Asymptotic Analysis of Multivariate Tail Conditional Expectations

Li Zhu\textsuperscript{1} Haijun Li\textsuperscript{2}

October 2011
Revision: May 2012

\textsuperscript{1}lzhu@math.wsu.edu, Department of Mathematics, Washington State University, Pullman, WA 99164, U.S.A.
\textsuperscript{2}lih@math.wsu.edu, Department of Mathematics, Washington State University, Pullman, WA 99164, U.S.A. This author is supported by NSF grants CMMI 0825960 and DMS 1007556.
Abstract

Tail conditional expectations refer to the expected values of random variables conditioning on some tail events and are closely related to various coherent risk measures. In the univariate case, the tail conditional expectation is asymptotically proportional to the value-at-risk, a popular risk measure. The focus of this paper is on asymptotic relations between the multivariate tail conditional expectation and value-at-risk for heavy-tailed scale mixtures of multivariate distributions. Explicit tail estimates of multivariate tail conditional expectations are obtained using the method of regular variation. Examples involving multivariate Pareto and elliptical distributions, as well as application to risk allocation are also discussed.

Key words and phrases: Tail risk allocation, tail conditional expectation, coherent risk, regular variation, multivariate Pareto distribution, elliptical distribution.

JEL classification: C02, G32.
MSC2000 classification: 91B30, 60F05.
1 Introduction

The tail conditional expectation (TCE) used in risk analysis describes the expected amount of risk that could be experienced given that risk factors exceed some threshold values. TCEs are closely related to various coherent risk measures that are preferable than the Value-at-Risk (VaR), a risk measure that is widely used but fails to satisfy the coherency principle. In this paper, we study the asymptotic relations between the multivariate TCEs and VaR, and show that for a large class of continuous risk factors that follow multivariate heavy-tailed distributions, the tail conditional expectation given that aggregated risk exceeds a large threshold is asymptotically proportional to the value-at-risk of aggregation.

The expectation of a random variable $X$ conditioning on a tail event $\{X > t\}$ has a variety of interpretations in reliability and risk modeling. In reliability modeling, $E(X - t \mid X > t)$ for a non-negative lifetime $X$ is known as the mean residual lifetime [22]. In insurance and finance, $E(X - t \mid X > t)$ is known as the mean excess loss of a loss variable $X$ [20], and a risk measure for right-tailed losses can be described by

$$TCE_p(X) := E(X \mid X > \text{VaR}_p(X)), \text{ for } 0 < p < 1,$$

where $\text{VaR}_p(X) := \sup\{x \in \mathbb{R} : \mathbb{P}\{X > x\} \geq 1 - p\}$ is known as the VaR with confidence level $p$ (i.e., $p$-quantile). In this paper, risk factors are interpreted as loss variables.

It is known that for continuous loss $X$, the TCE equals the worst conditional expectation (WCE), which is defined as the supremum of all expectations of $X$ conditioning on various tail events with probability at least $1 - p$. The WCE, and thus TCE for continuous losses, arise naturally via the duality theory from coherent risk measures that satisfy four fundamental operating axioms: (1) monotonicity, (2) subadditivity, (3) positive homogeneity and (4) translation invariance (see [3, 8, 20] for details). In the univariate case, a coherent risk measure $\varphi(X)$ for loss $X$ corresponds to the amount of extra capital requirement that has to be invested in some secure instruments so that the resulting position $\varphi(X) - X$ is acceptable to regulators/supervisors. The coherent risk measures, such as TCE, overcome the shortcomings of VaR that violates the subadditivity principle and often underestimates tail risk. It can be shown that for continuous losses, TCE is the average of VaR over all confidence levels greater than $p$, focusing more than VaR does on extremal losses. Thus, TCE is more conservative than VaR at the same level of confidence (i.e., $TCE_p(X) \geq \text{VaR}_p(X)$) and provides an effective tool for analyzing tail risks.

For light-tailed loss distributions, such as normal and phase-type distributions [5], TCE and VaR at the same level $p$ of confidence are asymptotically equal as $p \to 1$. It is precisely the heavy-tailedness of loss distributions that differentiates TCE and VaR in analyzing tail risks. Formally, a non-negative random variable $X$ with distribution function $F$ has a heavy
or regularly varying right tail at $\infty$ with tail index $\alpha > 0$ if its survival function is of the following form (see, e.g., [4] for details),

$$F(t) := P\{X > t\} = t^{-\alpha}L(t), \quad t > 0, \quad \alpha > 0,$$

(1.2)

where $L$ is a slowly varying function; that is, $L$ is a positive function on $(0, \infty)$ with property $\lim_{t \to \infty} L(ct)/L(t) = 1$, for every $c > 0$. We use RV$^{-\alpha}$ in this paper to denote the class of all regularly varying functions with tail index $\alpha$. For any random variable $X$, write $X = X_+ - X_-$, where $X_+ := \max\{X, 0\}$ and $X_- := \max\{-X, 0\}$. In this situation, the notion of regular variation (1.2) can be applied to $X_+$ (right tail of $X$) or $X_-$ (left tail of $X$).

Note that a regularly varying function behaves as a power function asymptotically, and in particular, any regularly varying function integrates in the way as that of a power function, as is shown in the well-known Karamata’s theorem (see, e.g., [21], page 25).

**Proposition 1.1.** If $U(t) \in$ RV$^{-\alpha}$ with tail index $\alpha > 1$, then $\int_t^\infty U(x)dx \in$ RV$^{-\alpha+1}$ with tail index $\alpha - 1 > 0$ and

$$\int_t^\infty U(x)dx \sim \frac{t}{\alpha - 1}U(t), \text{ for sufficiently large } t.$$

(1.3)

Here and hereafter the tail equivalence notation “$f(t) \sim g(t)$ as $t \to a$” means that $f(t)/g(t) \to 1$ as $t \to a$. An immediate consequence of applying Karamata’s theorem to TCE for a loss variable $X$ having a regularly varying right tail with tail index $\alpha > 1$ is illustrated as follows. For any $p$ sufficiently close to $1$ so that $\text{VaR}_p(X) > 0$, we have that $\{X > \text{VaR}_p(X)\} = \{X_+ > \text{VaR}_p(X)\}$ due to the fact that $X = X_+$ if $X \geq 0$, and thus

$$\text{TCE}_p(X) = \frac{\mathbb{E}(XI\{X > \text{VaR}_p(X)\})}{\mathbb{P}\{X > \text{VaR}_p(X)\}} = \frac{\mathbb{E}(X_+I\{X_+ > \text{VaR}_p(X)\})}{\mathbb{P}\{X_+ > \text{VaR}_p(X)\}}$$

$$= \frac{1}{\mathbb{P}\{X_+ > \text{VaR}_p(X)\}} \int_0^\infty \mathbb{P}\{X_+ I\{X_+ > \text{VaR}_p(X)\} > x\}dx$$

$$= \frac{1}{\mathbb{P}\{X_+ > \text{VaR}_p(X)\}} \left(\text{VaR}_p(X) \mathbb{P}\{X_+ > \text{VaR}_p(X)\} + \int_{\text{VaR}_p(X)}^\infty \mathbb{P}\{X_+ > x\}dx\right)$$

$$\sim \frac{\alpha}{\alpha - 1} \text{VaR}_p(X), \text{ as } p \to 1,$$

(1.4)

where $I\{B\}$ denotes the indicator function of set $B$. That is, TCE for any heavy-tailed loss distribution is asymptotically proportional to its VaR with asymptotic constant that depends on its tail index, in a manner similar to that for the Pareto loss distribution $F(t) = (1 + t)^{-\alpha}$, $t \geq 0$.

