 0.$$
Therefore, the tail density of $F$ is

$$f(tx) \sim c \left( 1 - \frac{\mathcal{F}_1(tx_1)}{\mathcal{F}_1(t)} \mathcal{F}_1(t), \ldots, 1 - \frac{\mathcal{F}_d(tx_d)}{\mathcal{F}_1(t)} \mathcal{F}_1(t) \right) \prod_{i=1}^d f_i(tx_i) \quad \frac{t^{-d}}{\mathcal{F}_1(t)^{\kappa_U \ell(F_1(t))}}.$$

Since the copula density $c(\cdot)$ is continuous in a small open neighborhood of $1 = (1, \ldots, 1)$, and (2.2) holds locally uniformly, the upper tail density $\lambda_U(\cdot)$ is continuous. For any small $\epsilon > 0$, it follows from the local uniform convergence that there exists a sufficiently large number $N_1$ such that for all $t > N_1$,

$$c \left( 1 - \frac{\mathcal{F}_1(tx_1)}{\mathcal{F}_1(t)} \mathcal{F}_1(t), \ldots, 1 - \frac{\mathcal{F}_d(tx_d)}{\mathcal{F}_1(t)} \mathcal{F}_1(t) \right) \frac{t^{-d}}{\mathcal{F}_1(t)^{\kappa_U \ell(F_1(t))}} - \lambda_U \left( \frac{\mathcal{F}_1(tx_1)}{\mathcal{F}_1(t)}, \ldots, \frac{\mathcal{F}_d(tx_d)}{\mathcal{F}_1(t)} ; \kappa_U \right) \leq \frac{\epsilon}{2}.$$

On the other hand, because of the continuity, for all $t > N_2$, where $N_2$ is sufficiently large,

$$\left| \lambda_U \left( \frac{\mathcal{F}_1(tx_1)}{\mathcal{F}_1(t)}, \ldots, \frac{\mathcal{F}_d(tx_d)}{\mathcal{F}_1(t)} ; \kappa_U \right) - \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha} ; \kappa_U) \right| \leq \frac{\epsilon}{2}.$$

Hence, as $t \to \infty$,

$$c \left( 1 - \frac{\mathcal{F}_1(tx_1)}{\mathcal{F}_1(t)} \mathcal{F}_1(t), \ldots, 1 - \frac{\mathcal{F}_d(tx_d)}{\mathcal{F}_1(t)} \mathcal{F}_1(t) \right) \frac{t^{-d}}{\mathcal{F}_1(t)^{\kappa_U \ell(F_1(t))}} \to \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha} ; \kappa_U),$$

from which, it follows that

$$\lambda(x) = \lim_{t \to \infty} \frac{f(tx)}{t^{-d} V_{\kappa_U}(t)} = \lim_{t \to \infty} c \left( 1 - \frac{\mathcal{F}_1(tx_1)}{\mathcal{F}_1(t)} \mathcal{F}_1(t), \ldots, 1 - \frac{\mathcal{F}_d(tx_d)}{\mathcal{F}_1(t)} \mathcal{F}_1(t) \right) \prod_{i=1}^d f_i(tx_i) \frac{t^{-d}}{\mathcal{F}_1(t)^{\kappa_U \ell(F_1(t))}}$$

$$= \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha} ; \kappa_U) \times \lim_{t \to \infty} \prod_{i=1}^d \left( \frac{f_i(tx_i)}{f_i(t)} \times t f_i(t) \times \frac{tx_i}{\mathcal{F}_1(t)} \right)$$

$$= \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha} ; \kappa_U) \times \alpha^d \left( \prod_{i=1}^d x_i^{-\alpha - 1} \right),$$

where Eq. (2.10) is due to Karamata’s theorem and the assumption that $f_i \in RV_{-\alpha - 1}$. Therefore, the tail density of $F$ exists and is given by (2.6). □

Note that if $w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d \setminus \{0\}$ with some components being zero, then $\lambda(w)$ in (2.6) is infinite. Note also that continuity of $c(\cdot)$ near $(1, \ldots, 1)$ in the absence of local uniform convergence does not imply continuity of $\lambda_U(\cdot)$, which is needed in the proof of (2.9). A counter example can be constructed (see page 87 of [4]) to show that local uniform convergence is needed in establishing (2.6).
Proposition 2.2 (the Gumbel case where $x_{F_i} = \infty$). Assume that marginal densities are right-tail equivalent in the sense of (2.5). Assume also that the marginal density $f_i$, $1 \leq i \leq d$, is differentiable and satisfies

$$f_i(t + m_i(t)x) \sim f_i(t)e^{-x}\sim f_1(t)e^{-x}, \ x \geq 0, \ \text{as} \ t \to \infty, \quad (2.11)$$

for some self-neglecting function $m_i(\cdot)$ whose derivative converges to 0; i.e., $m_i(t + m_i(t)x) \sim m_i(t)$, as $t \to \infty$, $i = 1, \ldots, d$. If the limit (2.2) holds locally uniformly for all $w \in [0, \infty)^d \setminus \{0\}$, then $F$ has a tail density $\lambda(\cdot)$ that is related to the upper tail density $\lambda_U(\cdot)$ of $C$ as follows: for any $w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d \setminus \{0\}$,

$$\lambda(w) := \lim_{t \to \infty} \frac{f(t + m(t))w_1, \ldots, t + m(t))w_d}{m^{-d}(t)V^{\kappa_U}(t)} = e^{-\sum_{i=1}^d w_i} \lambda_U(e^{-w_1}, \ldots, e^{-w_d}; \kappa_U) = \lambda_U(e^{-w_1}, \ldots, e^{-w_d}; \kappa_U) |J(e^{-w_1}, \ldots, e^{-w_d})|, \quad (2.12)$$

where $V(t)$ is a function satisfying that $V(t + m(t)x) \sim V(t)e^{-x}$ as $t \to \infty$ for some self-neglecting function $m(t)$, and $J(e^{-w_1}, \ldots, e^{-w_d})$ is the Jacobian determinant of the homeomorphic transform $y_i = e^{-w_i}, 1 \leq i \leq d$.

