Abstract

Disruptions and random supplies have been important sources of uncertainty that should be considered in the design and control of supply chains. There have been many real world examples in which a single catastrophic event has simultaneously degraded the capabilities of several suppliers leading to considerable erosion of profits and goodwill for a company. However, the literature on analytical models that account for the dependence nature of disruptions and its impact on supply chain performance is sparse. In this paper, we consider an \(m\)-manufacturer, 1-retailer, newsvendor inventory system with stochastically dependent manufacturing capacities, caused by random disruptions that may simultaneously inflict damages to the capacities of the manufacturers. We develop the structural/analytical properties of key performance measures and optimal inventory policies for the multi-source and assembly inventory systems. We show that stochastic dependence in disruptions can have opposite effects on system performance in the multi-source and assembly systems. While risk diversification is preferred in the multi-source system, risk concentration is preferred in the assembly system. Our results also suggest that, if the retailer ignores the effect of dependent disruptions, then in the multi-source structure, it would lead to underestimating the cost, overestimating the fill rate, and ordering more units than the optimum; however, in the assembly structure, the opposite would happen. We perform a comprehensive numerical study to validate our analytical results and generate useful managerial and operational insights for effective risk management of supply chains in the presence of dependent disruptions.

Keywords: Dependent supply disruptions, multi-source and assembly structures, supermodular and orthant dependence orders, copula, optimal inventory policy, performance bounds, numerical analysis
1. Introduction

Supply chain risk management is becoming an increasingly important research area. In the past several years, there has been a shift of focus from creating simply cost efficient supply chains to reliable and cost efficient supply chains (Snyder and Daskin, 2005). This shift is due to the large-scale negative impacts of supply disruptions and uncertainty in global supply chain networks. Many examples of these negative impacts have been reported in the literature. Pickett (2003) mentions earthquakes, severe weather, floods, fires, port explosions/disruptions, terrorist attacks, labor strikes, and financial distresses as the disruptive events that have happened in the past. In an empirical study, Hendricks and Singhal (2003) show that supply chain glitches (due to poor planning, part shortage, production problems, equipment breakdowns, etc.) are associated with an abnormal decrease of 10.28% in shareholder value.

Supply chain disruptions often start at a certain geographical region and immediately affect the suppliers located in the same region. In many situations they may spread or migrate through the network topology and have various potential cascading effects on other parts of the network (Thadakamalla et al., 2004). There had been many real world examples in which a single catastrophic event has significantly degraded the capabilities of several suppliers leading to considerable erosion of profits and good will for a firm. For example, an earthquake in Taiwan in 1999 damaged the manufacturing facilities of several major semiconductor suppliers and disrupted the flow of components to many PC and laptop manufacturers (Pickett, 2003). As another example, the outbreak of E. Coli in spinach in North America in 2006 had caused significant economic losses in several US food supply chains (The Economist, 2006). Despite the call for designing and operating resilient supply chains (NSF, 2004; Rice, 2003), there are few analytical supply chain disruption models available that account for the dependence nature of disruptions and its impact on supply chain performance. It is therefore essential to introduce analytical models that incorporate stochastic dependence among disruptions in supply chain risk analysis. In this paper, we introduce a two-echelon production-inventory model that explicitly take the dependence relation of disruptions into consideration, and study the structural properties of the key performance measures and the optimal ordering policy. The objective is to gain insights on effective management of such systems that can help firms make sound supplier selection decisions at the strategic level and better inventory management decisions at the operational level.

More specifically, we consider a single-period model for a two-echelon inventory system consisting of $m$ manufacturers and a single retailer. The manufacturers are subject to common-source disruptions so their production capacities, hence delivery quantities, are stochastically dependent. The cost components in our analysis include the holding cost for the leftover inventory and the shortage cost for unmet demand (Section 5 discusses an extension of the base model by including the purchase/production cost for delivered items in the cost function). The basic research question is how this dependence affects the retailer’s risk profile, performance measures and optimal ordering policy, and how to design and manage such a system to alleviate disruption risks. In order to understand the relationship between dependent disruption and supply chain structures, we study two supply chain structures, namely, the multi-
source and assembly supply chains. In the multi-source structure, the manufacturers supply the same product to the retailer, so the retailer’s replenishment quality is the sum of the items delivered by different manufacturers; in the assembly structure, each manufacturer produces a different part type, so the end units assembled by the retailer equals the minimum of the manufacturers’ delivery quantities. For each supply chain structure, we investigate the impact of the dependence in capacities, induced by correlated disruptions, on the key performances of the retailer (such as the expected cost, fill rate and service level) and the optimal ordering policy. Specifically, we ask the questions: as disruptions become more positively dependent in some stochastic sense (to be defined formally in Section 3), will the retailer’s cost increase or decrease? Will customer service, such as the fill rate and service level, improve or deteriorate? Should the retailer order more or less?

Using stochastic comparisons, in particular, dependence comparison theory (Joe, 1997; Li and Xu, 2000, 2001; Xu and Li, 2000), we investigate how the dependence strength of disruptions affect the retailer’s cost, the service measures such as the fill rate and service level, and the optimal ordering policy for both multi-source and assembly supply chains. We obtain the following analytical results: Stochastic dependence in disruptions can have opposite effects to supply chain performances with different supply chain structures. While risk diversification (via decreasing supply dependence) is preferred in the multi-source supply chain, risk concentration (via increasing supply dependence) is preferred in the assembly supply chain. We show that, if the firm ignores the dependence of supply chain disruptions and treats the disruptions as if they were independent, then in the multi-source structure, the firm would underestimate the cost, overestimate the fill rate, and order more units than the optimum (in the dual-source system); however, in the assembly structure, the opposite would happen. Therefore, disruption risks may be either alleviated or amplified by the interplay between the supply chain structure and the dependence in disruptions. The implication of our result is that, in the multi-source supply chain, the retailer can alleviate disruption risks by selecting the manufacturers who are less likely to be affected simultaneously by disruptions (e.g., the manufacturers in different geographical locations), whereas in the assembly supply chain, it is beneficial to select the manufacturers who have positively dependent capacities (e.g., the manufacturers with coordinated production plans).

Dependence analysis not only sheds light on the qualitative behavior of system performance, but also is a powerful tool to generate quantitative bounds. Specifically, we construct two benchmark systems: both systems being the same as the original system, except that one is subject to independent disruptions and another is subject to perfectly dependent disruptions. We show that the performances of the two benchmark systems constitute the upper and lower bounds of the corresponding performance in the original system. Both bounds are relatively easy to compute. In addition, a tighter range of the bounds for a performance measure suggests that the dependence effect of disruptions may be negligible; conversely, a wider range of the bounds means that this dependence effect may play a significant role in determining system performance.

We also perform a numerical study to validate our analytical results and quantify the performance and policy behavior of the two-echelon inventory system with dependent disruptions, in both the multi-source and assembly systems. Our experiment contains 120 problem scenarios generated by systematically varying several key system
characteristics including the dependence level of disruptions, dependence nature of disruptions, variability of manu-
facturing capacities and variability of customer demand. Our numerical analysis shows the following managerial and 
operational implications:

• In the multi-source supply chain, we observe that the disruptions with strong tail dependence (i.e., the random 
variables are dependent when they assume extreme values), often caused by catastrophic events, are more harm-
ful to the retailer than the disruptions with strong central dependence (i.e., the random variables are dependent 
when they assume values around their respective mean), often caused by the intermediate-level common-cause 
disruptive events such as transportation delays caused bad weather conditions. However, in the assembly supply 
chain, the disruptions with strong central dependence are more harmful to the retailer than the disruptions with 
strong tail dependence.

• In either supply chain, we observe that the retailer’s cost increase is significant if the retailer fails to select the 
suitable manufacturers at the strategic level, even if it makes the optimal ordering decision at the operational 
level. The average upper bound for the percentage cost increase is 30.6% for the multi-source system and 
89.8% for the assembly system, signifying the significant effect of stochastic dependence in disruptions on the 
retailer’s cost. At the operational level, we observe that if the retailer fails to recognize the effect of dependence 
in disruptions and orders the optimal quantity corresponding to independent disruptions, then its cost increase 
is moderate in the multi-source system (with an upper bound of 14.3%) and insignificant in the assembly 
system (with an upper bound of 9.5%). These results suggest that the consequence of making the strategic 
level, suboptimal decision in the assembly system is more severe than that in the multi-source system, but the 
consequence of making the operational level, suboptimal decision in the assembly system is less severe than 
that in the multi-source system.

• In the multi-source supply chain, highly variable capacities and highly dependent disruptions significantly in-
creases the retailer’s risk profile and deteriorates its performance. On the other hand, in the assembly structure, 
the combination of highly variable capacities and highly independent disruptions drastically amplifies the re-
tailer’s risk profit. In either supply chain, we find that the most effective tactic to alleviate this high risk is 
to reduce capacity variability, which can neutralize the effect of interdependencies among disruptions. If ca-
pacity variability cannot be reduced, then the retailer needs to decrease interdependencies of capacities in the 
multi-source supply chain and to increase interdependencies of capacities in the assembly supply chain.

In summary, this paper contributes to the supply chain risk management literature in the following aspects.

1. We propose new supply chain disruption models that explicitly capture the commonly observed phenomena that 
disruptions often simultaneously inflict damages to the capacities of the manufacturers.

2. Using dependence comparison theory, we develop the structural properties for the optimal ordering policy 
and the key performance measures in the two-echelon system with different sourcing structures. Our results
reveal when stochastic dependence in disruptions can positively or adversely affect the firm’s risk profile. We also develop performance bounds which can serve as the benchmarks for the performance of the system with dependent disruptions.

3. Our numerical study quantifies the individual and joint effects of ignoring stochastic dependence in disruptions in the firm’s strategic and/or operation decisions, and generates insights on how the dependence level of disruptions interacts with other key system characteristics.

The rest of the paper is organized as follows. Section 2 reviews the related literature on inventory models with supply uncertainty. Section 3 studies the two-echelon inventory systems with multi-source and assembly structures and develop analytical properties and bounds. Section 4 presents a numerical analysis of our model. Finally, Section 5 summarizes the paper, discusses extensions and proposes future research directions. The proofs of the major results can be found in the appendices.

2. Literature Review

In the supply chain risk management literature, one stream of research considers the optimal network design and facility location problems under capacity uncertainty (Snyder and Daskin, 2005; Snyder and Shen, 2006; Bundschuh et al., 2003; Santos et al., 2005; Vidal and Goetschalckx, 2000). The problems are typically formulated as mathematical and stochastic programs, and Snyder et al. (2006) provide a comprehensive review of these works. The objective of the models is to design or fortify supply chains by minimizing the expected cost, where failures of resources are treated as independent random events.

There is a rich body of literature studying 1-supplier, 1-retailer inventory models (Parlar, 1997; Karlin, 1958; Gregory and Beged-Dov, 1967; White, 1970; Panda, 1978; Ehrhardt and Taube, 1987; Schmitt et al., 2009; Wang and Gerchak, 1996; Gerchak et al., 1986; Gupta and Cooper, 2005; Ciarallo et al., 1994; Bollapragada and Morton, 1999; Yano and Lee, 1995) with uncertain supply. Wang and Gerchak (1996) study a periodic-review production planning problem with random demand and random capacity, where the capacities of different periods are assumed independent. Schmitt et al. (2009) consider both discrete-event uncertainty (disruptions) and continuous sources uncertainty (random yield).

Researchers also study the m-supplier, 1-retailer inventory models with random capacities, under both the newsvendor (Dada et al., 2007; Yang et al., 2007; Abdel-Maleka and Montanari, 2005; Tomlin and Wang, 2005; Federgruen and Yang, 2009a) and multi-period (Güler and Parlar, 1997; Schmitt and Snyder, 2009; Özekici and Parlar, 1999; Federgruen and Yang, 2009b) settings. In the newsvendor setting, Yang et al. (2007) propose a solution algorithm to find the optimal order quantities from the suppliers with random yields. Dada et al. (2007) study the structure of the optimal ordering policy of a profit maximizing newsvendor with unreliable suppliers, where the suppliers have different reliability and offer different prices for the same product. Federgruen and Yang (2009a) develop efficient
procedures to determine the order size to be placed by a newsvendor from a set of unreliable suppliers. They consider two models. In the first model, the orders must be such that the probability of satisfying demand is higher than a threshold. In the second model, the orders are determined so as to minimize the cost. Tomlin and Wang (2005) consider the value of flexibility and dual sourcing in unreliable newsvendor networks with \( N \) product types. In the multi-period setting, Güler and Parlar (1997) consider a two-supplier inventory model in which supply availability is modeled as a semi-Markov process. Özekicia and Parlar (1999) consider a periodic-review inventory models where the demand, supply and cost parameters change with respect to a random environment. Schmitt and Snyder (2009) argue that using the newsvendor model to study supply disruptions underestimates disruption risk.

Several 1-supplier, \( n \)-retailer inventory models with stochastic supply are also considered in the literature. Snyder and Shen (2006) study the differences between demand and supply uncertainties in two-echelon supply chains, where supply disruptions follow a Markov process. Corbett and Rajaram (2006) use dependence orders and copula to study the value of inventory pooling for a newsvendor with a multivariate demand distribution. They show that the value of inventory pooling decreases by increasing the dependence of the demand. This is one of the few works using copula in the OM research that we are aware of.

Our study differs from the existing literature in several aspects. First, in the context of the two-echelon inventory system with either the multi-source or assembly structure, we investigate the effect of the sourcing structure and the dependence of disruptions on system performance. Second, in addition to the tools commonly used inventory models, we utilize the methodologies from several active/new research areas, including multivariate dependence theory (Joe, 1996; Li and Xu, 2000, 2001; Xu and Li, 2000; Li et al., 2008) and copula (Joe, 1997; Kurowicka and Cooke, 2006) to develop analytical properties and simulation tools for performance evaluation and optimization. Finally, using a comprehensive numerical analysis, we systematically quantify the impact of dependent disruptions on the retailer’s strategic and operational level decisions. Our study generates useful guidelines for better management of supply chains with dependent disruptions.