The tail estimate (1.4) has been documented in the literature (see, e.g., page 283 of [20]), but the derivation in (1.4) illustrates how Karamata’s theorem and its multivariate...
extensions can be used to develop limiting results in asymptotic analysis of coherent risk measures, in the univariate case as well as in the multivariate case. A risk measure $\varrho(X)$ for a $d$-dimensional loss vector $X$ corresponds to a subset of $\mathbb{R}^d$ consisting of all the deterministic portfolios $x$ such that the modified positions $x - X$ is acceptable to regulators/supervisors. The coherency principles for multivariate risk measures that are similar to that in the univariate case, and multivariate TCEs were studied in [15]. Note, however, that multivariate TCEs are subsets of $\mathbb{R}^d$, which often lack tractable expressions. A multivariate regular variation method based on tail dependence function (see [13, 18]) was developed in [14] to derive tractable asymptotic bounds for multivariate TCEs, but these bounds are expressed in terms of univariate integrals of tail dependence functions and thus still cumbersome for loss distributions without explicit expressions of tail dependence functions, such as elliptical distributions. In this paper, we focus on the loss variables with heavy-tailed scale mixing:\n\[ X = (X_1, \ldots, X_d)^\top := (RT_1, \ldots, RT_d)^\top, \text{ and } R \geq 0, \] (1.5)\nwhere $R$ has a regularly varying survival function with tail index $\alpha$, and $(T_1, \ldots, T_d)^\top$, independent of $R$, is any random vector with some finite joint moments. Here and hereafter, “$\top$” denotes the matrix transpose. The class (1.5) of loss distributions is a sub-class of all multivariate regularly varying distributions that is discussed in [14], but it covers a variety of loss distributions, including multivariate Pareto distributions and multivariate elliptical distributions whose tail dependence functions are usually not explicit. Utilizing the regular variation property of $R$, we establish the explicit tail estimates of TCEs for the class (1.5) of loss distributions. In contrast to [14], a distinctive feature resulted from the approach used in this paper is that the tail estimates for the TCE given that aggregated loss exceeds a large threshold depend explicitly on tail index $\alpha$ and joint moments of variables $T_1, \ldots, T_d$.\n\nThe rest of the paper is organized as follows. In Section 2, we first prove, via Karamata’s theorem, a convergence theorem on the integral of the ratio of regularly varying functions, and then establish the tail estimate for multivariate conditional expectations of loss variables (1.5). In Section 3, we discuss the asymptotic properties of TCEs for multivariate elliptical distributions, and in particular, compare the tail estimates of TCEs obtained from our asymptotic approach and ones derived from the exact TCE formulas obtained in [16] for elliptical distributions.\n
2 Multivariate TCEs of Heavy-Tailed Scale Mixtures\n
In this section, we derive the tail estimate of TCE of loss variable $X_1$ given that another loss variable $X_2$ exceeds a larger threshold when $X_1, X_2$ are jointly distributed as that of (1.5) ($d = 2$), and as a consequence, tail estimates of various TCEs for aggregated risk factors can
be obtained. The following variants of Karamata’s theorem and Breiman’s theorem will be used in the proof of our main result.

Lemma 2.1. Let \( R \) be a non-negative random variable with regularly varying survival function \( U(t) := \mathbb{P}\{R > t\} = t^{-\alpha}L(t), \ t > 0, \ \alpha > 1 \), where \( L \) is a slowly varying function. Then we have
\[
\int_{ct}^{\infty} \mathbb{P}\{R > x\} \, dx \sim \frac{c^{1-\alpha}}{\alpha - 1} t \mathbb{P}\{R > t\}, \text{ as } t \to \infty, \text{ for every } c > 0.
\] (2.1)

Proof. According to Proposition 1.1, \( U(t) \in \text{RV}_{-\alpha} \) implies that
\[
\int_{ct}^{\infty} U(x) \, dx \sim \frac{ct}{\alpha - 1}, \text{ as } t \to \infty.
\] (2.2)

On the other hand, \( U(ct) \sim c^{-\alpha}U(t) \), as \( t \to \infty \). Plug this into (2.2) and we get the desired tail estimate (2.1).

Proposition 2.2. Let \( M(\cdot) \) be any finite non-negative measure on \( \mathbb{R}_+ \). If \( U(t) = t^{-\alpha}L(t) \in \text{RV}_{-\alpha} \) with tail index \( \alpha > 0 \) and \( \int_{0}^{\infty} x^{\alpha+\epsilon} M(dx) < \infty \) for some small \( \epsilon > 0 \), then
\[
\lim_{t \to \infty} \int_{0}^{\infty} \frac{U(t/x)}{U(t)} M(dx) = \int_{0}^{\infty} x^{\alpha} M(dx) = \int_{0}^{\infty} \lim_{t \to \infty} \frac{U(t/x)}{U(t)} M(dx).
\]

Proof. Let \( \tilde{R} \geq 0 \) be a random variable with survival function \( U(t) \). Since \( M(\cdot) \) is finite, we can construct a non-negative random variable \( X \), independent of \( \tilde{R} \), with distribution
\[
F(x) = \mathbb{P}\{X \leq x\} := \frac{M([0,x]) - M(\{0\})}{M(\mathbb{R}_+) - M(\{0\})}, \ x \in \mathbb{R}_+.
\]

Clearly, \( X \) has a finite moment of order greater than \( \alpha \). It follows from Breiman’s theorem (see, e.g., pages 231-232 of [21]) that
\[
\lim_{t \to \infty} \frac{\mathbb{P}\{RX > t\}}{\mathbb{P}\{R > t\}} = \mathbb{E}(X^\alpha).
\] (2.3)

That is, \( \lim_{t \to \infty} \int_{0}^{\infty} \frac{U(t/x)}{U(t)} dF(x) = \int_{0}^{\infty} x^\alpha dF(x) \), which yields the desired limit by canceling constant \( M(\mathbb{R}_+) - M(\{0\}) \) on both sides.

Proposition 2.2 allows us to pass the limit of tail ratio through integration, which facilitates asymptotic analysis of TCEs.