Proof. Since the derivative of $m_i(t)$ converges to 0, the derivative of $t + m_i(t)x$ with respect to $t$ converges to 1. Integrating both sides of (2.11) with respect to $t$ implies that

$$\overline{F}_i(t + m_i(t)x) \sim \overline{F}_i(t)e^{-x}, \ x \geq 0, \ \text{as} \ t \to \infty.$$  

That is, $F_i$ is in the max-domain of attraction of the Gumbel distribution, and thus the reciprocal hazard rate $\overline{F}_i(t)/f_i(t)$ can be taken as $m_i(\cdot)$ and $m_i(t) \sim m_1(t)$ as $t \to \infty$, $1 \leq i \leq d$. Let

$$m(t) := m_1(t) = \overline{F}_1(t)/f_1(t), \ V(t) := V(t)[\ell(\overline{F}_1(t))]^{1/\kappa_U}, \quad (2.13)$$

and obviously $V(t + m(t)x) \sim V(t)e^{-x}$ as $t \to \infty$ for the self-neglecting function $m(t)$. It follows from (2.13) that for any $x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d \setminus \{0\}$,

$$f(t + m(t)x_1, \ldots, t + m(t)x_d) = \frac{c(F_1(t + m(t)x_1), \ldots, F_d(t + m(t)x_d))}{m^{-d}(t)V^{\kappa_U}(t)} \prod_{i=1}^d f_i(t + m(t)x_i) \overline{F}_1(t)^{\kappa_U} \ell(\overline{F}_1(t))$$

Again, since the copula density $c(\cdot)$ is continuous in a small open neighborhood of $1 = (1, \ldots, 1)$, and (2.2) holds locally uniformly, the upper tail density $\lambda_U(\cdot)$ is continuous. Using similar arguments as that in Proposition 2.1 (see (2.9)), we have, as $t \to \infty$,

$$\frac{c\left(1 - \frac{F_1(t + m(t)x_1)}{\overline{F}_1(t)} \overline{F}_1(t), \ldots, 1 - \frac{F_d(t + m(t)x_d)}{\overline{F}_1(t)} \overline{F}_1(t)\right)}{[\overline{F}_1(t)]^{\kappa_U - d} \ell(\overline{F}_1(t))} \to \lambda_U(e^{-x_1}, \ldots, e^{-x_d}; \kappa_U).$$
Set \( u := \bar{F}_1(t) = m_1(t)f_1(t) \to 0 \) as \( t \to \infty \), and we have that for any \( x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d \setminus \{0\} \),

\[
\lambda(x) = \lim_{t \to \infty} \frac{f(t + m(t)x_1, \ldots, t + m(t)x_d)}{m^{-d}(t) V^\kappa_U(t)}
\]

\[
= \left( \prod_{i=1}^d e^{-x_i} \right) \lim_{u \to 0} \frac{c(1 - u e^{-x_1}, \ldots, 1 - u e^{-x_d})}{u e^{-d} \ell(u)}
\]

\[
= \left( \prod_{i=1}^d e^{-x_i} \right) \lambda_U(e^{-x_1}, \ldots, e^{-x_d}; \kappa_U),
\]

as desired. \( \square \)

**Remark 2.3.**

1. The formulations of tail densities in (2.6) and (2.12) are similar, except that the scaling functions are different (see (2.8) and (2.13)). Since \( m(t) \) is self-neglecting, \( m(t)/t \to 0 \) as \( t \to \infty \). We use a slower scaling \( m^d(t) \) in the Gumbel case (2.13) to accommodate the faster decay of the joint density tails.

2. It is seen from the proofs of Propositions 2.1 and 2.2 that the local uniform convergence assumption of the limit (2.2) is equivalent to assuming that the limit for the tail density in (2.6) or in (2.12) holds locally uniformly.

**Example 2.4** (Bivariate normal distribution with zero means and unit variances). For the univariate standard normal distribution, it is well known that \( \Phi(z) \sim \phi(z)/|z| \) as \( z \to -\infty \) and \( \Phi(z) \sim \phi(z)/z \) as \( z \to \infty \). For the upper tail, the auxiliary self-neglecting function can be taken as \( m(t) := \Phi(t)/\phi(t) \sim t^{-1} \). Obviously, \( \phi(t + z/t) \sim \phi(t) e^{-z} \) as \( t \to \infty \). For correlation \( \rho \) satisfying that \(-1 < \rho < 1\), the bivariate density is given by:

\[
f(z_1, z_2; \rho) = [2\pi(1 - \rho^2)]^{-1} \exp\left\{-\frac{1}{2} (1 - \rho^2)^{-1} [z_1^2 + z_2^2 - 2\rho z_1 z_2]\right\},
\]

so that, as \( t \to \infty \),

\[
f(t+y_1/t, t+y_2/t; \rho) = [2\pi(1 - \rho^2)]^{-1} \exp\left\{-\frac{1}{2} (1 - \rho^2)^{-1} [2t^2(1-\rho) + 2(y_1+y_2)(1-\rho) + O(t^{-2})]\right\},
\]

where the big O notation describes the upper growth (or decay) bound. Using the tail density \( \lambda(\cdot) \) in the Gumbel case (see the first equation of (2.12)), we match the above to

\[
f(t + m(t)y_1, t + m(t)y_2; \rho) \sim m^\kappa_U^{-2}(t) \phi^\kappa_U(t) \ell(\bar{F}(t)) \lambda(y_1, y_2)
\]

\[
= (2\pi)^{-\frac{\kappa_U}{4}} t^{-\kappa_U + 2} \exp\left\{-\frac{1}{2} t^2 \kappa_U \right\} \ell(\bar{F}(t)) \lambda(y_1, y_2), \ t \to \infty.
\]
Matching the $\frac{1}{2}t^2$ term in the exponent leads to the standard Gaussian tail density with tail order $\kappa_U = 2/(1 + \rho)$:

$$\lambda(y_1, y_2) = (1 + \rho)^{-2} \exp\{-y_1 y_2/(1 + \rho)\}.$$  

Note that $\lambda(\cdot)$ is continuous and the convergence to $\lambda(\cdot)$ is either locally increasing or locally decreasing, and thus the convergence is locally uniform due to Dini’s theorem. It then follows from (2.12) that the Gaussian copula tail density is given by

$$\lambda_U(w_1, w_2; \kappa_U) = \frac{1}{(1 + \rho)^2} (w_1 w_2)^{-\rho/(1 + \rho)} = \frac{\kappa_U^2}{4} (w_1 w_2)^{-\rho \frac{\kappa_U}{2}}, w_1 > 0, w_2 > 0. \quad (2.14)$$

Note also that

$$\ell(\Phi(t)) = t^{\kappa_U - 2} (2\pi)^{\frac{\kappa_U}{2} - 1} (1 + \rho)^{-2} (1 - \rho^2)^{-1} = O(t^{\kappa_U - 2}).$$  

Since

$$\Phi^{-1}(1 - u) \sim (-2 \log u)^{1/2} - [x(u)]^{-1} \log |x(u)|, \quad x(u) = (-2 \log u)^{1/2}, \quad u \to 0^+,$$

we have

$$\ell(u) = \ell(\Phi(\Phi^{-1}(1-u))) = O(\{\Phi^{-1}(1-u)^{\kappa_U - 2}\}) = O([(-2 \log u)^{1/2}]^{\kappa_U - 2}) = O((- \log u)^{(\kappa_U - 2)/2}).$$

This matches a previous approximation obtained in [5]. \hfill \Box

In the Weibull case, the univariate margins have supports that are bounded from above. For simpler notation, we assume that $x_{F_i} = 0$, or equivalently, we consider the situation in which $F$ concentrates on $(-\infty, 0]^d$ with continuous margins satisfying that $1 - F_i(-t^{-1}) \in \text{RV}_{-\beta}$, where $\beta > 0$. In this case (2.5) becomes

$$f_i(u) \sim f_1(u), \quad u \to 0^-, \quad i = 1, \ldots, d, \quad (2.15)$$

where $f_i$ is the density of the $i$th margin $F_i$, $i = 1, \ldots, d$. Note that (2.15) implies that $F_i(u) \sim F_1(u), 1 \leq i \leq d$, as $u \to 0^-$.  