3. A Single Period Analysis of an \( m \)-Manufacturer, 1-Retailer Model with Dependent Disruptions

In this section, we consider an \( m \)-manufacturer, 1-retailer, newsvendor inventory system with unreliable manufacturers. Demand occurs at the retailer and is denoted by \( D \). We denote manufacturer \( i \) by \( M_i \), \( i = 1, 2, \ldots, m \). Let \( K = [K_1, \ldots, K_m] \) denote the manufacturers’ production capacity vector, where \( K_i \) represents the random capacity of \( M_i \). The capacity of \( M_i \) in the period is a function of \( Z_i \), i.e., \( K_i = g_i(Z_i) \), \( i = 1, 2, \ldots, m \), where \( Z_i \) represents the random effect of a disruptive event on the capacity of \( M_i \). We assume that \( g_i \) is an increasing function for each \( i \), i.e., a larger value of \( Z_i \) implies a higher value of capacity \( K_i \). For example, we may let \( K_i = g_i(Z_i) = Z_i C_i \), where \( C_i \) is the full capacity of \( M_i \), and \( Z_i \) is the value of the random yield with support [0, 1]. If \( Z_i \) is a binary random variable, then it represents an extreme scenario that \( M_i \) is either completely reliable (\( Z_i = 1 \)) or completely unreliable \( Z_i = 0 \). Hereafter, we will call \( Z = [Z_1, \ldots, Z_m] \) the disruption vector or yield vector interchangeably. To capture
the phenomena that a common-source disruption may simultaneously inflicts damage to several manufacturers, we assume that vector \( Z \) has stochastically dependent elements. Because \( Z \) is a dependent random vector, the capacity vector \( K \) is also a dependent random vector.

At the beginning of the period, the retailer places an order to each of the \( m \) manufacturers. Let \( q_i \) be the order quantity placed to \( M_i \) and \( q = [q_1, \ldots, q_m] \) the corresponding order quantity vector. Due to uncertain capacities, the delivery quantity of \( M_i \) may be less than \( q_i \). We assume that the delivery quantity of \( M_i \), denoted by \( S_i(K, q_i) \), is a function of its capacity \( K \) and the received order quantity \( q_i \). Suppose \( S_i(K, q_i) \) is increasing in each of its arguments, and \( S_i(K, q_i) \leq \min(K, q_i) \), i.e., the delivery quantity of \( M_i \) is no more than its received order quantity, and also is no more than its available capacity. Because production capacities \( K \) are dependent, the deliveries \( S(K, q) = [S_1(K_1, q_1), \ldots, S_m(K_m, q_m)] \) are also dependent. We assume that the system incurs the unit holding cost \( h \) for leftover inventory and the unit shortage cost \( p \) for unmet demand.

To understand how different supply chain structures may be affected by dependent disruptions, we study two structures, namely, the multi-source and assembly structures. In the multi-source supply chain, the manufacturers produce the same product for the retailer, so the total replenishment received by the retailer is \( \sum_{i=1}^{m} S_i(K, q_i) \), the sum of the delivery quantities from all manufacturers. In the assembly supply chain, each manufacturer produces a different part type, so the end units assembled by the retailer is \( \min(S_i(K, q_i)) \).

A fundamental issue is whether stochastic dependence in disruptions alleviates or amplifies the risk profile of the retailer in a given structure. To understand this, we use the notions of supermodular and orthant dependence orders to compare the performances of two inventory systems that have different disruption vectors \( Z_+ = (Z_{1,+}, \ldots, Z_{m,+}) \) and \( Z_- = (Z_{1,-}, \ldots, Z_{m,-}) \), respectively. We first review several relevant notions of stochastic orders. Recall a function \( f \) is called supermodular if \( f(x \land y) + f(x \lor y) \geq f(x) + f(y) \), where \( x \land y \equiv \min{x_1, y_1}, \ldots, \min{x_m, y_m} \) and \( x \lor y \equiv \max{x_1, y_1}, \ldots, \max{x_m, y_m} \). In this work the terms increasing and decreasing mean non-decreasing and non-increasing, respectively.

**Definition 3.1.** (Müller and Scarsini, 2000; Mosler and Scarsini, 1991)

a. \( Z_- \) is said to be smaller than \( Z_+ \) in the upper orthant order, denoted as \( Z_- \prec_{uo} Z_+ \) if

\[
P(Z_{1,-} > z_1, \ldots, Z_{m,-} > z_m) \leq P(Z_{1,+} > z_1, \ldots, Z_{m,+} > z_m).
\]

b. \( Z_- \) is said to be smaller than \( Z_+ \) in the supermodular order, denoted as \( Z_- \prec_{sm} Z_+ \), if for every supermodular function \( f \),

\[
E(f(Z_-)) \leq E(f(Z_+)).
\]

c. \( Z_- \) is said to be smaller than \( Z_+ \) in the usual stochastic order, denoted as \( Z_- \prec_{st} Z_+ \), if \( P(Z_- \in \mathcal{U}) \leq P(Z_+ \in \mathcal{U}) \), for all upper sets \( \mathcal{U} \) in \( \mathcal{R}^m \), where a subset \( \mathcal{U} \subseteq \mathcal{R}^m \) is called upper if \( z \in \mathcal{U} \) and \( z \leq z' \) imply \( z' \in \mathcal{U} \).
Intuitively, $Z_- \prec_{uo} Z_+$ means that the elements of $Z_+$ are more likely to simultaneously realize larger values than the elements of $Z_-$. On the other hand, $Z_- \prec_{sm} Z_+$ means that $Z_{d,-}$ and $Z_{d,+}$ are stochastically identical for all $i$, but $(Z_{1,-}, \ldots, Z_{m,-})$ are more likely to simultaneously realize large values or to simultaneously realize small values compared to $(Z_{1,+}, \ldots, Z_{m,+})$. Because the focus of this study is to compare the impact of the dependence, rather than the magnitude, of disruptions on system performance, we assume that $Z_-$ and $Z_+$ have the same marginals. In contrast, the usual stochastic order, given in Definition 3.1.c, compares the magnitudes of two random vectors, that is, $Z_- \prec_{st} Z_+$ implies that the marginal of $Z_{i,-}$ is stochastically smaller than the marginal of $Z_{i,+}$ for each $i$. The following result is well-known and will be used to derive the properties of the performance measures of the retailer.

**Lemma 3.2.** (Müller and Scarsini, 2000; Shaked and Shanthikumar, 1994)

1. If $Z_- \prec_{uo} (\prec_{sm}) Z_+$, then
   \[
   [f_1(Z_{1,-}), \ldots, f_m(Z_{m,-})] \prec_{uo} (\prec_{sm}) [f_1(Z_{1,+}), \ldots, f_m(Z_{m,+})],
   \]
   where $f_i : \mathbb{R} \mapsto \mathbb{R}$, $i = 1, \ldots, m$, are increasing functions.

2. $Z_- \prec_{sm} Z_+$ implies marginals of $Z_-$ and $Z_+$ are the same and $Z_- \prec_{uo} Z_+$.

3. $Z_- \prec_{uo} Z_+$ implies $\min\{Z_-\} \prec_{st} \min\{Z_+\}$.

Lemma 3.2 shows that the supermodular and upper-orthant orders are invariant under increasing transformations. Also, while both supermodular and orthant dependence orders emphasize the comparison of dependence strengths of the two vectors, the supermodular order is a stronger notion of dependence compared with the orthant order.

### 3.1. Multi-source Supply Chain

Without loss of generality, let the initial inventory be zero. Fixing the order quantities $q$, the expected cost of the retailer is given by

$$g(q) = \mathbb{E}[p(D - I(K, q))] + h[I(K, q) - D]_+,$$

where $I(K, q) = \sum_{i=1}^m S_i(K_i, q_i)$ is the total delivery quantity. The minimum expected cost is then expressed as $g(q^*) = \min_{q \in A} g(q)$, where $q^* \in A^*$, and $A^*$ is the set of optimal order quantity vectors. In Section 5, we show that our results remain valid if we include the unit purchase cost $b_i$ from $M_i$ in the model.

We define the **fill rate** and **service level** under the fixed order quantity vector $q$, respectively, by

\[
FR(q) = 1 - \frac{E[D - I(K, q)]_+}{E(D)}, \quad (3.1)
\]

\[
SL(q) = P(I(K, q) \geq D). \quad (3.2)
\]

Next, we investigate how the performance measures in the multi-source supply chain are affected by the dependence nature and strength of the disruption vector $Z$. 
3.1.1. Effects of Dependent Disruptions on Performance Measures

When we compare two inventory systems with disruption vectors $Z_+$ and $Z_-$, we denote their corresponding capacities by $K_+$ and $K_-$, the cost functions by $g_+(q)$ and $g_-(q)$, the sets of the optimal order quantity vectors by $A^+_i$ and $A^-_i$, the fill rates by $FR_+(q)$ and $FR_-(q)$ and the service levels by $SL_+(q)$ and $SL_-(q)$.

**Theorem 3.3.** If $Z_+$ is more supermodular dependent than $Z_-$, then for the multi-source system,

a. For any order quantity vector $q$, $g_+(q) \geq g_-(q)$. Furthermore, $g_+(q^*_i) \geq g_-(q^*_i)$, for any $q^*_i \in A^+_i$ and $q^*_i \in A^-_i$.

b. For any order quantity vector $q$, $FR_+(q) \leq FR_-(q)$.

The above theorem states that the performance of the multi-source system improves if the supermodular dependence level of $Z$ decreases. Loosely speaking, it says that a multi-source supply chain with less dependent disruptions has a lower cost and a higher fill rate as compared to their counterparts in a multi-source supply chain with more dependent disruptions. The reason that we prefer less dependent capacities across different manufacturers is that lower dependence of $K$ can reduce the variability of the aggregate delivery quantity $I(K, q) = \sum_{i=1}^{m} S(K_i, q_i)$ while keeping its mean fixed, which subsequently can lower the cost. Furthermore, because the variability order is preserved under the increasing and convex transformation $[d - x]^+$, the low variability of $I(K, q)$ can reduce the expected unfilled demand $E[D - I(K, q)]^+$, leading to a higher fill rate as defined in Eq. (3.1). In order to induce less dependent capacities, we need the disruption vector $Z$ to be less dependent (in the sense of the supermodular dependence order).

In general, $Z_- \prec_m Z_+$ implies neither $SL_-(q) \geq SL_+(q)$ nor $SL_-(q) \leq SL_+(q)$. In other words, the service level of the multi-source system cannot be improved simply by altering the supermodular dependence level of the disruption vector. To increase the service level of the multi-source system, the aggregate delivery quantity $I(K, q)$ needs to become stochastically larger, which would happen if the random yield vector $Z$ becomes stochastically larger.

**Theorem 3.4.** If $Z_+$ is stochastically larger than $Z_-$, then in the multi-source system, $SL_-(q) \leq SL_+(q)$.

3.1.2. Optimal Ordering Policy

In this section, we investigate how the interdependencies of the capacity vector $K$, induced by the dependent disruption vector $Z$, affect the retailer’s inventory policy. We show that when $m = 2$ (dual-source) and $S_i(K_i, q_i) = \min\{q_i, K_i\}$, the optimal order quantity vector becomes smaller as the capacity vector $K$ becomes more supermodular dependent.

The dependence strength of a random vector is often reflected by a set of dependence parameter $\theta \in \Theta$ in the joint CDF, where $\Theta$ is the set of parameter values. As the dependence parameter varies, the random vector becomes more or less dependent in some stochastic sense. For example, the dependence nature of a multivariate normal random vector is determined by its covariance matrix $\Sigma = \{\sigma_{ij}\}$. It turns out that the supermodular dependence order of random variables in some distribution families can be established as we systematically vary relevant dependence parameters. To reflect this, we write a random vector, say $Z$, as $Z(\theta) = [Z_1(\theta), Z_2(\theta), \ldots, Z_m(\theta)]$. 

We need the following lemma, which is due to Porteus (2002) and Topkis (1978), to compare the optimal ordering policies of the two inventory systems with different disruption vectors.

**Lemma 3.5.** Let $A(\theta)$ be the set of feasible actions for $\theta \in \Theta$, where $\Theta$ is a set of parameter values. For any $\theta_+, \theta_- \in \Theta$ such that $\theta_- \leq \theta_+$, let $a_+ \in A(\theta_+)$ and $a_- \in A(\theta_-)$ The action set $A(\cdot)$ is called ascending on $\Theta$ if $a_+ \land a_- \in A(\theta_+)$ and $a_+ \lor a_- \in A(\theta_-)$ and is called descending if $a_+ \land a_- \in A(\theta_+)$ and $a_+ \lor a_- \in A(\theta_-)$. If the cost function $g(a, \theta)$ is supermodular (submodular) in $(a, \theta)$, the action space $A(\cdot)$ is descending (ascending) on the parameter set $\Theta$, and the optimal action space $A^*(\cdot)$ is non-empty for every $\theta$, then $A^*(\cdot)$ is descending (ascending) on $\Theta$.

If the set of optimal actions $A^*(\cdot)$ has only one member for each $\theta$, that is, $A^*(\theta) = \{a^*(\theta)\}$, then the above lemma implies that, when $g(a, \theta)$ is supermodular, $a^*(\theta)$ becomes smaller as $\theta$ increases. When $A^*(\theta)$ has more than one member, the lemma implies that if $\theta_- \leq \theta_+$, then for every $a^*_+ \in A^*(\theta_-)$ there exists $a^*_+ \in A^*(\theta_+)$, such that, $a^*_+ \leq a^*_-$.