Proposition 2.3. Let \( (X_1, X_2)^\top = (RT_1, RT_2)^\top \) be a bivariate random vector, where \( (T_1, T_2)^\top \), independent of a random variable \( \tilde{R} \geq 0 \), has finite moments \( \mathbb{E}(T_1), \mathbb{E}(T_1T_2^\alpha) \) and \( \mathbb{E}(T_2^\alpha) \) for some \( \epsilon > 0 \), where \( T_{2+} := \max\{T_2, 0\} \). If \( \tilde{R} \) has the survival function \( U(t) := \mathbb{P}\{R > t\} = t^{-\alpha}L(t) \in \text{RV}_{-\alpha} \) with tail index \( \alpha > 1 \), then we have
\[
\frac{\mathbb{E}(X_1 | X_2 > t)}{t} \sim \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}[T_1T_{2+}^\alpha]}{\mathbb{E}[T_{2+}^\alpha]}, \text{ as } t \to \infty.
\] (2.4)
Proof. Observe first that $\mathbb{E}(T_1) < \infty$ and $\mathbb{E}(T_1 T_2^\alpha) < \infty$ imply that $\mathbb{E}(T_1 T_2^{\alpha-1}) < \infty$. Since $X_2 > t$ if and only if $X_{2+} := \max\{X_2, 0\} > t$ for any $t > 0$, we have, for $t > 0$,

$$
\mathbb{E}(X_1 \mid X_2 > t) = \mathbb{E}(X_1 \mid X_{2+} > t) = \mathbb{E}(X_1 \mid X_{2+} > t) - \mathbb{E}(X_{1-} \mid X_{2+} > t)
$$

where $X_{1+} = \max\{X_1, 0\}$ and $X_{1-} = \max\{-X_1, 0\}$. Consider, for $t > 0$,

$$
\mathbb{E}(X_{1+} \mid X_{2+} > t) = \frac{\mathbb{E}(X_{1+} \mathbb{I}\{X_{2+} > t\})}{\mathbb{P}\{X_{2+} > t\}} = \int_0^\infty \frac{\mathbb{P}\{X_{1+} > x, X_{2+} > t\}}{\mathbb{P}\{X_{2+} > t\}} dx.
$$

It follows from Breiman’s theorem (see, e.g., pages 231-232 of [21]) and $\mathbb{E}(T_2^{\alpha+\epsilon}) < \infty$ for some $\epsilon > 0$ that $X_{2+} = RT_{2+}$ has a regularly varying survival function with tail index $\alpha$ and

$$
\mathbb{P}\{X_{2+} > t\} \sim \mathbb{E}(T_{2+}^{\alpha}) \mathbb{P}\{R > t\}, \text{ for sufficiently large } t.
$$

Let $x = tw$, we have

$$
\mathbb{E}(X_{1+} \mid X_{2+} > t) = \frac{t}{\mathbb{P}\{X_{2+} > t\}} \int_0^\infty \mathbb{P}\{X_{1+} > tw, X_{2+} > t\} dw
$$

$$
= \frac{t}{\mathbb{P}\{X_{2+} > t\}} \int_{\mathbb{R}_+^2} \int_0^\infty \mathbb{P}\{R > tw, R > \frac{t}{t_2}\} dwdF(t_1, t_2)
$$

$$
= \frac{t}{\mathbb{P}\{X_{2+} > t\}} \int_{\mathbb{R}_+^2} \int_0^\infty \mathbb{P}\{R > t \max\left\{\frac{w}{t_1}, \frac{1}{t_2}\right\}\} dw dF(t_1, t_2),
$$

where $F$ denotes the joint distribution of $(T_{1+}, T_{2+})^\top$, where $T_{i+} = \max\{T_i, 0\}, i = 1, 2$. For the inner integral, we have

$$
\int_0^\infty \mathbb{P}\{R > t \max\left\{\frac{w}{t_1}, \frac{1}{t_2}\right\}\} dw
$$

$$
= \int_0^{\frac{t_1}{t_2}} \mathbb{P}\left\{R > \frac{t}{t_2}\right\} dw + \int_{\frac{t_1}{t_2}}^\infty \mathbb{P}\left\{R > \frac{tw}{t_1}\right\} dw
$$

$$
= \frac{t_1}{t_2} \mathbb{P}\left\{R > \frac{t}{t_2}\right\} + \int_{\frac{t_1}{t_2}}^\infty \mathbb{P}\left\{R > \frac{tw}{t_1}\right\} dw.
$$

(2.8)

For the first summand in (2.8), observe that for any $t_1 > 0, t_2 > 0$,

$$
\lim_{t \to \infty} \frac{t_1}{t_2} \mathbb{P}\left\{R > \frac{t}{t_2}\right\} = \frac{t_1}{t_2} \mathbb{E} t^{-\alpha} L(t^{-1}t_1) = \frac{t_1}{t_2} \mathbb{E} \frac{t^{-\alpha-1} - t^{-\alpha} L(t^{-1}t_1)}{t^{-\alpha-1} L(t)} = \lim_{t \to \infty} \frac{\mathbb{P}\left\{R > \frac{t}{t_2}\right\}}{\mathbb{P}\{R > t\}} = t_1 t_2^{\alpha-1},
$$

where $R_*$ has the survival function $U_*(t) := t^{-\alpha+1} L(t) \in RV_{-\alpha+1}$ with $\alpha - 1 > 0$. Let $M_2(B) := \int_B \int_{\mathbb{R}_+} t_1 dF(t_1, t_2)$, for any Borel subset $B \subseteq \mathbb{R}_+$, denote the marginal mean
measure induced by \( T_{1+} \), and \( M_2(\cdot) \) is a finite measure due to the fact that \( \mathbb{E}(T_1) < \infty \). Thus by Proposition 2.2 with \( \mathbb{E}(T_1T_{2+}^\alpha) < \infty \),

\[
\lim_{t \to \infty} \frac{t_1}{t_2} \int_{\mathbb{R}^2_+} \frac{\mathbb{P}\left\{ R > \frac{t}{t_2} \right\}}{\mathbb{P}\{ R > t \}} dF(t_1, t_2) = \lim_{t \to \infty} \frac{\mathbb{P}\left\{ R > \frac{t}{t_2} \right\}}{\mathbb{P}\{ R > t \}} M_2(dt_2) = \mathbb{E}(T_{1+}T_{2+}^{\alpha-1}). \tag{2.9}
\]

For the second summand in (2.8), let \( x = \frac{tw}{t_1} \), and it follows from (2.1) that

\[
\int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{tw}{t_1} \right\} dt = t_1 \int_{t_2}^{\infty} \mathbb{P}\{ R > x \} dx \sim \frac{t_2^{\alpha-1}}{\alpha - 1} t_1 \mathbb{P}\{ R > t \}, \quad \text{as } t \to \infty.
\]

Let \( U^*(t) := \int_t^{\infty} \mathbb{P}\{ R > x \} dx = \int_t^{\infty} U(x) dx \in \text{RV} - \alpha+1 \) with \( \alpha - 1 > 0 \). Observe from (1.3) that

\[
\int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{tw}{t_1} \right\} dt = t_1 \left( \int_t^{\infty} U(x) dx \right) \frac{U^*(t/t_2)}{U^*(t)} \rightarrow \frac{t_1}{\alpha - 1} t_2^{\alpha-1}, \quad \text{as } t \to \infty,
\]

and it follows from Proposition 2.2 with \( \mathbb{E}(T_1T_{2+}^\alpha) < \infty \) that

\[
\lim_{t \to \infty} \frac{t_1}{t_2} \int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{tw}{t_1} \right\} dt = t_1 \left( \int_t^{\infty} U(x) dx \right) \frac{U^*(t/t_2)}{U^*(t)} \lim_{t \to \infty} \frac{t_1}{t} \int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{tw}{t_1} \right\} dt = (\alpha - 1)^{-1} \mathbb{E}(T_{1+}T_{2+}^{\alpha-1}). \tag{2.10}
\]