**Proposition 2.5** (the Weibull case where $x_{F_i} = 0$). Assume that marginal densities are right-tail equivalent in the sense of (2.15). If the marginal density $f_i(-t^{-1}) \in \text{RV}_{-\beta+1}$, $i = 1, \ldots, d$, $\beta > 0$, and the limit (2.2) holds locally uniformly for all $w \in [0, \infty)^d \setminus \{0\}$, then $F$ has a tail density $\lambda(\cdot)$ that is related to the upper tail density $\lambda_U(\cdot)$ of $C$ as follows: for any $w = (w_1, \ldots, w_d) \in \mathbb{R}_+^d \setminus \{0\}$,

$$\lambda(w) := \lim_{u \to 0^+} \frac{f(-uw)}{u^{-d} V^{\kappa_U}(-u)} = \beta^d (w_1 \cdots w_d)^{\beta - 1} \lambda_U(w_1^\beta, \ldots, w_d^\beta; \kappa_U)$$

$$= \lambda_U(w_1^\beta, \ldots, w_d^\beta; \kappa_U), J(w_1^\beta, \ldots, w_d^\beta), \quad (2.16)$$
where $V(-t^{-1}) \in \text{RV}_{-\beta}$ and $J(w_1^\beta, \ldots, w_d^\beta)$ is the Jacobian determinant of the homeomorphically transform $y_i = w_i^\beta$, $1 \leq i \leq d$.

**Proof.** Let $(X_1, \ldots, X_d)$ be distributed with distribution $F$ and density $f$ on $(-\infty, 0]^d$. Let $Y_i := -X_i$, $i = 1, \ldots, d$, and so the distribution of $(Y_1, \ldots, Y_d)$ concentrates on $\mathbb{R}_+^d$. The density $g$ of $(Y_1, \ldots, Y_d)$ is given by

$$g(y_1, \ldots, y_d) = f(-y_1, \ldots, -y_d), \quad (y_1, \ldots, y_d) \in \mathbb{R}_+^d.$$ 

Observe that $(Z_1, \ldots, Z_d) := (Y_1^{-1}, \ldots, Y_d^{-1})$ has the same copula $C$ as that of $(X_1, \ldots, X_d)$, and has a density given by

$$h(z_1, \ldots, z_d) = f(-z_1^{-1}, \ldots, -z_d^{-1})\left(\prod_{i=1}^d z_i^{-2}\right), \quad (z_1, \ldots, z_d) \in \mathbb{R}_+^d.$$ 

Observe also that

$$\mathbb{P}(Z_i > t) = \mathbb{P}(X_i > -t^{-1}) = \overline{F}_i(-t^{-1}), \quad i = 1, \ldots, d,$$ 

and thus the density of $Z_i$ is $f_i(-t^{-1})t^{-2} \in \text{RV}_{-\beta-1}$, and $\mathbb{P}(Z_i > t) \in \text{RV}_{-\beta}$, $i = 1, \ldots, d$, due to Karamata’s theorem (see Theorem 2.1 in [24]). Furthermore, it follows from (2.15) that the densities of $Z_i$’s are right-tail equivalent at $\infty$. Applying Proposition 2.1 to the distribution of $(Z_1, \ldots, Z_d)$, we have

$$\lim_{t \to \infty} \frac{h(tw)}{t^{-d}W_{\kappa_U}(t)} = \beta^d\left(\prod_{i=1}^d w_i^{-\beta-1}\right)\lambda_U(w_1^{-\beta}, \ldots, w_d^{-\beta}; \kappa_U)$$ 

where $W(t) = \overline{F}_1(-t^{-1})\ell(\overline{F}_1(-t^{-1}))^{1/\kappa_U}$ (see (2.8)). Translating all variables back to the set-up in terms of $u = t^{-1}$ and $x_i = w_i^{-1}$, $1 \leq i \leq d$, leads to

$$\lambda(x) = \lim_{u \to 0} \frac{f(-ux)}{u^{-d}V_{\kappa_U}(-u)} = \lim_{t \to \infty} \frac{h(tw)}{t^{-d}W_{\kappa_U}(t)}\left(\prod_{i=1}^d w_i^2\right)$$

$$= \beta^d(w_1 \cdots w_d)^{-\beta+1}\lambda_U(w_1^{-\beta}, \ldots, w_d^{-\beta}; \kappa_U)$$

$$= \lambda_U(x_1^{-\beta}, \ldots, x_d^{-\beta}; \kappa_U)|J(x_1^{-\beta}, \ldots, x_d^{-\beta})|,$$

where $V(-u) := W(u^{-1}) = \overline{F}_1(-u)\ell(\overline{F}_1(-u))^{1/\kappa_U}$. \hfill $\square$

**Remark 2.6.** 1. The assumption that $f_i(-t^{-1}) \in \text{RV}_{-\beta+1}$, $i = 1, \ldots, d$, $\beta > 0$, implies that $\overline{F}_i(-t^{-1}) \in \text{RV}_{-\beta}$, $i = 1, \ldots, d$; that is, for any $x > 0$,

$$\frac{\overline{F}_i(-(tx)^{-1})}{\overline{F}_i(-t^{-1})} \to x^{-\beta}, \quad i = 1, \ldots, d, \quad \text{as } t \to \infty.$$ 

This is the condition for the margins $F_i$’s to be in the Weibull max-domain of attraction when the right-end point of the support is zero.

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2. In Propositions 2.1, 2.2 and 2.5, we need to make assumptions directly on marginal density tails because we deal with the tails of joint densities. Accordingly, these assumptions on marginal density tails imply that corresponding marginal distributions are in the max-domain of attraction of Fréchet, Gumbel or Weibull distribution. Note that it is more subtle in general to infer tail behavior of a density from its cumulative distribution tail.