In the following theorem we assume that capacity $K_i$ are continuous random variables for each $i$. In Section 5, we discuss several extensions of our results, including a more general assumption on the distributions of the capacities.

**Theorem 3.6.** (For the dual-source system $(m = 2)$) Let $\theta_-, \theta_+ \in \Theta$, where $\Theta$ is the set of parameter values. Suppose $\theta_- \leq \theta_+$ implies $Z_i(\theta_-) =_\mu Z_i(\theta_+)$, $i = 1, 2, and [Z_1(\theta_-), Z_2(\theta_-)] \prec_{st} [Z_1(\theta_+), Z_2(\theta_+)]$. Let $S_i(g_i(Z_i(\theta), q_i)) = \min\{g_i(Z_i(\theta), q_i)\}$ and let $A^*(\theta) \subset \mathbb{R}^2$ be the set of optimal order quantity vectors associated with $\theta$. Then,

1. The optimal action space $A^*(\cdot)$ is descending on $\Theta$. In other words, for every $q^*_+ \in A^*(\theta_-)$ there exists $q^*_+ \in A^*(\theta_+)$ such that $q^*_+ \leq q^*_-$.

2. For every $q^*_+ \in A^*(\theta_-)$ there exists $q^*_+ \in A^*(\theta_+)$ such that $q^*_+ \leq q^*_-$ and $FR_i(q^*_+) \leq FR_i(q^*_-)$. 

The above theorem states that, in a dual-source supply chain, less dependent disruptions result in a larger optimal order quantity to each manufacturer and a higher fill rate. We can understand the monotonicity of the optimal order quantity vector with respect to dependence parameter $\theta$ as follows: when the disruptions, and thus capacities, are less dependent, the value of risk diversification is significant, providing the retailer with the incentive to place a larger order to each manufacturer. However, as the capacities become more dependent, the benefit of risk diversification becomes smaller, discouraging the retailer to place a larger order.

Theorems 3.3 and 3.6 have the following implications. At the strategic level, the results suggest that the retailer can reduce disruption risk by selecting the manufacturers that are less likely to be affected simultaneously by disruptions (e.g., by selecting the manufacturers in different geographical locations and/or with diverse suppliers). At the operational level, the results suggest that if the firm ignores interdependencies in supply disruptions and treats random as independent, it would underestimate the cost and overestimate the fill rate. Although differing in the problem settings and motivations, these results complement the results of Schmitt et al. [47] who find that in a 1-warehouse and $n$-retailer system with stochastic supply, a decentralized inventory design (in our problem context, a decentralized sourcing design) reduces cost variance through risk diversification effect.
Remark 3.7. Unfortunately, the objective function \( g(q, \theta) \) in general is not a supermodular function of \((q, \theta)\) when \( m > 2 \). This can be seen as follows. Analogous to the proof of Theorem 3.6, we can show
\[
\frac{\partial g(q, \theta)}{\partial q_i \partial \theta} = (p + h) \frac{\partial}{\partial \theta} P \left( \sum_{j=1}^{m} \min(q_j, K_j(\theta)) \geq D - q_i, K_i(\theta) \geq q_i \right) .
\]
The sufficient condition for the monotonicity of the optimal order quantity vector in \( \theta \) requires that \( g(q, \theta) \) be supermodular in \((q, \theta)\), that is, Eq.(3.3) is nonnegative. While Eq.(3.3) is indeed nonnegative for \( m = 2 \), it is not necessarily true for \( m > 2 \), because \( \sum_{j=1}^{m} \min(q_j, K_j(\theta)) \) does not follow the supermodular dependence order when \( \theta \) increases. However, when \( q \) is close to \( q^* \), our numerical test shows in all example cases \( \frac{\partial g(q, \theta)}{\partial q_i \partial \theta} \geq 0 \). This means, that at the neighborhood of the optimal order quantity vector \( q^* \), cost function \( g(q, \theta) \) is supermodular in \((q, \theta)\) and therefore the optimal action set is descending in \( \theta \). In Section 4, we report 120 problem scenarios and show, numerically, that this property holds for each scenario.

3.1.3. Bounds

Dependence analysis not only sheds lights on qualitative behavior of system performance, but also is a powerful tool to generate quantitative bounds. For each performance measure, the range between the upper and lower bounds can provide us with indications on the sensitivity of the performance to the dependence level of the disruption vector.

To develop the upper and lower bounds for the cost function \( g(q^*) \), let \( U = [U_1, \ldots, U_m] \) be the random vector with uniform marginals with support \([0,1]\), such that \( Z_i = U_i \) for all \( i \), where \( F^{-1}(\cdot) \) is the inverse of the CDF of \( Z_i \), and \( Z = [F^{-1}(U_1), F^{-1}(U_2), \ldots, F^{-1}(U_m)] \). The CDF of \( U = [U_1, \ldots, U_m] \) is known as the copula function of \( Z \), and we provide a further discussion of this notion in Section 4. In addition, let \( \hat{U} = [\hat{U}_1, \ldots, \hat{U}_m] \) and \( \hat{U} = [U_1, \ldots, U_1] \) be the independent and perfectly dependent random vectors with support \([0,1]\), respectively. It is known that if \( U \) is positively supermodular dependent, then \( \hat{U} <_{sm} U <_{sm} \hat{U} \), where the upper bound on \( U \) is due to the Frechet upper bound and the lower bound is the direct consequence that \( U \) is positively supermodular dependent (Müller and Scarsini, 2000; Shaked and Shanthikumar, 1994).

Since \( F^{-1}(\cdot) \) is an increasing function for all \( i \), Lemma 3.2 implies that, if \( U \) is positively supermodular dependent, then
\[
\hat{Z} = [F^{-1}(\hat{U}_1), F^{-1}(\hat{U}_2), \ldots, F^{-1}(\hat{U}_m)]
\]
\[
<_{sm} Z = [F^{-1}(U_1), F^{-1}(U_2), \ldots, F^{-1}(U_m)]
\]
\[
<_{sm} \hat{Z} = [F^{-1}(U_1), F^{-1}(U_1), \ldots, F^{-1}(U_1)].
\]

The random vector \( \hat{Z} = [F^{-1}(U_1), F^{-1}(U_1), \ldots, F^{-1}(U_1)] \) is known in the literature to be comonotonic or perfectly dependent (Kaas et al., 2001). For example, in the symmetric disruption case, if \( Z_i =_{st} F^{-1}(U_i) \) for all \( i \), then the random vector \( \hat{Z} = [F^{-1}(U_1), F^{-1}(U_1), \ldots, F^{-1}(U_1)] \) will be comonotonic. Using Theorem 3.3, we conclude that \( \hat{U} <_{sm} U <_{sm} \hat{U} \) implies \( \hat{Z} <_{sm} Z <_{sm} \hat{Z} \), and the latter further implies that the cost function \( g \) with disruption vector \( Z \) admits the following upper and lower bounds:
\[
\hat{g}(\hat{q}^*) \leq g(q^*) \leq \hat{g}(\hat{q}^*).
\]
where $\hat{g}(\hat{q}^*)$ is the minimum cost and $\hat{q}^*$ is the optimal order quantity in the system with independent disruption vector $\hat{Z}$, and $\tilde{g}(\tilde{q}^*)$ is the minimum cost and $\tilde{q}^*$ is the optimal order quantity in the system with comonotonic disruption vector $\tilde{Z}$. Similarly, Theorem 3.6 implies that, for a dual-source system, if $U$ is positively supermodular dependent, then $\tilde{q}^*$ and $\hat{q}^*$ are lower and upper bounds of the optimal order quantity $q^*$ in the system with disruption vector $Z$, respectively, and $\tilde{FR}(\tilde{q}^*)$ and $\hat{FR}(\hat{q}^*)$ are the lower and upper bounds of the fill rate $FR(q^*)$, respectively.

Since the comonotonic or independent random vector depends only on marginal distributions of its component random variables, evaluating system performance becomes computationally manageable as compared to that with dependent disruptions. These easily computable bounds can serve as the benchmarks for the system with dependent disruptions. The upper and lower bounds for a given performance measure will be reported in Section 4.

3.2. Assembly Supply Chain

We again assume the system has zero initial inventory. Given the order quantity vector $q$, the available inventory of the final product to meet demand is $I(K, q) = \min\{S_1(K_1, q_1), \ldots, S_m(K_m, q_m)\}$. Let $h_i$ be the unit holding cost of part type $i$. The cost, fill rate and service level at the retailer are defined, respectively, by

$$g(q) = E\left[p(D - I(K, q))^+ + \sum_{i=1}^{m} h_i(S_i(K, q_i) - \min[I(K, q), D])\right],$$

(3.5)

$$FR(q) = 1 - \frac{E((D - I(K, q))^+)}{E(D)},$$

(3.6)

$$SL(q) = P(I(K, q) \geq D).$$

(3.7)

3.2.1. Effects of Dependent Disruptions on Performance Measures

In this section, we use the notion of the upper orthant dependence order, given in Definition 3.1, to compare system performance in two assembly supply chains. We use the same notations as in the analysis of the multi-source supply chain, and let $SL_+(q_+)$ and $SL_-(q_-)$ be the service levels when the disruption vectors are $Z_+$ and $Z_-$, respectively. Using Lemma 3.2 (c), the following result can be easily established and we omit the proof.

**Lemma 3.8.** $Z_- \prec_{uo} Z_+$ implies $I(K_-, q) \prec_{st} I(K_+, q)$.

**Lemma 3.8** states that the available inventory of the final product becomes stochastically larger as the disruption vector $Z$ becomes more upper orthant dependent. This is because the delivery quantities of different part types tend to take the larger values simultaneously when disruptions are more positively dependent.

**Theorem 3.9.** If $Z_- \prec_{uo} Z_+$ and $Z_{i-} =_{st} Z_{i+}$ for each $i$, then in the assembly supply chain,

a. For any order quantity vector $q$, $g_+(q) \leq g_-(q)$. Furthermore, $g_+(q^*_+) \leq g_-(q^*_-)\), for all $q^*_+ \in A^*_+$ and $q^*_- \in A^*_-$.

b. For any order quantity vector $q$, $FR_+(q) \geq FR_-(q)$ and $SL_+(q) \geq SL_-(q)$.
Briefly, this theorem states that the assembly supply chain with more dependent disruptions has a lower cost, a larger fill rate and a higher service level. The intuition behind these results is that, in an assembly system, we prefer the balanced production capacities and balanced part deliveries from the manufacturers, i.e., we would like the deliveries of different part types to be roughly the same. To induce balanced production capacities across different manufacturers, disruptions need to be more positively dependent (in the upper orthant ordering sense).

3.2.2. Optimal Ordering Policy

Assuming \( S_i(K_i, q_i) = \min(K_i, q_i) \), in this section we show that the optimal order quantity vector becomes larger as the upper orthant dependence level of the disruption vector increases. To prove this result, we need Lemma 3.10.

**Lemma 3.10.** If \( S_i(K_i, q_i) = \min(K_i, q_i) \), for all i, then \( q^*_1 = q^*_2 = \ldots = q^*_m \), i.e., the retailer should order the same amount from each manufacturer.

**Theorem 3.11.** Let \( \theta_-, \theta_+ \in \Theta \), where \( \Theta \) is the set of parameter values. Suppose \( \theta_- \leq \theta_+ \) implies \( Z_i(\theta_-) = \text{st} Z_i(\theta_+) \), for every \( i = 1, \ldots, m \), and \( [Z_1(\theta_-), \ldots, Z_m(\theta_-)] <_{\text{st}} [Z_1(\theta_+), \ldots, Z_m(\theta_+)] \). Let \( A^*(\theta) \subset R^m \) denote the set of optimal order quantities associated with \( \theta \). Also let \( S_i(g_i(Z_i(\theta), q)) = \min(g_i(Z_i(\theta)), q) \). Then,

1. The optimal action set \( A^*(\cdot) \) is ascending on \( \Theta \). In other words, for every \( q^*_+ \in A^*(\theta_-) \) there exists \( q^*_- \in A^*(\theta_+) \) such that \( q^*_+ \geq q^*_- \). (We have assumed \( A^*(\theta) \subset R^m \). And also in the proof we have assumed that we have a decision variable not a decision vector).

2. For every \( q^*_- \in A^*(\theta_-) \) there exists \( q^*_+ \in A^*(\theta_+) \) such that \( q^*_+ \geq q^*_- \) such that \( \text{FR}_+([q^*_+, \ldots, q^*_m]) \geq \text{FR}_-([q^*_-, \ldots, q^*_m]) \) and \( \text{SL}_+([q^*_+, \ldots, q^*_m]) \geq \text{SL}_-([q^*_-, \ldots, q^*_m]) \).

Based on this theorem, as stochastic dependence in disruptions increases, the retailer is encouraged to place a larger order to each manufacturer. A plausible explanation is as follows. More dependent disruptions result in more balanced production capacities and deliveries, i.e., these quantities tend to simultaneously assume the values of similar magnitudes. This leads to a stochastically larger number of assembled units, thus the retailer can satisfy a larger fill rate and a higher service level. The intuition behind these results is that, in an assembly system, we prefer the balanced production capacities and balanced part deliveries from the manufacturers, i.e., we would like the deliveries of different part types to be roughly the same. To induce balanced production capacities across different manufacturers, disruptions need to be more positively dependent (in the upper orthant ordering sense).
chain structure changes. While risk diversification (via decreasing supply dependence) is preferred in the multi-source supply chain, risk concentration (via increasing supply dependence) is preferred in the assembly supply chain.