Plugging (2.8) into (2.7), we have

\[
\mathbb{E}(X_{1+} \mid X_{2+} > t) = \frac{\mathbb{P}\{ R > t \}}{\mathbb{P}\{ X_{2+} > t \}} \int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{t}{t_2} \right\} dF(t_1, t_2)
\]

\[
+ \frac{\mathbb{P}\{ R > t \}}{\mathbb{P}\{ X_{2+} > t \}} \int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{tw}{t_1} \right\} dt = \int_{\mathbb{R}^2_+} \mathbb{P}\left\{ R > \frac{tw}{t_1} \right\} dt.
\]

Using (2.6), (2.9) and (2.10), we obtain that

\[
\lim_{t \to \infty} \frac{\mathbb{E}(X_{1+} \mid X_{2+} > t)}{t} = \lim_{t \to \infty} \frac{\mathbb{P}\{ R > t \}}{\mathbb{P}\{ X_{2+} > t \}} \mathbb{E}\left[ T_{1+}T_{2+}^{\alpha-1} \left( 1 + \frac{1}{\alpha - 1} \right) \right] = \frac{\alpha - 1}{\alpha - 1} \mathbb{E}(T_{1+}T_{2+}^{\alpha-1}).
\]

Using the similar arguments, we also have

\[
\lim_{t \to \infty} \frac{\mathbb{E}(X_{1-} \mid X_{2+} > t)}{t} = \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}(T_{1-}T_{2+}^{\alpha-1})}{\mathbb{E}T_{2+}^\alpha}.
\]

Observe that \( T_1 = T_{1+} - T_{1-} \), we have

\[
\lim_{t \to \infty} \frac{\mathbb{E}(X_1 \mid X_{2+} > t)}{t} = \lim_{t \to \infty} \frac{\mathbb{E}(X_{1+} \mid X_{2+} > t)}{t} - \lim_{t \to \infty} \frac{\mathbb{E}(X_{1-} \mid X_{2+} > t)}{t} = \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}(T_{1+}T_{2+}^{\alpha-1})}{\mathbb{E}T_{2+}^\alpha},
\]

as desired. \( \square \)
Remark 2.4. 1. Note that the moment condition in Proposition 2.3 is rather mild. For example, if \( T_1 \) and \( T_2 \) have finite marginal moments of any order, then the moment condition in Proposition 2.3 holds for any tail index \( \alpha > 1 \).

2. Take \( T_1 = T_2 \) in Proposition 2.3, and we have from (2.4) that, as \( t \to \infty \),

\[
\frac{\mathbb{E}(X_i \mid X_i > t)}{t} \sim \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}[T_1 T_1^{\alpha - 1}]}{\mathbb{E}[T_1^{\alpha}]} = \frac{\alpha}{\alpha - 1} - \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}[T_1 T_1^{\alpha - 1}]}{\mathbb{E}[T_1^{\alpha}]} = \frac{\alpha}{\alpha - 1},
\]

due to the fact that \( T_1 T_1^{\alpha - 1} = 0 \) almost surely. This is exactly the same as (1.4).

The tail estimates of various TCEs can be obtained immediately from Proposition 2.3. Let \( \psi : \mathbb{R}^d \to \mathbb{R} \) be a strictly increasing homogeneous function with \( \psi(c \mathbf{x}) = c \psi(\mathbf{x}) \) for \( c \geq 0 \) and \( \mathbf{x} \in \mathbb{R}^d \). For example, any linear function of the form \( \psi(\mathbf{x}) = \sum_{i=1}^d a_i x_i \), defined on \( \mathbb{R}^d \), where \( a_i > 0 \), \( 1 \leq i \leq d \), is strictly increasing and homogeneous. Note that such a function must satisfy that \( \psi(0) = 0 \) and \( \text{VaR}_p(\psi(\mathbf{X})) \to \infty \) for heavy-tailed \( \mathbf{X} \), as \( p \to 1 \). For any random vector \( (X_1, \ldots, X_d)^\top = (RT_1, \ldots, RT_d)^\top \) with \( R \geq 0 \), observe that \( (X_i, \psi(\mathbf{X}))^\top = (RT_i, R\psi(T))^\top \), and thus the following result follows from Proposition 2.3.

**Theorem 2.5.** Let \( \mathbf{X} = (X_1, \ldots, X_d)^\top = (RT_1, \ldots, RT_d)^\top \) be a random vector, where the survival function of \( R, U(t) \in \text{RV}_\alpha \) with \( \alpha > 1 \), and \( \mathbf{T} = (T_1, \ldots, T_d)^\top \), independent of \( R \), has finite marginal moments \( \mathbb{E}(T_i^k) \), \( k \geq 1 \), \( 1 \leq i \leq d \). Then, for any \( 1 \leq i \leq d \), we have

\[
\mathbb{E}(X_i \mid \psi(\mathbf{X}) > \text{VaR}_p(\psi(\mathbf{X}))) \sim \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}[T_i \psi_+^{\alpha - 1}(\mathbf{T})]}{\mathbb{E}[\psi_+^{\alpha}(\mathbf{T})]} \text{VaR}_p(\psi(\mathbf{X})), \quad \text{as } p \to 1,
\]

where \( \psi_+(\mathbf{T}) := \max\{\psi(\mathbf{X}), 0\} \).

For example, if \( \psi(\mathbf{x}) = \sum_{i=1}^d x_i \) represents the linear aggregation of losses, then we have that for any \( 1 \leq i \leq d \), as \( p \to 1 \),

\[
\mathbb{E}(X_i \mid \sum_{i=1}^d X_i > \text{VaR}_p(\sum_{i=1}^d X_i)) \sim \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}\left(T_i \left(\sum_{i=1}^d T_i\right)^{\alpha - 1}\right)}{\mathbb{E}\left(\sum_{j=1}^d T_j\right)^{\alpha}} \text{VaR}_p\left(\sum_{i=1}^d X_i\right),
\]

where \( \left(\sum_{j=1}^d T_j\right)^+ := \max\{\sum_{j=1}^d T_j, 0\} \). As another example, let \( X(d) \) and \( T(d) \) denote the largest order statistics of \( (X_1, \ldots, X_d)^\top \) and \( (T_1, \ldots, T_d)^\top \) respectively, and we have for any \( 1 \leq i \leq d \),

\[
\mathbb{E}(X_i \mid X(d) > \text{VaR}_p(X(d))) \sim \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}(T(d) X(d)^{\alpha - 1})}{\mathbb{E}(T(d)^{\alpha})} \text{VaR}_p(X(d)), \quad \text{as } p \to 1,
\]

where \( T(d^+) := \max\{T(d), 0\} \). In fact, as indicated in Proposition 2.3 and Theorem 2.5, the TCE of a homogeneous function of \( X_1, \ldots, X_d \) given that another homogeneous function
of the variables exceeds its VaR can be also asymptotically expressed in terms of the joint moments of these functions. For example, let \( X_{(k)} \) denote the \( k \)-th largest order statistic of \( X_1, \ldots, X_d \), then the tail estimates of \( \mathbb{E}(X_i|X_{(k)} > t) \) and \( \mathbb{E}(X_{(i)}|X_{(k)} > t) \) can be obtained using Proposition 2.3.

