Hit-or-Miss Dependence of Random Closed Sets*

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Abstract

The notion of hit-or-miss dependence is introduced to describe spatial dependence of random closed sets, and its closure properties under various set operations are derived. The hit-or-miss dependence is then applied to Boolean germ-grain models to study local interactions within each grain. The dependence comparison results obtained not only show how local interactions of Boolean germ-grain models would affect system performance, but also yield various bounds for spatial reliability indexes, which can be calculated explicitly.

Key Words and Phrases: Random set, hit-or-miss dependence, Boolean model with local dependence, spatial damage.

1 Introduction

A random closed set is a random element whose values are closed subsets of a basic setting space. The spatial systems featuring random closed sets find applications in diverse fields ranging from imaging analysis to epidemiology [9, 5]. Recent studies on spatial reliability modeling in nanoelectronics [11, 2] show that spatial defects, which can be modeled as random sets, are often clustered and exhibit local interactions. In risk management of spatial supply chains, business shocks, including natural disasters, accidents and intentional attacks, often induce highly correlated spatial damages that can be also modeled as dependent random sets.

The distinct feature of flexible geometric shapes is crucial in the development of the theory of random sets. A main difficulty in applying random sets to spatial modeling is the shortage of convenient, flexible models of random sets that can provide various geometric shapes [5]. Lack of dependence concepts in the current literature aggravates this problem. The goal of this note is to develop tools to study the dependence properties of random closed sets.

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Some natural random sets, such as the union-stable set and Gaussian random set, emerge as the limits of properly normalized set operations of i.i.d. random sets [5]. Another widely used model of random sets is the Boolean germ-grain process, where germs are modelled by a spatial Poisson process (or a Markov point process for germs’ interactions, see, e.g., [10]), and the grain associated with a generic germ is a random closed set. While the Boolean model provides suitable examples of random sets [9, 6], the local dependence structure within each grain are left unexplored. Our motivation for dependence analysis of random closed sets stems from modeling spatial defects/damages in the yield analysis for nanoelectronics and risk management of supply-chain networks, where data show that spatial defects/damages, which are modeled by random sets, are dependent not only among their locations but also in their shapes and sizes [1, 2, 8, 11].

In this note, we introduce several notions to compare dependence strengths of two vectors of random closed sets and derive their closure properties under various set operations. Roughly speaking, a vector of random closed sets is said to be more hit-dependent (miss-dependent) than another vector of random closed sets if the former random closed sets are more likely than the latter random closed sets to simultaneously hit (simultaneously miss) a given set of compact sets. These dependence notions and their properties are natural extensions of the upper orthant and lower orthant dependence orders between two vectors of random variables, which have shown to be a powerful tool to study dependence structure of multivariate stochastic systems [7, 12, 4]. We then introduce several spatial reliability measures in a Boolean germ-grain model in which each grain (aggregated defect) is comprised of a set of dependent sub-grains (sub-defects). Using the hit/miss dependence notions and properties, we show how the local interactions of sub-grains affect spatial reliability measures and obtain their performance bounds, which can be calculated explicitly. Our results contribute to the literature as it is the first attempt to introduce dependence notions for spatial systems featuring dependent random closed sets and study their properties.

The rest of the paper is organized as follows. Section 2 reviews relevant backgrounds. Section 3 introduces several notions of dependence orders of random closed sets and studies their closure properties under monotone set operations. Section 4 studies how local dependence would affect the spatial reliability indexes of Boolean germ-grain models and develops the upper and lower bounds for these performance indexes. Some comments in Section 5 conclude the paper. For expositional convenience, all the proofs are presented in Appendix. Throughout this paper, the terms “increasing” and “decreasing” are used in the weak sense, the measurability of a function or a set is often assumed without explicit mention, and \( I\{B\} \) denotes the indicator function of set \( B \).