3.2.3. Bounds

As discussed in the multi-source structure, Theorems 3.9 and 3.11 can also be used to generate computable bounds for the performance measures and optimal ordering policy in the assembly structure. Using the notation developed in Section 3.1.3, it is easy to show, if \( U \), the copula of \( Z \), is positively supermodular dependent, then \( \hat{U} \prec_{sm} U \prec_{sm} \tilde{U} \) and \( \hat{Z} \prec_{sm} Z \prec_{sm} \tilde{Z} \). Consequently,

\[
\tilde{q}^* \geq q^* \geq \hat{q}^*,
\]

\[
\tilde{g}(\tilde{q}^*) \leq g(q^*) \leq \hat{g}(\hat{q}^*),
\]

\[
\tilde{FR}(\tilde{q}^*) \geq FR(q^*) \geq \hat{FR}(\hat{q}^*),
\]

\[
\tilde{SL}(\tilde{q}^*) \geq SL(q^*) \geq \hat{SL}(\hat{q}^*).
\]

Here, a tighter range of the bounds for a performance measure suggests that the dependence effect of disruptions may be negligible; conversely, a wider range of the bounds means that this effect may play a significant role in determining system performance. We will report these bounds for the assembly system in the next section.

4. Numerical Study

In this section, we perform a numerical study to quantify system performance and policy behavior of the two-echelon inventory systems with dependent disruptions, under both multi-source and assembly structures. Since system performance depends on several key problem characteristics such as the dependence level of disruptions, dependence nature of disruptions, variability of manufacturer capacities and variability of customer demand, we design a comprehensive set of problem scenarios to draw managerial and operational insight for the following questions.

- What are the individual and joint effects of ignoring the dependence in disruptions in the firm’s strategic and/or operation decisions?

- For each supply chain, when is the effect of dependence of disruptions negligible and when is it significant on the retailer’s performance and the optimal ordering policy? What is the range of the lower and upper bounds for each performance measure?

- How does the dependence strength of disruptions interact with other system characteristics, such as the dependence nature of disruptions (e.g., tail dependence vs. central dependence), manufacturer capacity variability and customer demand variability, and what is the compound effect of such interactions on system performance?
What is the impact of the interplay between the nature of disruption dependence (tail dependence vs. central dependence) and the system structure (multi-source vs. assembly) on system performance and ordering decisions?

What kind of combinations would most adversely affect system performance?

In the next section, we discuss a method to generate a sample of a dependent random vector, which facilitates our numerical analysis. In Sections 4.2 and 4.3, we outline our problem generation approach and discuss computational results for the multi-source and assembly supply chains, respectively.

4.1. Copula Functions

In our experiment, we need to generate random vectors that follow the supermodular or orthant dependence order when we systematically vary dependence parameters. Toward this end, we propose to use the copula function. Copula is a tool of transforming a multivariate distribution to a multivariate uniform distribution, where the obtained uniform distribution retains the dependence structure of the originals distribution (Joe, 1997; Kurowicka and Cooke, 2006). Next, we briefly introduce the concept of copula and the two copula families to be used in our numerical study.

Let us define \( F_1, \ldots, F_m \) as the marginal cumulative distribution functions (CDF) of \( X_1, \ldots, X_m \), respectively, and let \( F \) be their joint CDF. For continuous marginals, there exists a unique function \( C : [0, 1]^m \mapsto [0, 1] \) such that

\[
F(x_1, \ldots, x_m) = F(F_1^{-1}(u_1), \ldots, F_m^{-1}(u_m)) = C(u_1, \ldots, u_m),
\]

where \( F_i^{-1}(u_i) = \inf \{x : F_i(x) \geq u_i\} \), and \( C(u_1, \ldots, u_m) \) is the joint CDF of uniform random variables \( (U_1, \ldots, U_m) \) with support \([0, 1]\). In the literature, \( C(u_1, \ldots, u_m) \) is refereed to as the copula function.

Copula can be better understood if we explain the simulation procedure for sampling random vector \( X = [X_1, \ldots, X_m] \).

To do so, we first use copula \( C(u_1, \ldots, u_m) \) to generate uniformly distributed random vector \([u_1, \ldots, u_m]\), which is a realization of \([F_1(X_1), \ldots, F_m(X_m)]\). We transform each element of \([u_1, \ldots, u_m]\), using the inverse of its CDF \( F_i \), to generate \([x_1, \ldots, x_m]\), that is, \( x_i = F_i^{-1}(u_i) \) for all \( i \). We claim that \([x_1, \ldots, x_m]\) is a realization of \([X_1, \ldots, X_m]\). This procedure is analogous to the sampling procedure for a single random variable \( X \) with CDF \( F \), where a uniform random number \( u \) is first generated, and an observation of \( X \) is then obtained via transformation \( x = F^{-1}(u) \).

Similar to distribution functions, various families of copulas are defined in the literature. Sampling algorithms are available for several commonly used families (e.g., Whelan, 2004). In this work, we use the following two copula families.

**Clayton Copula:** There are various ways to define a Clayton copula for more than two variables. The simplest definition is the following:

\[
C(u, \theta) = \frac{1}{\theta (u_1^{-\theta} + \ldots + u_m^{-\theta} - (m - 1))}, \quad (4.1)
\]
where \( \theta \in (0, \infty) \) is called the dependence parameter of the Clayton copula (Trivedi and Zimmer, 2005; Embrechts et al., 2003). This copula belongs to the family of exchangeable Archimedean copulas (Hofert, 2008). The Clayton copula has strong lower tail and weak upper tail dependence, i.e., the dependence of random variables tends to concentrate in the extremely small values (Kousky and Cooke, 2009).

**Normal Copula:** The Normal copula is defined by (Trivedi and Zimmer, 2005; Embrechts et al., 2003)

\[
C(u, \Sigma) = \Phi\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_m), \Sigma\right),
\]

where \( \Phi \) denotes the marginal CDF of the standard normal random variable and \( \Phi' \) denotes the joint CDF of the normal random vector with means \( \theta \) and covariance matrix \( \Sigma = \{\sigma_{ij}\} \). This copula is symmetric, has strong “central” dependence, i.e., it is strongly dependent for the values around the means and no dependence at the tails (Joe, 1997).

The dependence structure of the Normal copula is characterized by the covariance matrix \( \Sigma \). The next lemma states that, by increasing the dependence parameters of Clayton and Normal copulas, we can generate random vectors that follow supermodular or orthant dependence orders. Let \( \Sigma_\pm = \{\sigma_{ij}\pm\} \) and \( \Sigma_\pm = \{\sigma_{ij}\pm\} \).

**Lemma 4.1.** (Wei and Hu, 2002; Corbett and Rajaram, 2006)

a. Let \( Z(\theta_-) \) and \( Z(\theta_+) \) be two random vectors with the same marginals and with the exchangeable Clayton copulas, defined in Eq. (4.1), with the dependence parameters \( \theta_- \) and \( \theta_+ \), respectively. Then \( \theta_- < \theta_+ \) implies \( Z(\theta_-) \triangleleft \text{sm} \left(\triangleleft \text{sm}\right) Z(\theta_+). \)

b. Let \( Z(\Sigma_-) \) and \( Z(\Sigma_+) \) be two random vectors with the same marginals and with the Normal copulas, defined in Eq. (4.2), with the covariance matrices \( \Sigma_- \) and \( \Sigma_+ \), respectively. Then \( \Sigma_- \leq \Sigma_+ \) implies \( Z(\Sigma_-) \triangleleft \text{sm} \left(\triangleleft \text{sm}\right) Z(\Sigma_+). \)

Since dependence parameters, such as \( \theta \) in Clayton and \( \Sigma \) in Normal, are copula-specific, we need a measure to quantify the dependence strength of random vectors that is copula-independent. In order to do this, we use the dependence measure known as the rank correlation (a.k.a. the Spearman’s rho). For any given \( Z \), the rank correlation between the elements of \( Z, Z_i \) and \( Z_j \), denoted by \( r(Z_i, Z_j) \), is the linear correlation between \( F_i(Z) \) and \( F_j(Z) \), i.e.,

\[
r(Z_i, Z_j) = \sigma_{F_i(Z), F_j(Z)},
\]

where \( F_i \) and \( F_j \) are CDFs of \( Z_i \) and \( Z_j \), respectively. The rank correlation is a better dependence measure than the linear correlation \( \sigma_{Z, Z} \) since it is invariant in strictly increasing transformations, is equal to one when one variable is an increasing function of the other, and is equal to zero only when the two variables are independent (Joe, 1997). In addition, rank correlation \( r(Z_i, Z_j) \) is independent of the marginals of \( Z_i \) and \( Z_j \) and can be written in terms of their copula \( C_{Z_i, Z_j} \) as follows (Joe, 1997):

\[
r(Z_i, Z_j) = 12 \int_0^1 \int_0^1 u v \, dC_{Z_i, Z_j}(u, v) = 3. \quad (4.3)
\]

If \( Z \) has the exchangeable Clayton copula with parameter \( \theta \), then \( r(Z_i, Z_j) = r \), for all \( i, j \). The above expression allows us to compute rank correlation \( r \) for each given \( \theta \). It is known that rank correlation \( r \) is increasing in \( \theta \) and they have a one-to-one relation. Particularly, \( r = 0 \) if and only if \( \theta = 0 \) and, as \( \theta \) approaches infinity, \( r \) approaches one (Trivedi
and Zimmer, 2005). If \( Z \) has the Normal copula with the covariance matrix \( \Sigma = \{ \sigma_{ij} \} \), then \( \sigma_{ij} = 2 \sin \left( \frac{\pi}{6} r(Z_i, Z_j) \right) \), for all \( i, j \) (Kurowicka and Cooke, 2006).

Using the Normal and Clayton copula families, we investigate the effect of dependent disruptions on the performance of multi-source and assembly supply chains. We believe that Clayton is a better candidate than Normal to model dependent disruptions caused by catastrophic events. In many situations, the capacities are independent across different manufacturers as long as no significant disruptive event occurs; however, after a severe disruption (e.g. an earthquake, a hurricane), the manufacturer capacities will simultaneously take small values and exhibit strong lower tail dependence. This type of dependence can be modeled by the Clayton copula due to its asymmetric dependence nature and its strong lower tail dependence. The Copulas with strong tail dependence are used in the fields of finance and insurance to model interdependencies among stock prices that are correlated only during the down periods or insurance payments that are correlated after a catastrophic event. For example, Kousky and Cooke (2009) explain that while wind damage and water damage are generally insured separately, since they are often independent, wind and water insurance payments may be tail dependent in a hurricane-prone state such as Florida. In contrast, the Normal copula, which has stronger dependence for the values around the means, is a better candidate to model less severe, normal disruptions.

4.2. Multi-source Supply Chain

We first outline our experiment design. We study a 3-manufacturer, 1-retailer inventory system. We use the notations \( X \sim U(a, b) \) to denote a uniform random variable \( X \) with support \([a, b]\), and \( X \sim N(\mu, \sigma) \) a normal random variable with mean \( \mu \) and standard deviation \( \sigma \). We assume that the random disruption \( Z_i \) has the marginal \( Z_i \sim U(0, 1) \), \( i = 1, 2, 3 \). We set other system parameters for the multi-source system as follows.

1. The Clayton and Normal copulas of \( (Z_1, Z_2, Z_3) \): We let the dependence level between any pair of disruption variables \( (Z_i, Z_j) \), \( i, j = 1, 2, 3 \), be the same, and denote \( r_{ij} = r \) as the rank correlation of any pair. We consider the following 10 cases, with one independent case \( (r = 0) \), one comonotonic case \( (r = 1) \), and four dependent cases with different rank correlation values \( (0 < r < 1) \) for each copula family.
   a. Rank correlation \( r = 0 \): this means disruptions \( (Z_1, Z_2, Z_3) \) are independent uniforms with support \([0,1]\).
   b. Rank correlation \( r = 1 \): in this case \( Z = [Z_1, Z_1, Z_1] \), where \( Z_1 \sim U(0, 1) \).
   c. The Clayton copula with rank correlation values \( r = 0.2, 0.4, 0.6 \) and 0.8. The dependence parameter values \( \theta \) corresponding to the rank correlation values are \( \theta = 0.31, 0.76, 1.5 \) and 3.15 respectively.
   c. The Normal copula with rank correlation values \( r = 0.2, 0.4, 0.6 \) and 0.8. The dependence parameter values \( \sigma = \sigma_{ij} = 2 \sin \left( \frac{\pi}{6} r \right) \) associated with the rank correlation values are \( \sigma = 0.21, 0.42, 0.62 \) and 0.81, respectively.

Note that when \( r = 0 \) or \( r = 1 \), \( Z \) is copula family-independent. We use the MATLAB® toolbox developed by Kopocinski (2007) to sample disruption vectors with the Clayton and Normal copulas.
2. Marginal distributions of capacities $K_i = g_i(Z_i)$: We assume the function form of $g_i$ are independent of $i$, i.e., $g_i = g$, $i = 1, 2, 3$. We take three forms of the increasing function $g$:

a. $K_i = g(Z_i) = 20Z_i \sim U(0, 20)$, $i = 1, 2, 3$. The mean capacity of each manufacturer is 10 and standard deviation is $\sqrt{33.33} = 5.77$;

b. $K_i = g(Z_i) = 8 + 4Z_i = U(8, 12)$, $i = 1, 2, 3$. The mean capacity of each manufacturer is 10 and standard deviation is $\sqrt{1.15} = 1.15$;

c. $K_i = 0$ if $Z_i \leq 0.1$ and $K_i = 11.11$ if $0.1 < Z_i \leq 1$, $i = 1, 2, 3$. This works out as $P(K_i = 0) = 0.1$ and $P(K_i = 11.11) = 0.9$. The mean capacity of each manufacturer is again 10 and standard deviation is $\sqrt{11.11} = 3.33$.

In each case we vary the standard deviation of the capacity while keeping its mean fixed. In Cases (a) and (b) the disruptions are continuous, whereas in Case (c) the disruptions are discrete, i.e., in Case (c) a manufacturer is completely unreliable with probability 0.1 and completely reliable with probability 0.9. We set the standard deviation of the capacity in Case (c) between the standard deviations in Cases (a) and (b).