3. As illustrated in Propositions 2.1, 2.2 and 2.5, different tail behavior of Fréchet, Gumbel and Weibull distributions makes direct description of joint distribution tail densities cumbersome. In contrast, joint distribution tail densities can be analyzed via copula tail densities in a unified fashion. Starting from a joint density, one can determine via (2.2) whether the tail density of its copula exists. If the copula tail density exists, then one can find the tail density of the joint distribution using Proposition 2.1, 2.2 or 2.5 after appropriate rescaling to equalize margins in their right tails. On the other hand, starting from a copula with upper tail density (2.2), one can incorporate univariate marginal distributions that are in the max-domain of attraction of the Fréchet, Gumbel or Weibull distribution, so that various multivariate distributions with tail densities can be constructed.

It is also worth mentioning that Propositions 2.1, 2.2 and 2.5 establish the relationships between tail densities of copulas and tail densities of joint distributions whose univariate margins are standardized in the tails, so that one can work with most convenient univariate margins for statistical tail inference. For example, it is sometimes not informative to visualize the data by transforming them to uniform margins, but rather working with normal margins may become more convenient, and in this situation the result such as Proposition 2.2 becomes relevant.

3 Analyzing hidden regular variation via copula tail densities

It was shown in [18, 16] that a multivariate regularly varying distribution can be constructed from a copula with tail density of order $\kappa_U = 1$, coupled with marginal univariate regularly varying distributions with tail index $-\alpha$. In this section, we extend this result to hidden regular variation.

Let $X = (X_1, \ldots, X_d)$ be a random vector having the distribution $F$ with continuous margins $F_1, \ldots, F_d$ and their respective densities $f_1, \ldots, f_d$. Again, we assume that margins
are right-tail equivalent in the sense of (2.5) with \( x_{F_1} = \infty \). The distribution \( F \) or the random vector \( X \) is said to be multivariate regularly varying (MRV, see [24]) at \( \infty \) with intensity measure \( \nu \) if there exists a scaling function \( h(t) \to \infty \) and a non-zero Radon measure \( \nu(\cdot) \) with tail index \(-\alpha\) such that as \( t \to \infty \), the following vague convergence holds,

\[
\lim_{t \to \infty} t P \left( \frac{X}{h(t)} \in \cdot \right) \rightarrow \nu(\cdot), \quad \text{in cone } \mathbb{R}_+^d \setminus \{0\} := [0, \infty]^d \setminus \{0\};
\]

that is, \( t P(X/h(t) \in B) \to \nu(B) \), for any relatively compact set \( B \subseteq \mathbb{R}_+^d \setminus \{0\} \), with \( \nu(\partial B) = 0 \). Since the margins are tail-equivalent, (3.1) can be expressed equivalently as

\[
\lim_{t \to \infty} \frac{P(X \in tB)}{F_1(t)} = \nu(B), \quad \forall \text{ relatively compact set } B \subseteq \mathbb{R}_+^d \setminus \{0\},
\]

satisfying that \( \nu(\partial B) = 0 \). The distribution \( F \) satisfying (1.2) is MRV with intensity measure \( \nu(\cdot) \) that has the Radon-Nikodym derivative \( \lambda(\cdot) \) with respect to the Lebesgue measure.

The regular variation property (3.2) defined on the cone \( \mathbb{R}_+^d \setminus \{0\} \) employs the relatively faster scaling \( h(t) \) that is necessary for convergence on the margins, but such a coarse normalization fails to reveal the finer dependence structure that may be present in the interior. A scaling of smaller order is necessary for any regular variation properties resided or hidden in a subcone of \( \mathbb{R}_+^d \setminus \{0\} \) (see Section 9.4 of [24]). We focus on the following commonly used subcone:

\[
C_0 = \{ x \in \mathbb{R}_+^d : x_i > 0, x_j > 0, \text{ for some } 1 \leq i < j \leq d \} = \{ x \in \mathbb{R}_+^d : x[2] > 0 \},
\]

where \( x[2] \) denotes the second largest component of vector \( x \). That is, \( C_0 \) consists of all points in \( \mathbb{R}_+^d \setminus \{0\} \) such that at most \( d - 2 \) coordinates are 0. The discussion on other subcones is similar [19, 8]. A random vector \( X \) is said to have hidden regular variation (HRV) on \( \mathbb{R}_+^d \) if (3.1) holds on \( \mathbb{R}_+^d \setminus \{0\} \) and there exists a scaling function \( h_0(t) \to \infty \) such that \( h(t)/h_0(t) \to \infty \) and a non-zero Radon measure \( \nu_0 \) with tail index \(-\alpha_0 \) \( (\alpha_0 \geq \alpha) \), such that

\[
\lim_{t \to \infty} t P \left( \frac{X}{h_0(t)} \in \cdot \right) \rightarrow \nu_0(\cdot), \quad \text{in } C_0, \quad \text{as } t \to \infty.
\]

Various examples and applications can be found in [24], and in particular, HRV implies asymptotic independence in the sense that \( \nu(C_0) = 0 \).

Multivariate regular variation and hidden regular variation have been also studied in the context of copulas [17, 16, 5, 8]. Let \( (U_1, \ldots, U_d) \) denote a random vector with copula \( C \) and standard uniform margins. The upper exponent function and tail order function of copula
\[ a_U(w; \kappa_U) := \lim_{u \to 0^+} \frac{\mathbb{P}\left( \bigcup_{i\neq j} \{ U_i > 1 - uw_i, U_j > 1 - uw_j \} \right)}{u^\ell(u)}, \ w \in C_0, \quad (3.4) \]
\[ b_U(w; \kappa_U) := \lim_{u \to 0^+} \frac{\overline{C}(1 - uw_i, 1 \leq i \leq d)}{u^\ell(u)}, \ w \in (0, \infty)^d, \quad (3.5) \]

provided that the limits exist for \( \kappa_U \geq 1 \) and some non-negative function \( \ell(u) \) that is slowly varying at 0. Clearly, the two functions are homogeneous of order \( \kappa_U \); that is, \( a_U(tw; \kappa_U) = t^{\kappa_U} a_U(w; \kappa_U) \) and \( b_U(tw; \kappa_U) = t^{\kappa_U} b_U(w; \kappa_U) \) for any \( t > 0 \). If \( \kappa_U = 1 \) and \( \ell(u) \to k \neq 0 \), then the functions (3.4) and (3.5), coupled with regularly varying margins, completely characterize multivariate regular variation (3.2) \([17, 16]\). It was shown in \([8]\) that any distribution with HRV (3.3) can be constructed using a copula \( C \) with exponent and tail order functions (3.4) and (3.5), coupled with regularly varying marginal distributions with tail index \(-\alpha\). Specifically, under (3.2) and (3.3), the limiting functions in (3.4) and (3.5) exist and

\[ a_U(w; \kappa_U) = v_0\left( \{ x \geq 0 : x_i > w_i^{-1/\alpha}, x_j > w_j^{-1/\alpha}, \text{ for some } i \neq j \} \right) = v_0\left( \{ x \geq 0 : (w^{1/\alpha} x)_{[2]} > 1 \} \right), \text{ for all continuity points } w \in C_0, \quad (3.6) \]
\[ b_U(w; \kappa_U) = v_0\left( \prod_{i=1}^d (w_i^{-1/\alpha}, \infty] \right), \text{ for all continuity points } w \in (0, \infty)^d. \quad (3.7) \]

Note that the tail order function \( b_U(\cdot; \kappa_U) \) is often used in estimating tail risk \([12]\), but the exponent function \( a_U(\cdot; \kappa_U) \) is mathematically convenient in deriving limiting results \([8]\). For simpler exposition on hidden regular variation \([3.3]\), we assume that \( 1 < \kappa_U < d \).