2 Preliminaries

While the basic setting space can be quite general, we restrict our attention to the Euclidean space \( \mathbb{R}^d \). Let \( \mathcal{S} \) denote the class of all closed sets of \( \mathbb{R}^d \) with the hit-or-miss topology \( T_\mathcal{S} \) (or equivalently the Fell topology, see [5]). Let \( \Sigma_\mathcal{S} \) denote the \( \sigma \)-algebra generated by the sets in \( T_\mathcal{S} \). A random
Thus, two random closed sets $\mathcal{Y}$ is a random element taking values in $(3, \Sigma_3)$. Examples of random closed sets include random points, random closed balls, among many others. The distribution of a random closed set $\mathcal{Y}$ is described by the corresponding probability measure induced on $\Sigma_3$, or equivalently, by its Choquet capacity functional defined by

$$T_{\mathcal{Y}}(U) := \Pr\{\mathcal{Y} \cap U \neq \emptyset\}, \text{ for any compact set } U \subset \mathbb{R}^d. \tag{2.1}$$

Thus, two random closed sets $\mathcal{Y}$ and $\mathcal{Y}'$ have the same distribution if and only if $\Pr\{\mathcal{Y} \cap U \neq \emptyset\} = \Pr\{\mathcal{Y}' \cap U \neq \emptyset\}$ for any compact set $U$. In the case where $d = 1$ and $\mathcal{Y} = (\infty, R]$ ($\mathcal{Y} = [R, \infty)$) for some random variable $R$, the capacity functional of $\mathcal{Y}$ is simply the survival (distribution) function of $R$: $T_{\mathcal{Y}}(U) = \Pr\{\mathcal{Y} \cap U \neq \emptyset\} = \Pr\{R \geq \inf U\} \Pr\{R \leq \sup U\}$, for any compact set $U \subset \mathbb{R}$.

The correlation of two random closed sets $\mathcal{F}_1$ and $\mathcal{F}_2$ can be described by using the spatial covariance function $\text{Cov}(x, y) := \Pr\{x \in \mathcal{F}_1, y \in \mathcal{F}_2\}$ $\mathbb{R}^d$. But, similar to the shortcomings in using the covariance of two random variables to analyze their dependence, the covariance function of two random closed sets can be misleading in dependence analysis. A more comprehensive approach is to extend the dependence notions of random vectors to random sets. Consider the simplest concept of orthant dependence order for random vectors $\mathbb{R}$. A random vector $R = (R_1, \ldots, R_n)$ is said to be more positively upper (lower) orthant dependent than random vector $R' = (R'_1, \ldots, R'_n)$, denoted by $R \geq_{\text{pod}} (\geq_{\text{plod}}) R'$, if $R_i$ and $R'_i$ have the same marginal distribution, $1 \leq i \leq n$, and for all real numbers $r_1, \ldots, r_n$ in $\mathbb{R}$,

$$\Pr\{R_1 > r_1, \ldots, R_n > r_n\} \geq \Pr\{R'_1 > r_1, \ldots, R'_n > r_n\}, \tag{2.2}$$

$$\Pr\{R_1 \leq r_1, \ldots, R_n \leq r_n\} \geq \Pr\{R'_1 \leq r_1, \ldots, R'_n \leq r_n\}. \tag{2.3}$$

If both (2.2) and (2.3) hold with identical margins, then we say $R$ is more positively orthant dependent than $R'$, denoted by $R \geq_{\text{pod}} R'$. For example, if $R$ and $R'$ have standard multivariate normal distributions with correlation matrices $(\rho_{ij})$ and $(\tau_{ij})$ respectively, then $\rho_{ij} \geq \tau_{ij}$ for all $i, j$ implies that $R \geq_{\text{pod}} R'$. It follows from the Pickands representation (see, e.g., Theorem 6.3 in $\mathbb{R}$) that any multivariate extreme value distribution is positively orthant dependent, and thus the orthant dependence naturally emerges as a limiting dependence property of multivariate extremes of properly scaled vector-valued random samples. In addition, orthant dependence order between two random vectors is preserved under marginal monotone transforms, as stated as follows. Its proof can be found in Chapter 9 of $\mathbb{R}$.

**Lemma 2.1.** 1. If $R \geq_{\text{pod}} R'$, then $(f_1(R_1), \ldots, f_n(R_n)) \geq_{\text{pod}} (f_1(R'_1), \ldots, f_n(R'_n))$ for all increasing (decreasing) functions $f_1, \ldots, f_n$.