3. Demand distribution: We consider low and high demand variability while keeping the mean demand the same, with $D \sim N(27, 4.5)$ and $D \sim N(27, 9)$.

4. Cost parameters: We consider the low and high lost sales costs, with $p = 2$ and $p = 3$, while fixing the holding cost $h = 1$.

The combination of the above parameter settings generates $10 \times 3 \times 2 \times 2 = 120$ problem scenarios. It is easy to show that in a symmetric system, the optimal order quantities are the same for all manufacturers, i.e., $q^*(r) = [q^*(r), q^*(r), q^*(r)]$.

For each scenario, we apply a local search method with multiple starting points to find the optimal order quantity $q^*(r)$ that minimizes the expected cost. The expected cost is estimated using a sampling method. For the remaining section, we denote $g(q, r)$, $FR(q, r)$ and $SL(q, r)$ as the retailer’s expected cost, fill rate and service level, given the rank correlation $r$ and ordering quantity $q$. In our discussion to follow, we may think of $r$ as the strategic-level decision variable and $q$ as the operational-level decision variable.

The results for the multi-source system are shown in Figures 1 and 2 and Tables 1 and 2. Figure 1 shows, for each copula family, the average optimal order quantity $q^*(r)$, the average minimum expected cost $g(q^*(r), r)$, the average fill rates $FR(q^*(r), r)$ and the average service level $SL(q^*(r), r)$ against rank correlation $r$, where each average value is taken over 12 scenarios specified in items (2), (3) and (4) in our experiment design. We first observe that Figure 1 supports our claims in Theorems 3.3 and 3.6, which state that increasing the dependence parameter $r$ decreases the optimal order quantity (for $m = 2$), increases the minimum expected cost, and lowers the fill rate (for $m = 2$). While Figures 1 (a) and 1 (c) only show that, on average, the optimal order quantity $q^*(r)$ and fill rate $FR(q^*(r), r)$ decrease as rank correlation $r$ increases for our $m = 3$ manufacturer system, we actually found that these properties hold true in each scenario, suggesting the robustness of Theorem 3.6 for the system with more than 2 manufacturers.
Figure 1: Multi-source systems: The average values of the optimal order quantity, cost, fill rate and service level against rank correlation of disruptions.

In addition, regardless of the copula family used, Figure 1 (a)-(c) provides the average upper and lower bounds for system performance, with $q^*(r) \in [9.48, 10.90]$, $g(q^*(r), r) \in [7.7, 10.0]$ and $FR(q^*(r), r) \in [0.87, 0.92]$.

To understand how the manufacturer selection decision at the strategic level may affect the retailer’s performance in the multi-source supply chain, we treat $g(q^*(0), 0)$ as the benchmark cost and $FR(q^*(0), 0)$ as the benchmark fill rate, which represent, respectively, the minimum expected cost and fill rate, given the optimal decisions are made at both the strategic level ($r = 0$) and operational level ($q = q^*(0)$). Thus, $g(q^*(r), r)$ and $FR(q^*(r), r)$ represent the retailer’s cost and fill rate, respectively, given the suboptimal decision $0 < r \leq 1$ is made at the strategic level but the optimal ordering quantity $q^*(r)$ corresponding to the chosen $r$ is made at the operational level. We compute the
following percentage performance gaps:

\[
\Delta g_2^s(r) = \frac{g(q^s(r), r) - g(q^s(0), 0)}{g(q^s(0), 0)} \times 100%,
\]

(4.4)

\[
\Delta FR_2^s(r) = \frac{FR(q^s(r), r) - FR(q^s(0), 0)}{FR(q^s(0), 0)} \times 100%,
\]

(4.5)

which are, respectively, the percentage cost increase and the percentage fill rate decrease due to the increase of rank correlation \( r \), given the optimal ordering policy is carried out at the operational level. When \( r = 1 \), the above expressions give the upper bounds of the percentage cost increase and fill rate decrease when the disruptions are comonotonic. According to Tables 1, the average upper bound on the percentage cost increase is \( \Delta g_2^s(1) = 30.6\% \).

The large magnitude of \( \Delta g_2^s(1) \) signifies the significant risk that the retailer faces if it does not consider the effect of disruptions at the strategic level, even if it takes this effect into account at the operational level. Even for a low rank correlation value \( r = 0.2 \), the percentage cost increase \( \Delta g_2^s(0.2) \) is 10.3\% for the disruptions of the Clayton type and is 8.5\% for the disruptions of the Normal type. The implication is that even “little” dependence in disruptions goes a long way to increase the retailer’s cost. This underscores that dependence of disruptions should be an important factor to consider in supplier selection decision in the multi-source supply chain. Tables 1 also shows that the percentage fill rate can go down by 5\% when \( r = 1 \). Overall, the dependence effect of disruptions on the fill rate appears moderate.

To study the impact of dependent disruptions on operational level decisions, in Table 1, we also report

\[
\Delta g_2^o(r) = \frac{g(q^o(0), r) - g(q^o(r), r)}{g(q^o(r), r)} \times 100%,
\]

(4.6)

\[
\Delta q^o(r) = \frac{q^o(0) - q^o(r)}{q^o(r)} \times 100%,
\]

(4.7)

\[
\Delta FR_2^o(r) = \frac{FR(q^o(0), r) - FR(q^o(r), r)}{FR(q^o(r), r)} \times 100%,
\]

(4.8)

where \( g(q^o(0), r) \) is the retailer’s cost when the rank correlation is \( r \) but the retailer orders the suboptimal quantity \( q^o(0) \). As such, \( \Delta g_2^o(r) \) captures the percentage cost increase, if the retailer fails to recognize the interdependencies among disruptions and orders the optimal quantity assuming independent disruptions. On the other hand, \( \Delta q^o(r) \) and \( \Delta FR_2^o(r) \) capture, respectively, the percentage order quantity increase over the optimal ordering quantity and the percentage fill rate increase if the retailer ignores the dependence in disruptions when determining the optimal order quantity. Table 1 shows that, on average, the percentage cost increase can be up to 14.3\% and the percentage order quantity increase can be up to 15\%, achieved at \( r = 1 \). Since \( q^o(0) \geq q^o(r) \), by ordering the suboptimal quantity \( q^o(0) \), the fill rates increase; however, the percentage increase \( \Delta FR_2^o(r) \) is insignificant for all values of \( r \). Table 1 suggests that the impact of dependence among disruptions at the operational level is insignificant when rank correlation is low to moderate (e.g., \( r \leq 0.4 \)), but becomes considerable when it is high (e.g., \( r \geq 0.6 \)). Overall, the cost of the suboptimal decision at the operational level is less severe than that at the strategic level.

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Finally, Table 1 reports the following performance gaps:

\[
\Delta g_{S+O}(r) = \frac{g(q^*(0), r) - g(q^*(0), 0)}{g(q^*(0), 0)} \times 100%,
\]

(4.9)

\[
\Delta FR_{S+O}(r) = \frac{FR(q^*(0), r) - FR(q^*(0), 0)}{FR(q^*(0), 0)} \times 100%.
\]

(4.10)

which measure the percentage cost increase and the percentage fill rate decrease due to the *compound effect* of making suboptimal decisions at both the strategic level \((r > 0)\) and operational level \((q = q^*(0))\). As seen, on average, the percentage cost increase reaches the upper bound 49.3\% and the percentage fill rate decrease reaches the upper bound 4.3\%, when \(r = 1\). Even for a low rank correlation value \(r = 0.2\), the performance gap of the cost is \(\Delta g_{S+O}(0.2) = 11.3\%\) for the disruptions of the Clayton type and is \(\Delta g_{S+O}(0.2) = 9.3\%\) for the disruptions of the Normal type. These results, again, testify our claim that even little correlation in disruptions can have a substantial impact on the retailer’s bottom line. In practice, firms often fail to take the dependence level of capacities into account when making supplier selection and inventory ordering decisions, and estimate system performance by assuming independent capacities. Our results demonstrate that, in doing so, it would considerably underestimate the cost and overestimate customer service even when the dependence level is weak; When this dependence level is strong, the retailer would be subject to a much high cost than anticipated, with a poorer customer service.

<table>
<thead>
<tr>
<th>Performance gap (%)</th>
<th>Copula</th>
<th>Clayton</th>
<th>Normal</th>
<th>Clayton</th>
<th>Normal</th>
<th>Clayton</th>
<th>Normal</th>
<th>---</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategic level</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Delta g_{S}(r))</td>
<td>10.3</td>
<td>8.5</td>
<td>18.8</td>
<td>15.4</td>
<td>23.5</td>
<td>20.6</td>
<td>27.5</td>
<td>24.8</td>
</tr>
<tr>
<td>(\Delta FR_{S}(r))</td>
<td>-1.6</td>
<td>-1.3</td>
<td>-3.0</td>
<td>-2.4</td>
<td>-3.9</td>
<td>-3.2</td>
<td>-4.6</td>
<td>-4.1</td>
</tr>
<tr>
<td>Operational level</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Delta g_{O}(r))</td>
<td>0.9</td>
<td>0.7</td>
<td>2.6</td>
<td>2.7</td>
<td>6.8</td>
<td>5.8</td>
<td>10.7</td>
<td>9.9</td>
</tr>
<tr>
<td>(\Delta FR_{O}(r))</td>
<td>0.6</td>
<td>0.5</td>
<td>1.2</td>
<td>0.8</td>
<td>1.1</td>
<td>0.9</td>
<td>1.1</td>
<td>0.9</td>
</tr>
<tr>
<td>(\Delta q^*(r))</td>
<td>4.7</td>
<td>4.1</td>
<td>8.9</td>
<td>7.1</td>
<td>12.0</td>
<td>9.8</td>
<td>13.9</td>
<td>12.6</td>
</tr>
<tr>
<td>Strategic &amp; operational levels</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(\Delta g_{S+O}(r))</td>
<td>11.3</td>
<td>9.3</td>
<td>21.8</td>
<td>18.5</td>
<td>31.8</td>
<td>27.6</td>
<td>41.2</td>
<td>37.1</td>
</tr>
<tr>
<td>(\Delta FR_{S+O}(r))</td>
<td>-1.0</td>
<td>-0.8</td>
<td>-1.9</td>
<td>-1.6</td>
<td>-2.8</td>
<td>-2.4</td>
<td>-3.6</td>
<td>-3.2</td>
</tr>
</tbody>
</table>

Table 1: Multi-source systems: Effect of dependent disruptions on system performance at strategic and/or operational levels

Figure 2 illustrates why the higher the dependence of disruptions, the lower the optimal order quantity. By ordering a larger, suboptimal quantity \(q^*(0)\), the retailer faces a significant downside risk of the increased overage cost, yet an insignificant upside benefit of the reduced underage cost. The tradeoff between the two costs results in a reduced optimal order quantity as the dependence level of disruptions increases.

Next, we investigate how the dependence strength of disruptions, measured by rank correlation \(r\), interacts with each of the four other factors described in our experiment design, i.e., the dependence nature of disruptions, the manufacturers’ capacity variability, customer demand variability, and the relative difference between the lost sales cost and the holding cost (with the holding cost fixed at \(h = 1\)). Figure 1 shows how the dependence nature of disruptions (Clayton vs. Normal) interacts with the dependence strength \((r)\) of disruptions. We observe that the
system with the disruptions of the Clayton type underperforms the system with the disruptions of the Normal type for each value of $r$, i.e., the former system is more costly and has a poorer service than the latter system. We thus conclude, in the multi-source system, the disruptions with strong tail dependence, often caused by catastrophic events, are more harmful than the disruptions with strong central dependence, often caused by the intermediate-level, common-cause disruptive events.

Let us now consider how the interplay between rank correlation $r$ and capacity variability (reflected by $g$) affects system performance. The first part of Table 2 shows the cost $g(q^*(r), r)$ and order quantity $q^*(r)$ for different values of $r$, as well as the percentage cost increase $\Delta g^*_S(r)$ over the minimum cost $g(q^*(0), 0)$ and the percentage order quantity increase over the optimal order quantity $q^*(r)$ with a fixed pair of $r$ and $g$, averaged over two copula families, two demand variability levels and two lost sales cost levels. We observe that a higher capacity variability increases each performance gap ($\Delta g^*_S(r)$ or $\Delta q^*(r)$) for a fixed $r$; similarly, a higher rank correlation increases each performance gap for a fixed $g$. We also observe that a moderate reduction of capacity variability (see the discrete case) can significantly reduce the retailer’s cost; in fact, rank correlation $r$ no longer has a significant impact on system performance when capacity variability is significantly low ($K_i \sim U(8, 12)$). This latter case may correspond to the situation that the retailer facing high capacity variability, $K_i \sim U(0, 20)$, uses a new supply strategy that secures 8 units of supply from a completely reliable source and orders the rest from an unreliable source with capacity $U(0, 4)$, resulting in the aggregate capacity $K_i \sim U(8, 12)$. This new strategy is effective because, by reducing capacity variability, the retailer can neutralize the dependence effect of disruptions. In contrast, the compound effect of high capacity variability ($K_i = U(0, 20)$) and high disruption correlation ($r = 0.8$) is detrimental to the retailer. As seen from Table 2, the average cost associated with the pair $r = 0.8$ and $K_i \sim U(8, 12)$ is 3.6, whereas the average cost associated with the pair $r = 0.8$ and $K_i \sim (0, 20)$ is 16.3, i.e., the latter cost is more than quadrupled of the former cost.
for each given $Z$

mances of the multi-source system and the assembly system, we use the same experiment design parameters whenever

4.3. Assembly Supply Chain

We again consider the 3-manufacturer, 1-retailer inventory system. To make fair comparisons between the performances of the multi-source system and the assembly system, we use the same experiment design parameters whenever possible. In particular, the assumptions for the distribution of $Z = [Z_1, Z_2, Z_3]$, the copula families of $Z$, the values