To establish the relations between the tail density of \( C \) and tail order or exponent function, we need to impose some bounding condition in order to control the variation of limiting processes moving from ray to ray originated from the open boundaries of the subcone \( C_0 \). Define a unit envelope of \( C_0 \cup C_\infty \):

\[ E_0 = \{ x \in C_0 \cup C_\infty : x_{[2]} = 1 \}, \]

where \( C_\infty = \{ x \geq 0 : x_i > 0, x_j > 0, i \neq j, x_k = \infty, \text{ for some } k \} \). Note that \( E_0 \) is compact within \( C_0 \cup C_\infty \) \([21, 8]\). As \( \epsilon \to 0^+ \), \( \epsilon E_0 \) approaches the open boundaries of \( C_0 \cup C_\infty \).

**Theorem 3.1.** Let \( X = (X_1, \ldots, X_d) \) be a non-negative random vector having a multivariate regularly varying distribution \( F \) in the sense of (3.2). Assume that \( F \) has a continuous density \( f \) which is regularly varying on \( C_0 \) in the sense that for any \( w \in C_0 \),

\[ \frac{f(tw)}{t^{-d} V_{\kappa_U}(t)} \to \lambda(w) > 0, \text{ as } t \to \infty, \quad (3.8) \]
where $V \in RV_{-\alpha}$ and $\kappa_U \geq 1$. If for sufficiently large $t \geq t_0$
\[ \sup_{w \in E_0} \frac{f(tw)}{t^{-dV^{\kappa_U}(t)}} \leq M < \infty, \]
then $F$ has hidden regular variation on $C_0$ with intensity measure
\[ \nu_0(B) = \lim_{t \to \infty} \frac{\mathbb{P}(X \in tB)}{V^{\kappa_U}(t)} = \int_B \lambda(w) dw, \]
for any relatively compact $B \subset C_0$.

**Proof.** Let $h(t) = t^{-dV^{\kappa_U}(t)} \in RV_{-\alpha \kappa_U -d}$. For any relatively compact $B \subset C_0$, there exists a small $\epsilon > 0$ such that $B \subseteq \{x : ||x|| \geq \epsilon\} \subset C_0$. For any $x \in B$, we have $x^{-1}_1 x \in E_0$. Consider, for any $x \in B$,
\[ \frac{f(tx)}{t^{-dV^{\kappa_U}(t)}} = \frac{f(tx_1 x^{-1}_1 x)}{h(tx_1 x^{-1}_1 x)} \frac{h(tx_1 x^{-1}_1 x)}{h(t)} \leq M \frac{h(tx_1 x^{-1}_1 x)}{h(t)}, \]
for sufficiently large $t$.

It follows from the Karamata representation for univariate regular variation (see page 29 of [24]) that for any $\delta > 0$,
\[ \frac{h(tx_1 x^{-1}_1 x)}{h(t)} \leq x^{-\alpha \kappa_U -d+\delta}_1, \]
for sufficiently large $t$.

We choose $\delta < \alpha \kappa_U$ so that $x^{-\alpha \kappa_U -d+\delta}_1$ is integrable over $\{x : ||x|| \geq \epsilon\}$. The dominated convergence theorem implies that
\[ \lim_{t \to \infty} \frac{\mathbb{P}(X \in tB)}{V^{\kappa_U}(t)} = \lim_{t \to \infty} \int_B \frac{f(tx)}{V^{\kappa_U}(t)} d(tx) = \lim_{t \to \infty} \int_B \frac{f(tx)}{t^{-dV^{\kappa_U}(t)}} dtx = \int_B \lambda(x) dx, \]
for any relatively compact $B \subset C_0$. \hfill \Box

**Remark 3.2.** 1. Note that $E_0$ is just one of several sets that can be used to control the limiting process variation. For example, one can use $\{x : ||x|| = 1\} \cap C_0$ to replace $E_0$ in Theorem 3.1 where $|| \cdot ||$ denotes some norm on $\mathbb{R}^d$. The advantage of using $E_0$ is that $E_0$ is compact within $C_0 \cup C_\infty$. If $\lambda(\cdot)$ is continuous on $E_0$, then it is bounded on $E_0$, and in this case, the continuity of $f(\cdot)$ and compactness of $E_0$ ensure that $f(tx)/[t^{-dV^{\kappa_U}(t)}]$ is bounded on $E_0$ for sufficiently large $t$. An easy checkable sufficient condition for Theorem 3.1 is that $\lambda(\cdot)$ is continuous on the compact subset $E_0$.

2. We follow the approach used in [4] to derive the conditions for HRV in Theorem 3.1. A more popular uniformity condition was also introduced in [4] to control the limiting process variation moving from ray to ray originated from the open boundaries of a
subcone. Using the similar arguments as these in Theorem 2.1 of \[4\], we can show that if \(\lambda(\cdot)\) in (3.8) is bounded on \(E_0\) and the following uniformity condition holds on \(E_0\):

\[
\lim_{t \to \infty} \sup_{w \in E_0} \left| \frac{f(tw)}{t^{-d} V(t)} - \lambda(w) \right| = 0,
\]

then the uniform convergence can be propagated to \(C_0\):

\[
\lim_{t \to \infty} \sup_{w \in \mathbb{R}^2} \left| \frac{f(tw)}{t^{-d} V(t)} - \lambda(w) \right| = 0,
\]

for any small \(\epsilon > 0\). This is true because of the scaling property of \(\lambda(\cdot)\) and the uniform convergence for univariate regular variation. Under this uniformity condition, the conclusion of Theorem 3.1 also holds. Note that there are examples in \[4\] showing if this uniformity condition is violated, then \([1 - F(tw)]/V(t)\) may not even have a limit, as \(t \to \infty\).