It is worth emphasizing here that the asymptotic proportionality constants of the TCEs discussed in Theorem 2.5 depend on tail index \( \alpha \) and also on the dependence structure of \( T_1, \ldots, T_d \). For example, consider the bivariate case with \( T_1 \geq 0, T_2 \geq 0 \) and \( \alpha = 2 \), and we then have for \( i = 1 \),

\[
\mathbb{E}(X_1|X_1 + X_2 > \text{VaR}_p(X_1 + X_2)) \sim \frac{2\mathbb{E}T_1^2 + 2\mathbb{E}(T_1T_2)}{\mathbb{E}T_1^2 + 2\mathbb{E}(T_1T_2) + \mathbb{E}T_2^2} \text{VaR}_p(X_1 + X_2),
\]
as \( p \to 1 \). Let \( \rho \) be the correlation coefficient of \((T_1, T_2)\). For the fixed marginal distributions of \( T_1 \) and \( T_2 \), \( \rho \) is increasing if and only if \( \mathbb{E}(T_1T_2) \) is increasing. Thus, we have

\[
\frac{2\mathbb{E}T_1^2 + 2\mathbb{E}(T_1T_2)}{\mathbb{E}T_1^2 + 2\mathbb{E}(T_1T_2) + \mathbb{E}T_2^2} =
\begin{cases}
1 & \text{if } \mathbb{E}T_1^2 = \mathbb{E}T_2^2, \\
\text{is increasing in } \rho & \text{if } \mathbb{E}T_1^2 < \mathbb{E}T_2^2, \\
\text{is decreasing in } \rho & \text{if } \mathbb{E}T_1^2 > \mathbb{E}T_2^2.
\end{cases}
\]

Thus, even the marginal distributions are fixed, merely changing the dependence structure of multivariate risk factors could change the asymptotic proportion of the TCEs with respect to the VaR.

**Example 2.6.** Consider a bivariate Pareto distribution of Marshall-Olkin type for random vector \( (X_1, X_2)^\top = (RT_1, RT_2)^\top \). Let \( R \) have the survival function \( U(t) = \mathbb{P}\{R > t\} = (1 + t)^{-\alpha} \) for \( t \geq 0 \) and \( \alpha > 1 \), and let \((T_1, T_2)\) have a bivariate Marshall-Olkin exponential distribution function on \([0, \infty)^2\) [19], namely,

\[
T_1 = \min\{E_1, E_{12}\}, \quad T_2 = \min\{E_2, E_{12}\},
\]

where \( E_1, E_2, E_{12} \) are independent and have the exponential distributions with parameters \( \lambda_1, \lambda_2, \lambda_{12} \) respectively. All the joint moments \( \mathbb{E}(T_1^i T_2^j) \) for any non-negative integers \( i, j \) can be calculated explicitly.

It is easy to see that \( \text{VaR}_p(R) = (1 - p)^{-1/\alpha} - 1 \). It follow from Breiman’s theorem (see page 231-232 of [21]) that \( \mathbb{P}\{X_1 + X_2 > t\} \sim \mathbb{E}(T_1 + T_2)^\alpha \mathbb{P}\{R > t\} \) for sufficiently large \( t \), which implies that

\[
\text{VaR}_p(X_1 + X_2) \sim \left( \frac{\mathbb{E}(T_1 + T_2)^\alpha}{1 - p} \right)^{1/\alpha} - 1, \quad \text{as } p \to 1.
\]

It follows from (2.11) that the tail estimate of TCE for \( i = 1, 2 \) is given by,

\[
\mathbb{E}(X_i|X_1 + X_2 > \text{VaR}_p(X_1 + X_2)) \sim \frac{\alpha}{\alpha - 1} \frac{\mathbb{E}[T_1(T_1 + T_2)^{\alpha - 1}]}{\mathbb{E}(T_1 + T_2)^\alpha} \left[ \left( \frac{\mathbb{E}(T_1 + T_2)^\alpha}{1 - p} \right)^{1/\alpha} - 1 \right],
\]
as $p \to 1$. For integer-valued $\alpha$, these tail estimates can be evaluated analytically. For example, if $\alpha = 2$, then

$$
\mathbb{E}(X_1 \mid X_1 + X_2 > \text{VaR}_p(X_1 + X_2)) \sim \frac{2[ET_1^2 + \mathbb{E}(T_1T_2)]}{ET_1^2 + 2\mathbb{E}(T_1T_2) + ET_2^2} \text{VaR}_p(X_1 + X_2)
$$

$$
\text{VaR}_p(X_1 + X_2) \sim \left(\frac{ET_1^2 + 2\mathbb{E}(T_1T_2) + ET_2^2}{1 - p}\right)^{1/2} - 1.
$$

Since

$$
\mathbb{E}(T_1) = \frac{1}{\lambda_1 + \lambda_{12}}, \quad \text{var}(T_1) = \frac{1}{(\lambda_1 + \lambda_{12})^2}, \quad \mathbb{E}(T_2) = \frac{1}{\lambda_2 + \lambda_{12}}, \quad \text{var}(T_2) = \frac{1}{(\lambda_2 + \lambda_{12})^2},
$$

we obtain that

$$
\mathbb{E}(T_1^2) = \text{var}(T_1) + (\mathbb{E}T_1)^2 = \frac{2}{(\lambda_1 + \lambda_{12})^2}, \quad \mathbb{E}(T_2^2) = \frac{2}{(\lambda_2 + \lambda_{12})^2}.
$$

We also know from [19] that

$$
\mathbb{E}(T_1T_2) = \frac{1}{(\lambda_1 + \lambda_{12})\lambda} + \frac{1}{(\lambda_2 + \lambda_{12})\lambda},
$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Using all the above moment expressions, we obtain the tail estimates: as $p \to 1$,

$$
\mathbb{E}(X_1 \mid X_1 + X_2 > \text{VaR}_p(X_1 + X_2)) \sim \frac{2}{(\lambda_1 + \lambda_{12})^2} + \frac{1}{\lambda(\lambda_1 + \lambda_{12})} + \frac{1}{\lambda(\lambda_2 + \lambda_{12})} + \frac{1}{(\lambda_2 + \lambda_{12})^2},
$$

$$
\text{VaR}_p(X_1 + X_2) \sim \left(\frac{2}{(\lambda_1 + \lambda_{12})^2} + \frac{2}{\lambda(\lambda_1 + \lambda_{12})} + \frac{2}{\lambda(\lambda_2 + \lambda_{12})} + \frac{2}{(\lambda_2 + \lambda_{12})^2}}{1 - p}\right)^{1/2} - 1.
$$

Note that for the bivariate Pareto distribution of Marshall-Olkin type, the correlation of $X_1$ and $X_2$ is decreasing in $\lambda_{12}$. It is evident that $\text{VaR}_p(X_1 + X_2)$ is asymptotically increasing in the correlation of $X_1$ and $X_2$, whereas the asymptotic monotonicity of TCE with respect to the correlation is more subtle and also depends on the marginal parameters $\lambda_1$ and $\lambda_2$. □

**Example 2.7.** Consider the mixture model (1.5) where $T_1, \ldots, T_d$ are independently and exponentially distributed random variables with unit mean, and $R$ is a strictly positive random variable with the Laplace transform of $R^{-1}$ being given by $\varphi(t)$. The marginal survival function of $X_i$ is then given by $\overline{F}_i(t) = \mathbb{E}(e^{-t/R}) = \varphi(t)$, $t \geq 0$. Assume that $\varphi(t) \in \text{RV}_{-\alpha}$ with tail index $\alpha > 1$.