<table>
<thead>
<tr>
<th>Capacity variability $K_c$</th>
<th>Performance measure</th>
<th>Rank correlation $r$</th>
<th>Performance measure</th>
<th>Rank correlation $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_c$ = 5.77</td>
<td>$g(q'(r), r)$</td>
<td>11.7 13.5 14.8 15.7 16.5 17.0</td>
<td>$\Delta g_q(r)$</td>
<td>15.2% 26.2% 33.7% 39.3% 44.7%</td>
</tr>
<tr>
<td>$K_c$ = 3.33</td>
<td>$q'(r)$</td>
<td>13.1 11.8 10.9 10.2 9.8 9.5</td>
<td>$\Delta q'(r)$</td>
<td>11.1% 20.6% 29.4% 33.8% 38.1%</td>
</tr>
<tr>
<td>$K_c$ = 1.15</td>
<td>$g(q'(r), r)$</td>
<td>8 8.3 8.7 8.9 9.1 9.5</td>
<td>$\Delta g_q(r)$</td>
<td>3.8% 8.5% 10.8% 13.9% 18.0%</td>
</tr>
<tr>
<td>$K_c$ = 0.2</td>
<td>$q'(r)$</td>
<td>9.9 9.8 9.7 9.7 9.5 9.5</td>
<td>$\Delta q'(r)$</td>
<td>1% 2.1% 2.1% 4.2% 4.2%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Demand variability $\sigma_D$</th>
<th>Performance measure</th>
<th>Rank correlation $r$</th>
<th>Performance measure</th>
<th>Rank correlation $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_D = 4.5$</td>
<td>$g(q'(r), r)$</td>
<td>7.4 8.1 8.7 9 9.3 9.6</td>
<td>$\Delta g_q(r)$</td>
<td>9.7% 17.8% 22.1% 25.9% 30.2%</td>
</tr>
<tr>
<td>$\sigma_D = 9$</td>
<td>$q'(r)$</td>
<td>10.8 10.4 10.0 9.7 9.6 9.4</td>
<td>$\Delta q'(r)$</td>
<td>4.3% 8.5% 11.6% 13.1% 15.0%</td>
</tr>
</tbody>
</table>

| Lost sales cost $p$ = 2    | $g(q'(r), r)$       | 6.6 7.1 7.5 7.8 8.1 8.3  | $\Delta g_q(r)$      | 8.1% 13.9% 18.3% 22.2% 25.7% |
| $p$ = 3                    | $q'(r)$             | 10.5 10.1 9.9 9.7 9.5 9.3  | $\Delta q'(r)$       | 4% 7% 8.9% 11.0% 12.8% |

Table 2: Multi-source systems: Interactive effect between rank correlation of disruptions and other system characteristics

The second part of Table 2 shows the effect of the interaction between rank correlation $r$ and demand variability $\sigma_D$ on the retailer’s average cost and optimal order quantity. We observe that, while a higher $\sigma_D$ worsens system performance for a fixed $r$ and a higher $r$ worsens system performance for a fixed $\sigma_D$, the two factors seem to have limited interactions, evidenced by almost the same cost or order quantity gap between $\sigma_D = 4.5$ and $\sigma_D = 0.9$, for each given $r$. This means that, while both disruption correlation and demand variability adversely affect system performance, the former is more harmful than the latter to the retailer’s bottom line.

Finally, the last part of Table 2 shows the effect of the interplay between rank correlation $r$ and the lost sales cost $p$ on system performance, with holding cost fixed at $h = 1$. The results indicate that both the high lost sales cost and high disruption correlation can significantly affect system performance. In particular, for a high lost sales cost, the retailer may order significantly more items than necessary if it does not consider the dependence effect of disruptions.

4.3. Assembly Supply Chain

We again consider the 3-manufacturer, 1-retailer inventory system. To make fair comparisons between the performances of the multi-source system and the assembly system, we use the same experiment design parameters whenever possible. In particular, the assumptions for the distribution of $Z = [Z_1, Z_2, Z_3]$, the copula families of $Z$, the values
Figure 3: Assembly systems: The average values of the optimal order quantity, cost, fill rate and service level against rank correlation of disruptions of rank correlation \( r \) and the distributions of capacity \( K = [K_1, K_2, K_3] \) in the assembly system are the same as their counterparts in the multi-source system. We set the lost sales cost at \( p = 6 \) and \( p = 9 \) and fix the holding cost at \( h = 1 \). Here, we triple the lost sales cost compared these in the multi-source system, since each assembled final product consists three individual parts. We also change the demand distribution to two settings, \( D \sim N(9, 1.5) \) and \( D \sim N(9, 3) \), which will keep the total mean demand for different part types the same as that in the multi-source system. The above settings again generate 120 scenarios. Figure 3 plots, on average, the order quantity \( q^*(r) \), the minimum cost \( g(q^*(r), r) \), the fill rate \( FR(q^*(r), r) \) and service level \( SL(q^*(r), r) \), against the rank correlation \( r \). The results shown in Figure 3 validate Theorem 3.9 and Theorem 3.11, which state that a larger \( r \) increases the optimal order quantity and improves each of the three performance measures in the assemble system. These properties indeed hold in each individual scenario we tested. We observe that the effect of dependence of disruptions on each performance
measure is more drastic in the assembly system than in the multi-source system. On average, \( q'(r) \in [8.82, 10.53] \), \( g(q'(r), r) \in [14.5, 27.6] \), \( FR(q'(r), r) \in [0.67, 0.84] \), and \( SL(q'(r), r) \in [0.33, 0.56] \), for \( 0 \leq r \leq 1 \). The wide ranges of these intervals underscore the significant role played by rank correlation \( r \), that is, the retailer could significantly undermine its performance were it fail to select the appropriate manufacturers with balanced production capacities. Also, Figure 3 shows that, on average, the system with the Clayton copula outperforms the system with the Normal copula; indeed, we observe that this property holds true for each scenario. A plausible explanation is that, with the Clayton copula, severe disruptions of manufacturing capacities tend to occur simultaneously; consequently, the retailer will not pay a high inventory cost as the result of unbalanced deliveries of different part types.

Similar to the multi-source supply chain, we define several performance gaps to capture the effect of dependent disruptions on system performance in the assembly supply chain at the strategic and operational levels. First, we can define \( \Delta g_3^*(r) \) and \( \Delta FR_3^*(r) \), similar to their definitions in Eqs. (4.4) and (4.5), but using \( g(q'(1), 1) \) as the benchmark cost and \( FR(q'(1), 1) \) as the benchmark fill rate. We also define

\[
\Delta SL_3^*(r) = \frac{SL(q'(r), r) - SL(q'(1), 1)}{SL(q'(1), 1)} \times 100%.
\]

These measures are reported in the first part of Table 3. The results show that, when \( r \) decreases from 1 to 0, the cost increases by 89.8%, the fill rate decreases by 19.9% and the service level decreases by 39.5%. In fact, even little discordance in production capacities (e.g., \( r = 0.8 \)) goes a long way to increase the retailer’s cost and worsen its service performance. By comparing the first parts of Tables 1 and 3, we see that \( \Delta g_3^*(r) \) in the assembly system is much higher than that in the multi-source system, for each fixed \( r \) and copula family. In other words, the effect of disruption dependence is even more drastic in the assembly structure than in the multi-source structure.

To capture the effect of disruption dependence on the operational level decisions, we use the same definitions of \( \Delta g_0^*(r) \), \( \Delta q'(r) \) and \( \Delta FR_0^*(r) \), given in Eqs. (4.6)-(4.8). We also define

\[
\Delta SL_0^*(r) = \frac{SL(q'(0), r) - SL(q'(r), r)}{SL(q'(r), r)} \times 100%.
\] (4.11)

The interpretation for each performance gap is similar to that in the multi-source system. For example, \( \Delta g_0^*(r) \) is the retailer’s percentage cost increase, when the rank correlation of disruptions is \( r \) but the retailer orders the suboptimal quantity \( q'(0) \) by assuming independent disruptions. The second part of Table 3 shows that, using the suboptimal policy \( q'(0) \), the percentage cost increase reaches the upper bound of 9.5%, while the percentage order quantity decrease reaches an upper bound of \( \Delta q'(1) = 16.2% \), both occurred when \( r = 1 \). Overall, we observe that the retailer’s cost is robust against the suboptimal order quantity \( q'(0) \). By comparing the values of \( \Delta g_0^*(r) \) in Table 3 with those in Table 1 we see that the operating cost increase by using the suboptimal policy \( q'(0) \) are less than those in the multi-source system, which implies that ignoring stochastic dependence of disruptions at the operational level is less costly in the assembly system. Table 3 also shows that, by placing suboptimal order quantity \( q'(r) \), the percentage of fill rate and service level decrease will reach the maximum values of 5.4% and 24.9%, respectively, for \( r = 1 \). Figure 4 shows why the cost function is robust to changing the
order quantity. In this figure, the dashed lines plot the lost sales and inventory holding costs associated with the optimal order quantity \( q^*(r) \), while the solid lines plot those costs for the suboptimal order quantity \( q^*(0) \). By ordering the suboptimal quantity, the retailer faces a higher lost sales cost and a lower inventory holding cost. Adding up the inventory holding and lost sales costs, the total cost associated with \( q^*(r) \) is always less than the cost associated with \( q^*(0) \); however, the gap between the two costs is often small. Hence, the cost is not very sensitive to the placement of the suboptimal ordering quantity \( q^*(0) \).

To measure the cost increase and the fill rate and service level decreases due to the compound effect of making suboptimal decisions at both the strategic level \((r < 1)\) and the operational level \((q = q^*(0))\), let \( \Delta g^*_S + \Delta g^*_O(r) \) be defined similarly as in Eqs. (4.9) and (4.10), using \( g(q^*(1), 1) \) as the benchmark cost and \( FR(q^*(1), 1) \) as the benchmark fill rate. In addition, let

\[
\Delta SL^*_S + \Delta SL^*_O(r) = \frac{SL(q^*(0), r) - SL(q^*(1), 1)}{SL(q^*(1), 1)} \times 100\%
\]  

be the percentage service level decrease if suboptimal decisions are made at the strategic level \((r < 1)\) and at the operational level \((q = q^*(0))\). The last part of Table 3 reports the results. Comparing the numbers with those in Table 1, we note that the percentage decrease in cost and the percentage increase in fill-rate by making the right decision at the strategic and operational levels are significantly more than those values in the multi-source system. That is, considering the dependence strength of disruptions in the assembly system is more important than that in the multi-source system.

<table>
<thead>
<tr>
<th>Performance gap (%)</th>
<th>Copula</th>
<th>——</th>
<th>Clayton</th>
<th>Normal</th>
<th>Clayton</th>
<th>Normal</th>
<th>Clayton</th>
<th>Normal</th>
<th>——</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategic level</td>
<td>( \Delta g^*_S (r) )</td>
<td>0</td>
<td>20</td>
<td>37</td>
<td>36.3</td>
<td>53.6</td>
<td>53.5</td>
<td>67.3</td>
<td>71.8</td>
</tr>
<tr>
<td></td>
<td>( \Delta FR^*_S (r) )</td>
<td>0</td>
<td>-4.8</td>
<td>-8.0</td>
<td>-8.4</td>
<td>-11.8</td>
<td>-12.1</td>
<td>-14.8</td>
<td>-16.1</td>
</tr>
<tr>
<td></td>
<td>( \Delta SL^*_S (r) )</td>
<td>0</td>
<td>-12.3</td>
<td>-15.9</td>
<td>-20.2</td>
<td>-24.9</td>
<td>-27.3</td>
<td>-30.6</td>
<td>-34.2</td>
</tr>
<tr>
<td>Operational level</td>
<td>( \Delta g^*_O (r) )</td>
<td>9.5</td>
<td>4.7</td>
<td>3.2</td>
<td>2.4</td>
<td>1.6</td>
<td>1</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>( \Delta FR^*_O (r) )</td>
<td>-5.4</td>
<td>-3.6</td>
<td>-3.5</td>
<td>-2.6</td>
<td>-2.3</td>
<td>-1.8</td>
<td>-1.5</td>
<td>-0.8</td>
</tr>
<tr>
<td></td>
<td>( \Delta SL^*_O (r) )</td>
<td>-24.9</td>
<td>-17.8</td>
<td>-17.6</td>
<td>-12.9</td>
<td>-11.2</td>
<td>-8.3</td>
<td>-7.2</td>
<td>-3.3</td>
</tr>
<tr>
<td></td>
<td>( \Delta q^*(r) )</td>
<td>-16.2</td>
<td>-12.3</td>
<td>-10.3</td>
<td>-8.3</td>
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<td>-5.5</td>
<td>-4.7</td>
<td>-2.6</td>
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<tr>
<td>Strategic &amp;</td>
<td>( \Delta g^<em>_S + \Delta g^</em>_O (r) )</td>
<td>9.5</td>
<td>25.6</td>
<td>41.4</td>
<td>39.5</td>
<td>56</td>
<td>55</td>
<td>68.4</td>
<td>72.2</td>
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<tr>
<td>operational levels</td>
<td>( \Delta FR^<em>_S + \Delta FR^</em>_O (r) )</td>
<td>-5.4</td>
<td>-8.3</td>
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<td>-10.8</td>
<td>-13.8</td>
<td>-13.6</td>
<td>-16.1</td>
<td>-16.7</td>
</tr>
<tr>
<td></td>
<td>( \Delta SL^<em>_S + \Delta SL^</em>_O (r) )</td>
<td>-24.9</td>
<td>-27.9</td>
<td>-30.7</td>
<td>-30.5</td>
<td>-33.3</td>
<td>-33.3</td>
<td>-35.6</td>
<td>-36.4</td>
</tr>
</tbody>
</table>

Table 3: Assembly systems: Effect of dependence of disruptions on system performance at strategic and/or operational levels

Table 4 studies the interplay between the rank correlation and capacity variability, demand variability or the lost sales cost. The first part of Table 4 shows the interaction between rank correlation and capacity variability. Similar to the multi-source system, the rank correlation has an insignificant impact on cost when \( K_i \sim U(8, 12), i = 1, 2, 3 \). That is, when each manufacturer guarantees the delivery of 8 units of its part type, the interdependence of the capacities
becomes a less important factor. Therefore, as in the multi-source supply chain, reducing capacity variability is an effective tactic to counter the adverse impact of dependent disruptions in the assembly supply chain. On the other hand, when the capacities are highly variable (\(K_i \sim U(0, 20)\)), decreasing \(r\) from one to zero increases the cost by 107.8\%, i.e., decreasing the rank correlation has a severe negative effect on the retailer’s cost when capacity variability is high. The implication is that, in an assembly supply chain, if capacity variability cannot be reduced, it is essential to increase the interdependencies among capacities. In addition, Table 4 shows that increasing \(r\) from 0 to 1 decreases the optimal order quantity by 25.1\% in the system with highly variable capacities (\(K_i \sim U(0, 20)\)), while it only decreases the optimal order quantity by 12.1\% in the system with low variable capacities (\(K_i \sim U(8, 12)\)). We also observe the highly adverse effect of independent discrete disruptions on the cost. For \(r = 0\), this percentage cost increase reaches 110\%, which is even higher than that in the continuous disruption case \(K_i \sim U(0, 20)\) with a higher capacity variability. A plausible explanation for this phenomenon is that, in the case of discrete disruptions, when some suppliers are completely disrupted and others fully operational, the part delivery is either 0 or \(q^*(0)\), resulting in the high inventory cost as well as the high lost sales cost.