**Corollary 3.3.** Let \(C\) be a \(d\)-dimensional copula with density \(c(\cdot)\). If (2.2) holds locally uniformly with tail density \(\lambda_U(\cdot; \kappa_U)\), \(\kappa_U \geq 1\), then

\[
a_U(w; \kappa_U) = \int_{\{x: (w/x)_{[2]} > 1\}} \lambda_U(x; \kappa_U) dx, \quad w \in C_0.
\]

**Proof.** We use the construction presented in Proposition 2.1 with regularly varying margins having continuous densities. It follows from (2.10) that

\[
\lambda(x) = \lim_{t \to \infty} \frac{f(tx)}{t^{-d} V(t)} = \alpha^d \left( \prod_{i=1}^{d} x_i^{-\alpha-1} \right) \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha}; \kappa_U),
\]

and thus \(\lambda(\cdot)\) is continuous on \(E_0\), and Theorem 3.1 implies that

\[
\nu_0(B) = \int_B \lambda(x) dx, \quad \text{for any relatively compact } B \subset C_0.
\]

Set \(B = \{x : (w^{1/\alpha} x)_{[2]} > 1\} \subset C_0\), and we obtain from (3.6) that

\[
a_U(w; \kappa_U) = \nu_0 \left( \{x \geq 0 : (w^{1/\alpha} x)_{[2]} > 1\} \right) = \int_{\{x : (w^{1/\alpha} x)_{[2]} > 1\}} \lambda(x) dx.
\]

By Proposition 2.1 \(\lambda(x) = \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha}; \kappa_U)|J(x_1^{-\alpha}, \ldots, x_d^{-\alpha})|\), and thus

\[
a_U(w; \kappa_U) = \int_{\{x : (w^{1/\alpha} x)_{[2]} > 1\}} \lambda_U(x_1^{-\alpha}, \ldots, x_d^{-\alpha}; \kappa_U)|J(x_1^{-\alpha}, \ldots, x_d^{-\alpha})| dx
\]

\[
= \int_{\{x : (w/x)_{[2]} > 1\}} \lambda_U(x; \kappa_U) dx
\]

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as desired. □

To derive the relation between the tail order function \( b_U(:, \kappa_U) \) and the tail density \( \lambda_U(:, \kappa_U) \), we express it in the differentiation form because in general \( \lambda_U(:, \kappa_U) \) may not be integrable near the open boundaries of the subcone \( C_0 \).

**Corollary 3.4.** Let \( C \) be a \( d \)-dimensional copula with density \( c(\cdot) \). If \( 2.2 \) holds locally uniformly with tail density \( \lambda_U(:, \kappa_U) \), \( \kappa_U \geq 1 \), then

1. \( D_w a_U(w; \kappa_U) = (-1)^d(d-1) \lambda_U(w; \kappa_U) \), \( w \in C_0 \);

2. \( D_w b_U(w; \kappa_U) = \lambda_U(w; \kappa_U) \), \( w \in (0, \infty)^d \).

where \( D_w = \partial^d/\partial w_1 \cdots \partial w_d \) denotes the \( d \)-th order partial differentiation operator with respect to \( w_1, \ldots, w_d \).

**Proof.** For any locally finite measure \( \mu(\cdot) \) on \( \mathbb{R}^d_+ \), the inclusion-exclusion principle implies that

\[
\mu\left( \bigcup_{i \neq j} \{ \mathbf{x} : x_i < w_i, x_j < w_j \} \right) = H_\mu(w_1, \ldots, w_d)
- (-1)^{d-1}(d-1) \mu(\{ \mathbf{x} : x_1 < w_1, \ldots, x_d < w_d \}),
\]

where \( H_\mu(w_1, \ldots, w_d) \) is a linear function of \( \mu(\{ \mathbf{x} : x_j < w_j, j \in I \}) \), \( I \subset \{1, \ldots, d\} \) with \( d > |I| \geq 2 \). Take the Radon measure \( \Lambda(B) = \int_B \lambda_U(\mathbf{x}; \kappa_U) d\mathbf{x} \) in \( 3.9 \), and we have,

\[
a_U(w; \kappa_U) = \int_{\{ \mathbf{x} : w/\mathbf{x} \geq 1 \}} \lambda_U(\mathbf{x}; \kappa_U) d\mathbf{x}
= H_\Lambda(w_1, \ldots, w_d) + (-1)^d(d-1) \int_{\{ \mathbf{x} : x_1 < w_1, \ldots, x_d < w_d \}} \lambda_U(\mathbf{x}; \kappa_U) d\mathbf{x}.
\]

Taking the derivative on the both sides and noticing that \( D_w H_\Lambda(w_1, \ldots, w_d) = 0 \), we obtain that \( D_w a_U(w; \kappa_U) = (-1)^d(d-1) \lambda_U(w; \kappa_U) \). On the other hand, take the probability measure \( \mathbb{P}(\cdot) \) in \( 3.9 \), and we have from \( 3.4 \) and \( 3.5 \) that

\[
a_U(w; \kappa_U) = H_\mathbb{P}(w_1, \ldots, w_d) + (-1)^d(d-1)b_U(w; \kappa_U).
\]

The fact that \( D_w H_\mathbb{P}(w_1, \ldots, w_d) = 0 \) leads to \( D_w a_U(w; \kappa_U) = (-1)^d(d-1) D_w b_U(w; \kappa_U) \). □

**Remark 3.5.**

1. The differentiation formulas presented in Corollary 3.4 justify the following heuristic: for any \( w = (w_1, \ldots, w_d) \in (0, \infty)^d \),

\[
\lambda_U(w; \kappa_U) = \lim_{u \to 0^+} \frac{c(1 - uw_i, 1 \leq i \leq d)}{u^{\kappa_U - d} \ell_U(u)} = \lim_{u \to 0^+} \frac{D_w \overline{c}(1 - uw_i, 1 \leq i \leq d)}{u^{\kappa_U} \ell_U(u)}
= D_w \left( \lim_{u \to 0^+} \frac{\overline{c}(1 - uw_i, 1 \leq i \leq d)}{u^{\kappa_U} \ell_U(u)} \right) \frac{\partial^d b_U(w; \kappa_U)}{\partial w_1 \cdots \partial w_d},
\]

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where the limits can be exchanged due to the regularity conditions, such as continuity or uniform convergence.

2. Similarly, the lower tail order function of copula \( C \) is defined as follows:

\[
b_L(w; \kappa_L) := \lim_{u \to 0^+} \frac{C(uw_i, 1 \leq i \leq d)}{u^{\kappa_L} \ell(u)}, \quad \forall \ w = (w_1, \ldots, w_d) \in (0, \infty)^d,
\]

(3.10)

if the limit exists for \( \kappa_L \geq 1 \) and some non-negative function \( \ell(u) \) that is slowly varying at 0 (see [5]). The function \( b_L(\cdot; \kappa_L) \) is related to the lower tail density \( \lambda_L(\cdot; \kappa_L) \) defined in (2.3) as follows:

\[
\lambda_L(w; \kappa_L) := \lim_{u \to 0^+} \frac{c(uw_i, 1 \leq i \leq d)}{u^{\kappa_L - d} \ell(u)} = \lim_{u \to 0^+} \frac{D_w C(uw_i; 1 \leq i \leq d)}{u^{\kappa_L} \ell(u)}, \quad \forall \ w = (w_1, \ldots, w_d) \in (0, \infty)^d.
\]

Again, the relation is best expressed in the differentiation form because the lower tail density \( \lambda_L(\cdot; \kappa_L) \) may not be integrable near the open boundaries of the subcone.