The scale mixture $X = (X_1, \ldots, X_d)^T$ has regularly varying margins and Archimedean copula dependence structure [17]. For example, if $R^{-1}$ has the gamma distribution with unit
scale parameter and shape parameter $\alpha > 0$, then the Laplace transform $\varphi(t) = (1 + t)^{-\alpha}$, leading to the so called Clayton copula dependence structure for $X$. The tail estimate of TCE given that the linear aggregation of losses exceeds a large threshold can be calculated explicitly via (2.11). For example, if $\alpha = 2$, then

$$
E(X_i \mid \sum_{i=1}^{d} X_i > \text{VaR}_p(\sum_{i=1}^{d} X_i)) \sim \frac{\alpha}{d(\alpha - 1)} \text{VaR}_p(\sum_{i=1}^{d} X_i).
$$

due to the fact that $\sum_{i=1}^{d} T_i$ has a gamma distribution with unit scale parameter and shape parameter $d$. In fact, since $T_1, \ldots, T_d$ are independently and identically distributed, we have

$$
E \left( T_i \left( \sum_{i=1}^{d} T_i \right)^{\alpha-1} \right) = \frac{1}{d} \sum_{i=1}^{d} E \left( T_i \left( \sum_{i=1}^{d} T_i \right)^{\alpha-1} \right) = \frac{1}{d} E \left( \sum_{i=1}^{d} T_i \right)^{\alpha}.
$$

Thus for any $\alpha > 1$,

$$
E(X_i \mid \sum_{i=1}^{d} X_i > \text{VaR}_p(\sum_{i=1}^{d} X_i)) \sim \frac{\alpha}{d(\alpha - 1)} \text{VaR}_p(\sum_{i=1}^{d} X_i).
$$

Note that the tail estimates for $\text{VaR}_p(\sum_{i=1}^{d} X_i)$ in terms of the marginal VaR are discussed in [2, 1, 9] and the references therein, and our tail estimate for TCE in terms of VaR complements these asymptotic results. □

## 3 TCE for Multivariate Elliptical Distributions

Let $\Sigma$ be a $d \times d$ positive-semidefinite matrix, and $U = (U_1, \ldots, U_m)^\top$ be uniformly distributed on the unit sphere in $\mathbb{R}^m$. Consider the stochastic representation

$$
X = (X_1, \ldots, X_d)^\top = (\mu_1, \ldots, \mu_d)^\top + RA(U_1, \ldots, U_m)^\top,
$$

where $A$ is a $d \times m$ matrix with $AA^\top = \Sigma$ and $R > 0$ is a random variable independent of $U$. The distribution of $X$ is known as a $d$-dimensional elliptical (contoured) distribution with dispersion matrix $\Sigma$ and is one of most widely used radially symmetric multivariate distributions [10]. The examples of elliptical distributions include the multivariate normal, t and logistic distributions. Since we are interested in tail behaviors of $X$, we may choose $(\mu_1, \ldots, \mu_d) = (0, \ldots, 0)$ without loss of generality:

$$
X = (X_1, \ldots, X_d)^\top = RA(U_1, \ldots, U_m)^\top.
$$
The characteristic function of $X$ in $(3.1)$ can be written in the form as $E(e^{it^\top X}) = \Psi(t^\top \Sigma t/2)$, where $\Psi : \mathbb{R}_+ \mapsto \mathbb{R}$ is called the characteristic generator. If $R$ has a density and $\Sigma$ is positive-definite, then the density of $X$ exists and can be written in the following form:

$$f(x) = c_d|\Sigma|^{-1/2}g_d(x^\top \Sigma^{-1}x/2),$$

where $g_d : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called the density generator and $c_d$ is the normalizing constant. For example, for the multivariate normal distribution, $g_d(x) = \exp\{-x\}$, $x \geq 0$. For the multivariate t distribution, $g_d(x) = (1 + xk_p)^{-p}$, $x \geq 0$, (3.2) where the parameter $p > d/2$ and $k_p$ is some constant that may depend on $p$ (see, e.g., page 81 of [10]). Note that these density generators do not depend on $d$, as is the case for many other elliptical distributions. It can be shown (see, e.g., Theorem 2.9 in [10]) that the density $h$ of $R$ and $g_d$ are related as follows:

$$h(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}r^{d-1}g_d(r^2), \quad r \geq 0,$$

and in particular, $h(r) = 2\sqrt{\pi}g_1(r^2)/\Gamma(1/2)$. If $\mathbb{E}(R^2) < \infty$ and $\Sigma$ is positive-definite, then the covariance matrix $\text{Cov}(X) = d^{-1}\mathbb{E}(R^2)\Sigma$.

Let $S := \sum_{i=1}^d X_i$ for an elliptically distributed random vector $(X_1, \ldots, X_d)^\top$ with dispersion matrix $\Sigma = (\sigma_{i,j})$. Since any affine transform of an elliptically distributed random vector is also elliptical, the random vector $(X_i, S)$ is elliptical with dispersion matrix

$$\Sigma_{i,S} = \begin{pmatrix} \sigma_i^2 & \sigma_{i,S} \\ \sigma_{i,S} & \sigma_S^2 \end{pmatrix}$$

where $\sigma_i^2 = \sigma_{i,i}$, $\sigma_{i,S} = \sum_{j=1}^d \sigma_{i,j}$, and $\sigma_S^2 = \sum_{i=1}^d \sum_{j=1}^d \sigma_{i,j}$. The explicit expression of TCE of $X_i$ given that $S$ exceeds a threshold is obtained in [16] as follows.

**Theorem 3.1.** Let $X = (X_1, \ldots, X_d)^\top$ be defined as that in $(3.1)$ with dispersion matrix $\Sigma$ and density generator $g_d$. If $\mathbb{E}(R) < \infty$ and $\Sigma$ is positive-definite, then, for $1 \leq i \leq d$,

$$\mathbb{E}(X_i|S > t) = \frac{\sigma_{i,S}}{\sigma_S} \frac{\int_t^{\infty} g_1(x) dx}{\int_{t/\sigma_S}^{\infty} g_1(x^2/2) dx},$$

where $t = \text{VaR}_p(S)$.