The interaction between rank correlation \(r\) and demand variability \(\sigma_D\) on the retailer’s average cost is shown in the second part of Table 4. Similar to the multi-source system, we observe that rank correlation and demand variability seem to have limited interactions on the cost. This is evident by almost the same cost and optimal order quantity gap between the two demand distributions. However, the interaction of the two factors has a significant impact on the optimal order quantity, that is, when \(r\) decreases from one to zero, for \(D = 4.5\), \(q^*(r)\) drops from 9.8 to 8.8, and for \(D = 9\), \(q^*(r)\) drops from 11.3 to 8.8.

The last part of Table 4 shows how the cost and the order quantity are affected by the interplay between the rank correlation and the lost sales cost. Comparing the last part of Tables 2 and 4, we again find that the effect of
dependence in disruptions is more drastic in the assembly structure than in the multi-source structure.

5. Concluding Remarks

We consider an m-manufacturer, 1-retailer newsvendor system subject to dependent disruptions, with either the multi-source or assembly structure. We show that stochastic dependence in disruptions has opposite effects on system performance in the two structures: while a higher level of dependence amplifies the disruption risk in the multi-source supply chain, it alleviates this risk in the assembly supply chain. In addition, as disruptions become more dependent, the retailer should order less in the multi-source supply chain but order more in the assembly supply chain. We also perform a comprehensive numerical study and draw managerial and operational implications on how dependent disruptions affect the retailer’s strategic and operational decisions.

For exposition simplicity, in this paper we have made some simplifying assumptions. However, our results remain valid under the following extensions.

1. We have assumed in Theorem 3.6 that $K_i$ is a continuous random variable for each $i$. In fact, the proof can be
extended to the case where \( K_i \) is a mixture of a discrete and a continuous random variables. For example, \( K_i = B_i K'_i \), for all \( i \), where \( B_i \) is a binary random variable and \( K'_i \) is a continuous random variable. This allows us to model the scenario that the manufacturer is either completely disrupted (\( B_i = 0 \)), or not disrupted but its capacity is a random yield of the full capacity (\( B_i = 1 \) and \( K'_i \) takes a positive realization).

2. We can extend Theorems 3.3 and 3.9 to the system with \( m \) manufacturers and \( n \) retailers. We assume that each retailer faces a random demand. We define the total cost as the sum of individual retailers’ costs, the total fill rate as the expected total satisfied demand divided by the total mean demand over different retailers, and the joint service level as the joint probability of meeting demands at all retailers. Using the same methodology presented in Section 3, we can show that increasing the supermodular dependence strength of the disruption vector increases the total costs and decreases the total fill rate in the multi-source system. In contrast, increasing the upper orthant dependence strength of disruption vector decreases the total cost and increases the total fill rate and the joint service level in the assembly system.

3. We have assumed that the unit purchase cost \( b_i = 0 \) for each \( i \). We can relax this assumption by including the expected purchase cost \( E[\sum_{i=1}^{m} b_i I_i(K_i, q_i)] \) in the cost function \( g \). All of our results in Section 3 still hold for \( b_i \geq 0, i = 1, \ldots, m \).

We conclude this paper by identifying several potential directions for future research. First, in this paper we consider simple expectation-based performance measures. We can study some conditional expectation-based performance measures when the system near capacity boundaries under distress. For example, we can consider expected shortfall, defined as the expected cost that exceeds a threshold (Gilli and Köllezi, 2006) or expected cost, fill rate and service level given manufacturers’ capacities are below some thresholds. These conditional performance measures under extreme distress can be estimated explicitly using the extreme value theory. We would like to investigate the usefulness of these conditional measures in the field of supply chains risk management. Second, in this paper we assume that the retailer has no advanced supply information when placing orders. In some cases, the forecast of possible disruptions is available and the retailer can take proactive actions in its inventory planning. For example, both Wal-marts and Home Depot leaned from past hurricane disasters and overstocked the needed items before Hurricane Katrina struck in August 2005, and their supply chains were able to recover quickly from the disruption of Hurricane Katrina. It will be interesting to incorporate advanced supply information in our models. Third, we would like to apply our methods to the inventory system in a discrete-time dynamic setting in order to capture the temporal effect of dependent disruptions on system performance and ordering policies. Finally, in Section 4, we discussed the intuition behind using tail dependent copulas to model disruptions that are due to extreme events; however, we are not aware of any analysis that has been performed on supply chain data to determine the tail dependence of disruptions. Such analysis has been performed on insurance data and has yielded this stylized fact that extreme losses are more tail-dependent (Kousky and Cooke, 2009). By conducting a similar analysis on supply chain data we can empirically justify using tail dependent copulas to model extremal dependence of disruptions.
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Appendix A. Proof of Theorem 3.3

First note that, by Lemma 3.2 (1), $Z_\prec \preceq_{st} Z_+$ implies $K_\prec \preceq_{st} K_+$, since $K_i = g_i(Z_i)$ and $g_i$ is an increasing function for each $i$.

Let us consider function $\phi(z) = f\left(\sum_{i=1}^{m} z_i\right)$. Corbett and Rajaram (2006) show that $\phi(z)$ is supermodular in $z$ for every convex function $f : \mathcal{R} \to \mathcal{R}$. Müller and Scarsini (2000) show that if $\phi(f_1(\cdot), \ldots, f_m(\cdot))$ is also supermodular for increasing functions $f_1, \ldots, f_m : \mathcal{R} \to \mathcal{R}$. Therefore, $\phi(f(z_1), \ldots, f(z_m)) = f(\sum_{i=1}^{m} f_i(z_i))$ is supermodular in $z_i$ for every convex function $f$. Let $k = (k_1, \ldots, k_m)$. Because $[x - d]^+$ and $[d - x]^+$ are convex in $x$, $S_i(k_i, q_i)$ is increasing in $k_i$ for each $i$ and a positive linear combination of supermodular functions is still supermodular (Proposition 2.3.5 in (Simchi-Levi et al., 2005)), function $g(q)D = d, K = k = p[d - \sum_{i=1}^{m} S_i(k_i, q_i)] + h[\sum_{i=1}^{m} S_i(k_i, q_i) - d]^+$ is supermodular in $k$ for any realized demand $D = d$. Hence $K \prec \preceq_{st} K_+$ implies

$$g_-(q)D = d = E \left[ p[d - \sum_{i=1}^{m} S_i(K_i, -q_i)] + h[\sum_{i=1}^{m} S_i(K_i, -q_i) - d]^+ \right]$$

$$\leq E \left[ p[d - \sum_{i=1}^{m} S_i(K_i, +q_i)] + h[\sum_{i=1}^{m} S_i(K_i, +q_i) - d]^+ \right]$$

Unconditioning on $D$ yields $g_-(q) \leq g_+(q)$. This result further implies $g_+(q^*_+) \geq g_-(q^*_-) \geq g_-(q^*_-)$, for $q^*_+ \in A^+_\infty$ and $q^*_- \in A^-\infty$. In words, it states that the minimum expected cost of the system, under the optimal ordering policy, increases as its disruption vector becomes more supermodular dependent. Finally,

$$FR_- (q)D = d = 1 - \frac{E(d - \sum_{i=1}^{m} S_i(K_i, -q_i))^+}{E(D)}$$

$$\geq 1 - \frac{E(d - \sum_{i=1}^{m} S_i(K_i, +q_i))^+}{E(D)} = FR_+(q)D = d.$$  

The result then follows after unconditioning on $D = d$.

Appendix B. Proof of Theorem 3.4

Let us define $\mathcal{U}(d, q) = \left\{ z \in \mathcal{R}^m : \sum_{i=1}^{m} S_i(g_i(z_i), q_i) \geq d \right\}$ as the set of realized disruption vectors for which the aggregate delivery quantity is greater than the realized demand $D = d$. For each $i$, $S_i(g_i(z_i), q_i)$ is increasing in $z_i$, since, by our assumption, $g_i$ is an increasing function of $z_i$ and $S_i$ is an increasing function of each of its arguments.
Therefore, \( z \in \mathcal{U}(d, q) \) and \( z' \geq z \) imply \( z' \in \mathcal{U}(d, q) \), that is, \( \mathcal{U}(d, q) \) is an upper set. Therefore, if \( Z_- \prec_u Z_+ \), then

\[
SL_-(q|D = d) = P \left( \sum_{i=1}^{n} S_i(g_i(Z_{-i}), q_i) \geq d \right)
\]

\[
= P(Z_- \in \mathcal{U}(d, q))
\]

\[
\leq P(Z_+ \in \mathcal{U}(d, q))
\]

\[
= P \left( \sum_{i=1}^{n} S_i(g_i(Z_{+i}), q_i) \geq d \right)
\]

\[
= SL_+(q|D = d).
\]

Unconditioning on \( D \), we obtain the desired result.

**Appendix C. Proof of Theorem 3.6**

1. We first recall the Leibniz rule, which states that, for differentiable functions \( f(x, y), a(y), \) and \( b(y) \),

\[
\frac{\partial}{\partial y} \int_{a(y)}^{b(y)} f(x, y)dx = \int_{a(y)}^{b(y)} \frac{\partial f(x, y)}{\partial y}dx + f(b(y), y) \frac{\partial b(y)}{\partial y} - f(a(y), y) \frac{\partial a(y)}{\partial y}.
\]

(C.1)

Using the Leibniz rule, for demand \( D \) with CDF \( F_D \), where \( F_D \) is continuous and differentiable, we have:

\[
\frac{dE_D[y - D]^{+}}{dy} = \frac{d}{dy} \int_{0}^{y} (y - x) f_{D}(x)dx
\]

\[
= yf_{D}(y) + \int_{0}^{y} f_{D}(x)dx - \frac{d}{dy} \int_{0}^{y} x f_{D}(x)dx
\]

\[
= \int_{0}^{y} f_{D}(x)dx = F_{D}(y)
\]

(C.2)

Using the above preliminary results, next we prove the theorem. Let \( [K_{i}(\theta), K_{z}(\theta)] \) denote the capacity vector associated with \([Z_{i}(\theta), Z_{z}(\theta)]\) for \( i = 1, 2 \). When \( S_i(K_{i}(\theta), q_i) = \min[K_{i}(\theta), q_i] \), the cost of the retailer is expressed as

\[
g(q_1, q_2, \theta) = pE \left[ D - \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] \right]^{+} + hE \left[ \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] - D \right]^{+},
\]

\[
= pE \left[ D - \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] \right]^{+} - \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] - D + (p + h)E \left[ \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] - D \right]^{+}
\]

\[
= pE \left[ D - \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] \right] + (p + h)E \left[ \sum_{i=1}^{2} \min[K_{i}(\theta), q_i] - D \right]^{+}.
\]

(C.3)

In our problem context, \( A(\theta) = \mathbb{R}^{2+} \), for any \( \theta \in \Theta \), and therefore is descending on \( \Theta \) and \( A^{*}(\theta) \) is non-empty. Therefore, by Lemma 3.5, it suffices to prove \( g(q_1, q_2, \theta) \) is supermodular in \((q_1, q_2, \theta)\). In the proof of Theorem 3.3, we have shown that that \( f \left( \sum_{i=1}^{n} f_i(z_i) \right) \) is supermodular in \( z \), for every convex function \( f \) and increasing function \( f_i \), for all \( i \). Let \( k = (k_1, k_2) \). Because \([x - d]^{+} + [d - x]^{+}\) is convex in \( x \) and \( S_i(k_i, q_i) \) is increasing in \( q_i \) for \( i = 1, 2 \), function \( g(q|D = d, K = k) = p[d - \sum_{i=1}^{2} S_i(k_i, q_i)]^{+} + h[\sum_{i=1}^{2} S_i(k_i, q_i) - d]^{+} \) is supermodular.
in \( q_1 \) and \( q_2 \) for any realized demand \( D = d \) and realized capacity vector \( K = k \). Since a linear combination of supermodular functions is still supermodular (Simchi-Levi et al., 2005, Proposition 2.3.5), \( g(q| D = d, K = k) \) is a supermodular function of \( q_1 \) and \( q_2 \).