3. If the tail density \( \lambda_U(\cdot; \kappa_U) \) is non-zero and differentiable, then it is well-behaved. Since \( \lambda_U(\cdot; \kappa_U) \) is homogeneous of order \( \kappa_U - d \), the Euler representation implies that

\[
\sum_{j=1}^d w_j \frac{\partial \lambda_U(w; \kappa_U)}{\partial w_j} = (\kappa_U - d)\lambda_U(w; \kappa_U).
\]

If \( \kappa_U < d \) (\( \kappa_U \geq d \)), then \( \lambda_U(\cdot; \kappa_U) \) is directionally decreasing (increasing) and directionally convex along all the rays originated from the open boundaries of the subcone \( C_0 \). If \( 1 \leq \kappa_U < d \) (or \( \kappa_U > d \)), then \( \lambda_U(\cdot; \kappa_U) \) goes to \( \infty \) (or zero) at the boundaries of the subcone and reaches to zero (or \( \infty \)) at \( \infty \) (see (2.14) for \( \kappa_U < 2 \) when \( 0 < \rho < 1 \) and \( \kappa_U > 2 \) when \( -1 < \rho < 0 \)).

**Example 3.6.** Consider again a bivariate normal distribution with zero means and unit variances (see Example 2.4). The tail order is \( \kappa_U = 2/(1 + \rho) \), and the Gaussian tail density is \( \lambda(y_1, y_2) = (1 + \rho)^{-2} \exp\{-(y_1 + y_2)/(1 + \rho)\} \), and the Gaussian copula tail density is given by

\[
\lambda_U(w_1, w_2; \kappa_U) = \frac{1}{(1 + \rho)^2} (w_1 w_2)^{-\frac{2}{1 + \rho}} = \frac{\kappa_U^2}{4} (w_1 w_2)^{-\rho \kappa_U}, \quad w_1 > 0, w_2 > 0.
\]

It follows from Corollary 3.4 that

\[
a_U(w_1, w_2; \kappa_U) = b_U(w_1, w_2; \kappa_U) = (w_1 w_2)^{\kappa_U}
\]
for any $-1 < \rho < 1$. It must be mentioned that the bivariate exponent function and tail dependence function are not identical in the tail dependence case where the tail order is 1 (see [13]).

**Example 3.7.** A bivariate Gumbel copula has the following form:

\[
C(u, v) = \exp\{-A(-\log u, -\log v)\}, \quad \text{where } A(x, y) = (x^\delta + y^\delta)^{1/\delta}, 1 \leq \delta < \infty.
\]

According to Example 2 of [5], the lower tail order $1 < \kappa_L = \frac{2^{1/\delta}}{\delta} \leq 2$, which implies intermediate lower tail dependence and exact lower independence for the Gumbel copula. The lower tail order function defined in (3.10) is given by

\[
b_L(w_1, w_2; \kappa_L) = w_2^{2^{1/\delta-1}} w_1^{2^{1/\delta-1}} = w_1^{\kappa_L/2} w_2^{\kappa_L/2}.
\]

By taking the partial derivatives, the lower tail order density is given by

\[
\lambda_L(w_1, w_2; \kappa_L) = \frac{\kappa_L^2}{4} w_1^{\kappa_L/2-1} w_2^{\kappa_L/2-1}.
\]

Observe that for $1 < \kappa_L < 2$, the tail order density $\lambda_L(\cdot; \kappa_L)$ is infinite at $(0,0)$ which indicates a relatively stronger (than independence) dependence in the lower tail. Note that for the first-order tail dependence case, the lower tail order density function is also infinite at the origin $(0,0)$ (see Proposition 2.2 of [18]). In contrast to the first-order tail dependence, the tail order density $\lambda_L(\cdot; \kappa_L)$ is also infinite on the axis that $w_1 = 0$ and on the axis that $w_2 = 0$; that is, $\lambda_L(\cdot; \kappa_L)$ is infinite on the open boundaries of the subcone $C_0$.

It is sometimes easier to work with the lower tail of a joint density function, and the upper tail density of its survival copula can be derived accordingly. For example, a $d$-dimensional Archimedean copula can be viewed as the survival copula of a scale mixture of the uniform simplex (see [20]). For the upper tail of such an Archimedean copula family, the following result is more convenient.

**Proposition 3.8.** Let $X = (X_1, \ldots, X_d)$ be non-negative with distribution $F$, density $f$ and continuous margins $F_1, \ldots, F_d$ that are left-tail equivalent in the sense that

\[
f_i(u) \sim f_1(u), \quad \text{as } u \to 0^+, \quad i = 1, \ldots, d,
\]

where $f_i$ is the density of $F_i, i = 1, \ldots, d$. If the marginal density $f_i(t^{-1}) \in \text{RV}_{-\alpha+1}, i = 1, \ldots, d, \alpha > 0$, and the limit (2.3) holds locally uniformly, then $F$ has a lower tail density $\lambda(\cdot)$ given by:

\[
\lambda(w) := \lim_{u \to 0^+} \frac{f(uw)}{-adV^{\kappa_L}(u)} = \lambda_L(w_1^\alpha, \ldots, w_d^\alpha; \kappa_L)|J(w_1^\alpha, \ldots, w_d^\alpha)|,
\]

for some function $V(t^{-1}) \in \text{RV}_{-\alpha}$.  

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Proof. Let \( Y := (X_1^{-1}, \ldots, X_d^{-1}) \). The copula \( C \) of \( X \) is the survival copula of \( Y \) and thus the lower tail density of \( F \) is the upper tail density of \( Y \). The density function of \( Y \) is given by
\[
f_Y(y) = f(y_1^{-1}, \ldots, y_d^{-1}) \left( \prod_{i=1}^{d} y_i^{-2} \right).
\]
Observe that the density of \( Y_i \) is given by
\[
f_i(t^{-1}) t^{-2} \in RV_{-\alpha-1}, \quad i = 1, \ldots, d,
\]
and the densities of \( Y_i \)'s are right-tail equivalent at \( \infty \). Let \( u := t^{-1} \), and Proposition 2.1 implies that
\[
\lambda(u) = \lim_{u \to 0^+} \frac{f(u \omega)}{u^{dV_{\kappa_L}(\omega)}} = \lim_{t \to \infty} \frac{f_Y(tw_1^{-1}, \ldots, tw_d^{-1})}{t^{-dV_{\kappa_L}(t^{-1})}} \left( \prod_{i=1}^{d} w_i^{-2} \right)
\]
\[
= \lambda_L(w_1^{\alpha}, \ldots, w_d^{\alpha}; \kappa_L)^{\alpha d} \left( \prod_{i=1}^{d} w_i^{\alpha+1} \right) \left( \prod_{i=1}^{d} w_i^{-2} \right)
\]
\[
= \lambda_L(w_1^{\alpha}, \ldots, w_d^{\alpha}; \kappa_L) |J(w_1^{\alpha}, \ldots, w_d^{\alpha})|
\]
as desired. \( \square \)