The tail estimate of TCE of $X_i$ given that $S$ exceeds a large threshold can now be obtained from Theorem 3.1 via Karamata’s theorem (see Proposition 1.1).
Corollary 3.2. Let \( X = (X_1, \ldots, X_d)^\top \) be defined as that in (3.1) with dispersion matrix \( \Sigma \) and density generator \( g_d \). If \( \Sigma \) is positive-definite and the density \( h(r) \) of \( R \) exists and is regularly varying with tail index \( \alpha + 1, \alpha > 1 \), then, for \( 1 \leq i \leq d \),

\[
\mathbb{E}(X_i | S > \text{VaR}_p(S)) \sim \frac{\alpha}{\alpha - 1} \frac{\sigma_{i,S}}{\sigma_S^2} \text{VaR}_p(S), \quad p \to 1.
\]  

(3.5)

Proof. Observe first that since \( \alpha > 1 \), \( \mathbb{E}(R) < \infty \). It follows from (3.3) that \( h(r) = 2\sqrt{\pi} g_1(r^2) / \Gamma(1/2) \), and thus the density generator \( g_1(r^2) \in \text{RV}_{-\alpha+1}^\alpha \) and \( g_1(r^2) \in \text{RV}_{-(\alpha+1)}^\alpha \). Applying Proposition 1.1 to (3.4), we have, as \( t \to \infty \),

\[
\frac{\int_{t^2/2\sigma_S^2}^\infty g_1(x) dx}{\int_{t/\sigma_S}^\infty g_1(x^2/2) dx} \sim \frac{2}{\alpha - 1} \frac{t^2}{2\sigma_S^2} \frac{g_1(t^2/2\sigma_S^2)}{g_1(t^2/2)} = \frac{\alpha}{\alpha - 1} \frac{t}{\sigma_S^2}.
\]

Plug this tail estimate into (3.4), we have

\[
\mathbb{E}(X_i | S > t) \sim \frac{\alpha}{\alpha - 1} \frac{\sigma_{i,S}}{\sigma_S^2} t
\]

where \( t = \text{VaR}_p(S) \to \infty \) as \( p \to 1 \). \( \square \)

Remark 3.3. 1. For any random vector \((X_1, X_2)^\top\) with a bivariate elliptical distribution, it follows from Lemma 2 in [16] and the similar arguments as these in the proof of Corollary 3.2 that as \( p \to 1 \),

\[
\mathbb{E}(X_1 | X_2 > \text{VaR}_p(X_2)) \sim \frac{\alpha}{\alpha - 1} \frac{\sigma_{1,2}}{\sigma_{2,2}} \text{VaR}_p(X_2).
\]  

(3.6)

In fact, the tail estimate (3.5) can be derived from (3.6) since \((X, S)\) has a bivariate elliptical distribution.

2. It follows from Proposition 1.1 that the density \( h(r) \in \text{RV}_{-(\alpha+1)}^\alpha \) implies that the survival function of \( R \), \( \mathbb{P}\{R > t\} = \int_t^\infty h(r) dr \in \text{RV}_{-\alpha} \), which is the condition used in Theorem 2.5. Conversely, the condition that \( \mathbb{P}\{R > t\} = \int_t^\infty h(r) dr \in \text{RV}_{-\alpha} \) implies that \( h(r) \in \text{RV}_{-(\alpha+1)}^\alpha \) provided that \( h(r) \) is asymptotically monotone. This result, due to E. Landau, has a relation with the von Mises condition (see Proposition 2.5 in [21]).

3. Note that the proofs of both Theorem 3.1 and Corollary 3.2 require the existence of the joint density function. In contrast, the tail estimates for elliptical distributions via Theorem 2.5 do not require such an assumption.

The tail estimate for \( \mathbb{E}(X_i | S > t) \), as \( t \to \infty \), can be also obtained using Theorem 2.5. Let \((T_1, \ldots, T_d)^\top = A(U_1, \ldots, U_m)^\top\), then \( X = (X_1, \ldots, X_d)^\top = R(T_1, \ldots, T_d)^\top \). If \( R \) has a
regularly varying survival function with tail index \( \alpha > 1 \), then \( \mathbf{X} \) has multivariate regularly varying tails. It follows from (2.11) that for \( 1 \leq i \leq d \), as \( p \to 1 \),

\[
E(X_i \mid S > \text{VaR}_p(S)) \sim \frac{\alpha}{\alpha - 1} \frac{E \left( T_i \left( \sum_{i=1}^{d} T_i \right)^{\alpha-1} \right)}{E(\sum_{j=1}^{d} T_j)^{\alpha}} \text{VaR}_p(S),
\]

In the bivariate case, this tail estimate becomes: for \( i = 1, 2 \),

\[
E(X_i \mid X_1 + X_2 > \text{VaR}_p(X_1 + X_2)) \sim \frac{\alpha}{\alpha - 1} \frac{E \left( T_i (T_1 + T_2)^{\alpha-1} \right)}{E(T_1 + T_2)^{\alpha}} \text{VaR}_p(X_1 + X_2). \quad (3.7)
\]

In contrast, (3.5) has the following expression in the bivariate case: for \( i = 1, 2 \),

\[
E(X_i \mid X_1 + X_2 > \text{VaR}_p(X_1 + X_2)) \sim \frac{\alpha}{\alpha - 1} \frac{\sigma_{i,1} + \sigma_{i,2}}{\sigma_{1,1} + \sigma_{1,2} + \sigma_{2,1} + \sigma_{2,2}} \text{VaR}_p(X_1 + X_2). \quad (3.8)
\]

To illustrate the fact that the asymptotic proportionality constants in (3.5) and (2.11) are approximately same for elliptical distributions, we simulated samples with simple size \( n = 100,000 \) from the bivariate t distribution and compared two proportionality constants in (3.7) and (3.8) via simulation. The bivariate t distributions are sampled for various degrees of freedom \( \nu > 2 \) (d.f.) and covariance matrices \( \Sigma' = (\sigma'_{i,j}) \) of the corresponding bivariate normal distributions. Note that for the t distributions,

1. the tail index \( \alpha = \nu \) for its joint survival function (see, e.g., [6]),
2. \( (T_1, \ldots, T_d)^\top \) in (2.11) has the normal distribution with zero mean vector and covariance matrix \( \Sigma' = (\sigma'_{i,j}) \), and
3. \( \Sigma' = \frac{\nu - 2}{\nu d} E(R^2) \Sigma \), where \( R^{1/2} \) has the inverse gamma distribution with parameter \( \nu/2 \).

Since \( \sigma'_{i,j} \) and \( \sigma_{i,j} \) are proportionally related, (3.5) and (3.8) can be also expressed in the exactly same way in terms of \( \sigma'_{i,j} \)s. The comparisons of the asymptotic proportionality constants in (3.5) and (2.11) for various parameters are listed in Table 1 and 2, where \( \sigma'_{1,1} = 1 \) and \( \sigma'_{2,2} = 1.5 \).