Next, we need to show that \( g(q_1, q_2, \theta) \) is a supermodular function of \( (q_i, \theta) \), or equivalently, \( \frac{\partial g(q_1, q_2, \theta)}{\partial q_i} \) is increasing in \( \theta \) for \( i = 1, 2 \). Without loss of generality, we prove the result for \( i = 1 \). First consider the partial derivative of the first term of Eq.(C.3) with respect to \( q_1 \). We have (we ignore the positive coefficient)

\[
\frac{\partial E}{\partial q_1} \left[ D - \sum_{i=1}^{2} \min[K_i(\theta), q_i] \right] = -\frac{\partial E}{\partial q_1} [\min[K_1(\theta), q_1]],
\]

which is independent of \( \theta \) since the marginals of \( K_1(\theta) \) does not depend on \( \theta \). Conditioning on \( K_1(\theta) = k_1 \), we can write the second term of Eq.(C.3) as (we ignore the positive coefficient)

\[
E \left[ \sum_{i=1}^{2} \min[K_i(\theta), q_i] - D \right]^+ = \int_{0}^{\infty} E \left[ (k_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = k_1 \right] dF_{K_1(\theta)}(k_1)
\]

\[
+ \int_{q_1}^{\infty} E \left[ (q_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = k_1 \right] dF_{K_1(\theta)}(k_1).
\]

Using Leibniz rule stated in Eq.(C.1), we obtain the partial derivative of the above expression with respect to \( q_1 \) as

\[
\frac{\partial E}{\partial q_1} \left[ \sum_{i=1}^{2} \min[K_i(\theta), q_i] - D \right]^+ = \int_{0}^{\infty} \frac{\partial}{\partial q_1} E \left[ (k_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = k_1 \right] f_{K_1(\theta)}(k_1) dk_1
\]

\[
+ E \left[ (q_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = q_1 \right] f_{K_1(\theta)}(q_1)
\]

\[
+ \int_{q_1}^{\infty} \frac{\partial}{\partial q_1} E \left[ (q_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = k_1 \right] f_{K_1(\theta)}(k_1) dk_1
\]

\[
- E \left[ (q_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = q_1 \right] f_{K_1(\theta)}(q_1)
\]

The first expression vanishes because the term \( E \left[ (k_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = k_1 \right] f_{K_1(\theta)}(k_1) \) is independent of \( q_1 \). The second and fourth terms are cancelled out with each other. Therefore, Therefore,

\[
\frac{\partial E}{\partial q_1} \left[ \sum_{i=1}^{2} \min[K_i(\theta), q_i] - D \right]^+ = \int_{q_1}^{\infty} \frac{\partial}{\partial q_1} E \left[ (q_1 + \min[K_2(\theta), q_2] - D)^+ | K_1(\theta) = k_1 \right] f_{K_1(\theta)}(k_1) dk_1
\]

\[
= \int_{q_1}^{\infty} P \left( q_1 + \min[K_2(\theta), q_2] \geq D \left| K_1(\theta) = k_1 \right. \right] f_{K_1(\theta)}(k_1) dk_1
\]

\[
= P \left( \min[K_2(\theta), q_2] \geq D - q_1 \left| K_1(\theta) \geq q_1 \right. \right) P(K_1(\theta) \geq q_1)
\]

\[
= P \left( \min[K_2(\theta), q_2] \geq D - q_1, K_1(\theta) \geq q_1 \right),
\]

where the second equality follows from Eq.(C.2). Now, because \([K_1(\theta_1), K_2(\theta_1)] \prec_{sm} [K_1(\theta_2), K_2(\theta_2)]\), for \( \theta_1 \leq \theta_2 \), and \( \min[x, y] \) is an increasing function of \( y \); by Lemma 3.2.a, we must have \([K_1(\theta_1), \min[q_2, K_2(\theta_2)]] \prec_{sm} [K_1(\theta_2), \min[q_2, K_2(\theta_2)]]\). Since, by Lemma 3.2.b, the supermodular order between two random vectors implies the upper orthant order between the two random vectors, we obtain \([K_1(\theta_1), \min[q_2, K_2(\theta_2)]] \prec_{sm} [K_1(\theta_2), \min[q_2, K_2(\theta_2)]]\). This further implies that Eq.(C.5) is an increasing function of \( \theta \). We thus have shown
that the derivatives of both terms of Eq. (C.3) with respect to \( q_1 \), given by Eqs. (C.4) and (C.5), are increasing in \( \theta \). This conclude our proof that \( g(q_1, q_2, \theta) \) is a supermodular function of \((q_1, q_2, \theta)\).

2. Let \( q^*_+ \in A^+(\theta) \) and \( q^*_- \in A^-(\theta) \) be the optimal order quantity vectors in the multi-source systems with the disruption vectors \( Z(\theta_-) \) and \( Z(\theta_+) \), respectively. Since \( A^+(\theta) \) is descending, \( q^*_+ \) = \( q^*_+ \& q^*_- \in A^+(\theta) \). Since \( q^*_- \leq q^* \), By Theorem 3.3, \( FR_+(q^*_-) \leq FR_+(q^*_+) \leq FR_-(q^*_+) \).

Appendix D. Proof of Theorem 3.9

a. The assumptions of Theorem 3.9 imply \( K_- <_{st} K_+ \) and \( K_{i,-} =_{st} K_{i,+} \) for all \( i \). This implies \( E[S_i(K_{i,-}, q_i)] = E[S_i(K_{i,+}, q_i)] \). Hence

\[
E \left[ \sum_{i=1}^{m} h_i S_i(K_{i,-}, q_i) \right] = E \left[ \sum_{i=1}^{m} h_i S_i(K_{i,+}, q_i) \right].
\]

Next consider the remaining terms of Eq. (3.5). For every realized demand \( D = d \), \( p[d-I(K, q)]^+ - \sum_{i=1}^{m} h_i \min[I(K, q), d] \) is a decreasing function of \( I(K, q) \). In addition, by Lemma 3.8, \( Z_- <_{st} Z_+ \) implies \( I(K_-, q) <_{st} I(K_+, q) \). Therefore, \( Z_- <_{st} Z_+ \) implies

\[
E \left[ p[d-I(K_+, q)]^+ - \sum_{i=1}^{m} h_i \min[I(K_+, q), d] \right] \leq E \left[ p[d-I(K_-, q)]^+ - \sum_{i=1}^{m} h_i \min[I(K_-, q), d] \right].
\]

Unconditioning on \( D \) results in

\[
E \left[ p[D-I(K_+, q)]^+ - \sum_{i=1}^{m} h_i \min[I(K_+, q), D] \right] \leq E \left[ p[D-I(K_-, q)]^+ - \sum_{i=1}^{m} h_i \min[I(K_-, q), D] \right].
\]

Combining Eqs. (D.1) and (D.2) we obtain \( g_-(q) \geq g_+(q) \). This further implies \( g_-(q^*_-) \geq g_+(q^*_+) \geq g_+(q^*_+) \).

b. Given \( D = d \), the conditional fill rate, \( FR(q|D = d) = 1 - \frac{|d-I(K, q)|^+}{E[D]} \), is increasing in \( I(K, q) \). By Lemma 3.8, \( I(K_-, q) <_{st} I(K_+, q) \), hence \( FR_+(q|D = d) \leq FR_+(q|D = d) \). By unconditioning on \( D \) we have \( FR_+(q) \leq FR_+(q) \). Finally, \( I(K_-) <_{st} I(K_+) \) implies \( P(I(K_-, q) \geq d) \leq P(I(K_+, q) \geq d) \), for a realized demand \( D = d \). Unconditioning on \( D \) results in \( SL_-(q) \leq SL_+(q) \).

Appendix E. Proof of Lemma 3.10

Let \( q = [q_1, \ldots, q_m] \) be any order quantity vector and let \( q_{(1)} = [q_{(1)}, \ldots, q_{(1)}] \), where \( q_{(1)} = \min[q_1, \ldots, q_m] \) is the minimum of the order quantities. We shall show \( g(q) \geq g(q_{(1)}) \), that is, the retailer can reduce its expected cost by ordering \( q_{(1)} = [q_{(1)}, \ldots, q_{(1)}] \) rather than \( q = [q_1, \ldots, q_m] \). Assuming \( S_i(K_i, q_i) = \min[K_i, q_i] \), it is easily seen that \( I(K, q) = I(K, q_{(1)}) \). Therefore, from Eq. (3.5),

\[
g(q) - g(q_{(1)}) = E \left[ \sum_{i=1}^{m} h_i \min[q_i, K_i] \right] - E \left[ \sum_{i=1}^{m} h_i \min[q_{(1)}, K_i] \right] \geq 0, \tag{E.1}
\]
Appendix F. Proof of Theorem 3.11

a. In the assembly system, $A(\theta) = \mathcal{R}^+$, for any $\theta \in \Theta$, and the optimal action space $A^*(\cdot)$ is non-empty. By Lemma 3.5, $A^*(\cdot)$ is ascending on $\Theta$ if $g(q, \theta)$ is submodular in $(q, \theta)$. This is equivalent to showing $g(q, \theta_–) - g(q, \theta_+)$ is increasing in $q$. We show this by proving that $g(q, \theta_-|D = d) - g(q, \theta_+|D = d)$ is increasing in $q$ by conditioning on $D = d$. From Eq. (3.5), for $q_i = q, i = 1, \ldots, m$, we can write

$$g(q, \theta|D = d) = E \left[ \sum_{i=1}^m h_i \min[q, K_i(\theta)] + p[d - \min[q, K_i(\theta)]]^+ - \sum_{i=1}^m h_i \min[q, K_i(\theta), d] \right].$$

where $K_i(\theta) = g_i(Z_i(\theta))$ and $K_{i1}(\theta) = \min(K_i(\theta), \ldots, K_m(\theta))$. The first term is independent of $\theta$ since the marginal of $K_i(\theta)$ is independent of $\theta$. Therefore, we can write

$$g(q, \theta_-|D = d) - g(q, \theta_+|D = d) = pL(q, d) + hM(q, d),$$

where

$$L(q, d) = E \left[ d - \min[q, K_{i1}(\theta_-)]^+ - E \left[ d - \min[q, K_{i1}(\theta_+)]^+ \right] \right]$$

$$M(q, d) = -E \left[ \min[q, K_{i1}(\theta_-), d] \right] + E \left[ \min[q, K_{i1}(\theta_+), d] \right].$$

We show that $L(q, d)$ and $M(q, d)$ are both increasing in $q$. To do so, we first use the expression $E(X) = \int_0^\infty P(X \geq x)dx$, where $X$ is a non-negative random variable, to express $L(q, d)$ as

$$L(q, d) = \int_0^\infty \left[ P \left( \left[ d - \min[q, K_{i1}(\theta_-)]^+ \right] \geq x \right) - P \left( \left[ d - \min[q, K_{i1}(\theta_+), d] \right] > x \right) \right] dx$$

$$= \int_0^d \left[ P \left( \min[q, K_{i1}(\theta_-)] > d - x \right) - P \left( \min[q, K_{i1}(\theta_+)] > d - x \right) \right] dx.$$  \hfill (F.3)

Differentiating two cases, $q > d - x$ and $q \leq d - x$, we can write the integrant of the above equation as

$$P \left( \min[q, K_{i1}(\theta_-)] > d - x \right) - P \left( \min[q, K_{i1}(\theta_+)] > d - x \right)$$

$$= \begin{cases} 
    P \left( K_{i1}(\theta_-) > d - x \right) - P \left( K_{i1}(\theta_+) > d - x \right) & \text{if } q > d - x, \\
    0 & \text{if } q \leq d - x.
\end{cases}$$

The first expression is non-negative since $K_{i1}(\theta_-) <_{st} K_{i1}(\theta_+)$ and is independent of $q$, as long as $q > d - x$. In other words, $P \left( \min[q, K_{i1}(\theta_-)] > d - x \right) - P \left( \min[q, K_{i1}(\theta_+)] > d - x \right)$ is an increasing step function of $q$ with a jump at $q = d - x$. As a result, $L(q, d)$, which is the integral of this increasing function of $q$ over $[0, d]$, is also increasing in $q$.

The next step is to prove $M(q, d)$ is increasing in $q$, using a similar approach that shows the same result for $L(q, d)$. We can rewrite $M(q, d)$ as

$$M(q, d) = \int_0^\infty \left[ P \left( \min[q, K_{i1}(\theta_+), d] \geq x \right) - P \left( \min[q, K_{i1}(\theta_-), d] \geq x \right) \right] dx$$

$$= \int_0^d \left[ P \left( \min[q, K_{i1}(\theta_+)] \geq x \right) - P \left( \min[q, K_{i1}(\theta_-)] \geq x \right) \right] dx.$$  \hfill (F.4)
where the last expression follows since the integrant equals 0 if $d < x$. Differentiating the cases $q > x$ and $q \leq x$, we can write the integrant of the above equation as

$$P (\min(q, K(1)(\theta_+)) \geq x) - P (\min(q, K(1)(\theta_-)) \geq x)$$

$$= \begin{cases} 
 P(K(1)(\theta_+) \geq x) - P(K(1)(\theta_+) \geq x) dx, & \text{if } q > x, \\
 0, & \text{if } q \leq x. 
\end{cases}$$

The first expression is non-negative since $K(1)(\theta_-) \prec_{st} K(1)(\theta_+)$ and is independent of $q$, for $q > x$.

Therefore, $P (\min(q, K(1)(\theta_+)) \geq x) - P (\min(q, K(1)(\theta_-)) \geq x)$ is an increasing step function of $q$, with a jump at point $q = x \geq 0$. Then, $M(q, d)$, the integral of this increasing function of $q$ over $x \in [0, d)$, is also increasing in $q$. After unconditioning on $D = d$, we conclude that $g(q, \theta) = g(q, \theta) = \int_0^\infty [pL(q, d) + hM(q, d)]dF(d)$ is increasing in $q$. That is, $g(q, \theta)$ is a submodular function of $(q, \theta)$.

b. This result is the direct consequence of Theorem 3.9 (b) and Theorem 3.11 (a).