In Propositions 2.1, 2.2, 2.5 and 3.8 we assume the existence of tail densities of copulas with a goal of deriving tail densities of multivariate distributions with various margins. In fact, for continuous univariate margins \( F_i \)'s, the homeomorphism between uniform random variables \( U_i \)'s and \( F_i^{-1}(U_i) \)'s makes these limiting results hold conversely. For example, similar to Proposition 3.8, under regularity conditions, one can derive the lower tail density \( \lambda_L(\cdot; \kappa_L) \) for the copula of \( X \), by assuming the existence of the lower tail density \( \lambda(\cdot) \) of \( X \), as follows:
\[
\lambda_L(\omega; \kappa_L) = \lambda(\omega^{1/\alpha}) |J(\omega_1^{1/\alpha}, \ldots, \omega_d^{1/\alpha})|.
\]

To illustrate (3.11), consider the following scale mixture:
\[
X = R(S_1, \ldots, S_d),
\]
where the random vector \( (S_1, \ldots, S_d) \) follows the uniform distribution on the \( d \)-dimensional unit closed simplex \( \{ x \geq 0 : ||x||_1 = 1 \} \) with \( l_1 \) norm \( ||x||_1 := \sum_{i=1}^{d} x_i \), and \( R \) is a radial random variable supported on \( [0, \infty) \) with \( P(R = 0) = 0 \). The survival copula induced by \( X \) is a \( d \)-dimensional Archimedean copula (see [20]). Since this type of Archimedean copula is constructed by Williamson’s \( d \)-transform of \( R \), here we refer to it as the WT-Archimedean copula. Let \( f_R \) be the density function of \( R \). By Theorem 5.5 of [2], the density function of \( X \) exists and is given by
\[
f_X(x) = \Gamma(d) ||x||_1^{d+1} f_R(||x||_1), \quad x \in [0, \infty)^d \setminus \{0\}.
\]
Clearly, depending on the density function of $R$, the tail density of WT-Archimedean copulas can be derived easily. A concrete example is given as follows.

**Example 3.9** (Generalized gamma and simplex mixture copula, or GGS copula). For $\alpha, \beta > 0$, let $X$ be defined as in (3.12), where $R^{1/\beta}$ follows Gamma($\alpha, 1$) with shape parameter $\alpha$, and unit scale parameter. The survival copula $\hat{C}$ of $X$ is referred to as the GGS copula, and when $\alpha > \beta$ the upper tail order of the GGS copula is $\kappa = \alpha/\beta$ (see Example 2 of [9]). The density function of $R$ is $f_R(r) = \left[\frac{\beta}{\Gamma(\alpha)}\right]^{-1} r^{\alpha/\beta - 1} \exp\{-r^{1/\beta}\}$, and the density function of $X$ is then given by

$$f_X(x) = \frac{\Gamma(d)}{\beta \Gamma(\alpha)} \|x\|^{\alpha/\beta - d} \exp\{-\|x\|_1^{1/\beta}\}, \quad x \in [0, \infty)^d \setminus \{0\}. \quad (3.14)$$

It can be shown (see the proof of Proposition 1 in [9]) that the marginal distribution of $X_i$ satisfies $F_{X_i}(t^{-1}) \in RV_1$ as long as $E[1/R] < \infty$, that is, when $\kappa = \alpha/\beta > 1$. Clearly, $f_X$ is (multivariate) regularly varying at 0 in the sense that, letting $V(u) = u$, the lower tail density of $X$ is given by

$$\lambda(w) = \lim_{u \to 0^+} \frac{f_X(uw)}{u^{-d} V^{\kappa}(u)} = \frac{\Gamma(d)}{\beta \Gamma(\alpha)} \|w\|_{1}^{\kappa - d}.$$

Observe from (2.1) that the upper tail density of the GGS copula is related to the lower tail density of the copula $C$ of $X$, and then by Proposition 3.8 the upper tail density of the GGS copula is given by the lower tail density of $C$:

$$\lambda_L(w; \kappa, C) = \frac{\Gamma(d)}{\beta \Gamma(\alpha)} \|w^{1/\alpha}\|_{1}^{\kappa - d} |J(w_1^{1/\alpha}, \ldots, w_d^{1/\alpha})|.$$

When $\kappa = \alpha/\beta > 1$, the upper tail order function of the GGS copula can be calculated as

$$\frac{\Gamma(d)}{\beta \Gamma(\alpha)} \int_{[0, w]} \|v\|_{1}^{\kappa - d} dv. \quad (3.15)$$

For example, taking $d = 2$ and considering $C_0$, the integral in (3.15) can be calculated explicitly as follows: for any $w_1 > 0$ and $w_2 > 0$,

$$\int_{[0, w_1] \times [0, w_2]} (v_1 + v_2)^{\alpha/\beta - 2} dv_1 dv_2 = \left[\left(w_1 + w_2\right)^{\kappa} - w_1^{\kappa} - w_2^{\kappa}\right]/(\kappa(\kappa - 1)) < \infty.$$

If follows from the particular form of the density in (3.14) that $f_X(uw)/[u^{-d} V^{\kappa}(u)]$ converges to $\lambda(w)$ monotonically, and thus the convergence is locally uniform due to Dini’s theorem. The condition in Corollary 3.3 is satisfied, and $Y = (X_1^{-1}, \ldots, X_d^{-1})$ has HRV on $C_0$. □
4 Concluding Remarks

Most multivariate distributions are specified by their densities, especially in high dimensional situations, and thus the density approach provides a tractable method in analyzing multivariate extremes [1]. We introduce higher order tail densities of copulas and show that tails of multivariate distributions with various margins can be expressed as tail densities of underlying copulas, coupled with marginal tail transforms of Fréchet, Gumbel or Weibull type.

It should be emphasized that we define tail densities of copulas by exploring local regular variation properties of multivariate distribution tails, whereas cumulative distributions themselves may not be regularly varying on any subcone of $\mathbb{R}^d_+$. As a matter of fact, local regular variation properties are especially useful for analyzing hidden regular variation that is accumulated over a subcone. In particular, we establish a multivariate copula version of Karamata’s theorem and show that under mild regularity conditions, local regular variation of the tail density of a copula implies hidden regular variation on a subcone. Our results extend the main results in [4, 18] to hidden regular variation.

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References