Note that the asymptotic proportionality constants in the third row of Tables 1-2 depend only on \( \Sigma' \) and are independent of d.f. \( \nu = \alpha \). We also numerically compared the asymptotic proportionality constants in (3.6) and (2.4). The comparisons for bivariate t distributions with various parameters are listed in Tables 3-4, where \( \sigma'_{1,1} = \sigma'_{2,2} = 1 \). In contrast to Tables 1-2, we included the factor \( \alpha/(\alpha - 1) \) in Tables 3-4. It is evident that the asymptotic proportionality constants in (3.5) and (2.11) are decreasing in tail index \( \alpha \) only through \( \alpha/(\alpha - 1) \) for elliptical distributions. On the other hand, the asymptotic proportionality constants in (3.6) and (2.4) are increasing as correlation increases.
\[
\begin{array}{cccccccccc}
\sigma'_{12} & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.55 & 0.6 & 0.7 & 0.75 & 0.8 \\
\frac{E(T_1(T_1+T_2)^{\alpha-1})}{E(T_1+T_2)^{\alpha}} & 0.406 & 0.416 & 0.419 & 0.424 & 0.429 & 0.433 & 0.434 & 0.436 & 0.437 & 0.438 \\
\frac{\sigma'_{1,1}+\sigma'_{1,2}}{\sigma'_{1,1}+\sigma'_{1,2}+\sigma'_{2,1}+\sigma'_{2,2}} & 0.407 & 0.414 & 0.419 & 0.424 & 0.429 & 0.431 & 0.432 & 0.436 & 0.438 & 0.439 \\
\text{Difference} & -0.001 & 0.002 & 0 & 0 & 0 & 0.002 & 0.002 & 0 & -0.001 & -0.001 \\
\end{array}
\]

Table 1: Comparison of Approximation for Bivariate t Distribution with d.f. = \( \alpha = 2.1 \)

\[
\begin{array}{cccccccccc}
\sigma'_{12} & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.55 & 0.6 & 0.7 & 0.75 & 0.8 \\
\frac{E(T_1(T_1+T_2)^{\alpha-1})}{E(T_1+T_2)^{\alpha}} & 0.407 & 0.411 & 0.420 & 0.425 & 0.426 & 0.431 & 0.434 & 0.436 & 0.439 & 0.440 \\
\frac{\sigma'_{1,1}+\sigma'_{1,2}}{\sigma'_{1,1}+\sigma'_{1,2}+\sigma'_{2,1}+\sigma'_{2,2}} & 0.407 & 0.414 & 0.419 & 0.424 & 0.429 & 0.431 & 0.432 & 0.436 & 0.438 & 0.439 \\
\text{Difference} & 0 & -0.003 & 0.001 & 0.001 & -0.003 & 0 & 0.002 & 0 & 0.001 & 0.001 \\
\end{array}
\]

Table 2: Comparison of Approximation for Bivariate t Distribution with d.f. = \( \alpha = 7 \)

4 Concluding Remarks

One motivation to study tail estimates of TCE of loss variables given that aggregated risk exceeds a large threshold is that such tail estimates can be applied to evaluate tail risk allocation/decomposition. That is, the tail estimate of \( \mathbb{E}(X_i \mid \sum_{i=1}^{d} X_i > \text{VaR}_p(\sum_{i=1}^{d} X_i)) \) provides the contribution to the total tail risk attributable to the \( i \)-th risk factor \( X_i \), as measured by TCEs. The risk allocation/decomposition with TCE for elliptically distributed loss vectors and for multivariate Pareto portfolios can be found in [16, 7, 24], and a general discussion on this topic can be found in [20]. The tail conditional variance for elliptical distributions is studied in [23].

We have shown that the TCE of a loss variable given that aggregated risk exceeds a large threshold is asymptotically proportional to the VaR of aggregated risk as the confidence level approaches 1. The proportionality constant can be expressed explicitly in terms of the tail index and joint moments of mixed variables with heavy-tailed scale mixing. Several examples of loss variables involving multivariate Pareto and elliptical distributions have been explicitly calculated to illustrate our results.

If the mixture model (1.5) fits data, then the model parameters can be estimated (see [12, 21] for details on statistical inference). These proportionality constants in Theorem 2.5 can be evaluated using numerical integration or simulation. For example, if loss variables follow a multivariate t distribution as is the case in the numerical examples discussed in Section 3, then the proportionality constants can be evaluated through numerical integration or simulation on multivariate normally distributed variables (see [11]).

Acknowledgments: The authors would like to sincerely thank anonymous referees for their
Table 3: Comparison of Approximation for Bivariate t Distribution with d.f. = $\alpha = 2.1$  

<table>
<thead>
<tr>
<th>$\sigma'_{1.2}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.55</th>
<th>0.6</th>
<th>0.7</th>
<th>0.75</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\alpha}{\alpha-1} \frac{E[T_1T_2^{\alpha-1}]}{E[T_2^{\alpha-1}]}$</td>
<td>0.197</td>
<td>0.380</td>
<td>0.566</td>
<td>0.770</td>
<td>0.963</td>
<td>1.045</td>
<td>1.145</td>
<td>1.341</td>
<td>1.434</td>
<td>1.524</td>
</tr>
<tr>
<td>$\frac{\alpha}{\alpha-1} \sigma'<em>{1.2} \sigma'</em>{2.2}$</td>
<td>0.191</td>
<td>0.382</td>
<td>0.573</td>
<td>0.764</td>
<td>0.955</td>
<td>1.05</td>
<td>1.145</td>
<td>1.336</td>
<td>1.432</td>
<td>1.527</td>
</tr>
<tr>
<td>Difference</td>
<td>0.006</td>
<td>-0.002</td>
<td>-0.007</td>
<td>0.006</td>
<td>0.008</td>
<td>-0.005</td>
<td>0</td>
<td>0.005</td>
<td>0.002</td>
<td>-0.003</td>
</tr>
</tbody>
</table>

Table 4: Comparison of Approximation for Bivariate t Distribution with d.f. = $\alpha = 7$  

<table>
<thead>
<tr>
<th>$\sigma'_{1.2}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.55</th>
<th>0.6</th>
<th>0.7</th>
<th>0.75</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\alpha}{\alpha-1} \frac{E[T_1T_2^{\alpha-1}]}{E[T_2^{\alpha-1}]}$</td>
<td>0.122</td>
<td>0.232</td>
<td>0.341</td>
<td>0.465</td>
<td>0.575</td>
<td>0.650</td>
<td>0.708</td>
<td>0.817</td>
<td>0.876</td>
<td>0.932</td>
</tr>
<tr>
<td>$\frac{\alpha}{\alpha-1} \sigma'<em>{1.2} \sigma'</em>{2.2}$</td>
<td>0.117</td>
<td>0.233</td>
<td>0.350</td>
<td>0.467</td>
<td>0.583</td>
<td>0.642</td>
<td>0.700</td>
<td>0.815</td>
<td>0.875</td>
<td>0.933</td>
</tr>
<tr>
<td>Difference</td>
<td>0.005</td>
<td>-0.001</td>
<td>-0.009</td>
<td>-0.002</td>
<td>-0.008</td>
<td>0.008</td>
<td>0.008</td>
<td>0.002</td>
<td>0.001</td>
<td>-0.001</td>
</tr>
</tbody>
</table>

detailed comments, which led to an improvement of the presentation of this paper.

References