2. If $R \geq_{\text{plod}} R'$, then $(f_1(R_1), \ldots, f_n(R_n)) \geq_{\text{plod}} (f_1(R'_1), \ldots, f_n(R'_n))$ for all increasing (decreasing) functions $f_1, \ldots, f_n$.

3. If $R \geq_{\text{pod}} R'$, then $(f_1(R_1), \ldots, f_n(R_n)) \geq_{\text{pod}} (f_1(R'_1), \ldots, f_n(R'_n))$ for any functions $f_1, \ldots, f_n$ that are either all increasing or all decreasing.

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Let $\mathcal{Y}_i = (-\infty, R_i]$, and $\mathcal{Y}_i' = (-\infty, R_i']$, $1 \leq i \leq n$. Then $R \geq_{\text{pod}} R'$ is equivalent to $\Pr\{\mathcal{Y}_i \cup U_i \neq \emptyset, 1 \leq i \leq n\} \geq \Pr\{\mathcal{Y}_i' \cup U_i \neq \emptyset, 1 \leq i \leq n\}$, for any compact set $U \subset \mathbb{R}$. In words, it is more likely for random sets $\mathcal{Y}_i$'s to hit any compact set simultaneously.

3 Hit-or-Miss Dependence of Random Sets

A random closed subset of $\mathbb{R}^d$ is denoted by a calligraphical letter, say $\mathcal{Y}$, and its realization is denoted by a regular capital letter, say $Y$. For any subset $B \subseteq \mathbb{R}^d$, $-B$, the reflection of $B$, is denoted by $\bar{B}$. Any spatial point, deterministic or random, is denoted by a lowercase letter, and for any subset $B \subseteq \mathbb{R}^d$, $\overline{B}$, the closed (open) ball in $\mathbb{R}^d$ centered at $x$ with radius $r > 0$ is represented by $\mathcal{B}[x, r]$ ($\mathcal{B}(x, r)$). Let $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_n)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_n)$ be two vectors of random closed subsets in $\mathbb{R}^d$.

**Definition 3.1.**
1. $\mathcal{F}$ is said to be more hit-dependent than $\mathcal{F}'$, denoted by $\mathcal{F} \geq_{\text{hit}} \mathcal{F}'$, if $\mathcal{F}_i$ and $\mathcal{F}'_i$ have the same distribution for $i = 1, \ldots, n$, and for all compact subsets $\{U_i\}$ of $\mathbb{R}^d$,
   $$\Pr\{\mathcal{F}_i \cap U_i \neq \emptyset, 1 \leq i \leq n\} \geq \Pr\{\mathcal{F}'_i \cap U_i \neq \emptyset, 1 \leq i \leq n\}.$$
2. $\mathcal{F}$ is said to be more miss-dependent than $\mathcal{F}'$, denoted by $\mathcal{F} \geq_{\text{miss}} \mathcal{F}'$, if $\mathcal{F}_i$ and $\mathcal{F}'_i$ have the same distribution for $i = 1, \ldots, n$, and for all compact subsets $\{U_i\}$ of $\mathbb{R}^d$,
   $$\Pr\{\mathcal{F}_i \cap U_i = \emptyset, 1 \leq i \leq n\} \geq \Pr\{\mathcal{F}'_i \cap U_i = \emptyset, 1 \leq i \leq n\}.$$
3. $\mathcal{F}$ is said to be more hit-or-miss dependent than $\mathcal{F}'$, denoted by $\mathcal{F} \geq_{\text{hit-miss}} \mathcal{F}'$, if $\mathcal{F} \geq_{\text{hit}} \mathcal{F}'$ and $\mathcal{F} \geq_{\text{miss}} \mathcal{F}'$.

Intuitively, $\mathcal{F} \geq_{\text{hit}} (\geq_{\text{miss}}, \geq_{\text{hit-miss}}) \mathcal{F}'$ means that the subsets of $\mathcal{F}$ are more likely to hit (miss, hit-or-miss) different parts of $\mathbb{R}^d$ simultaneously. When $d = 1$, let $\mathcal{F}_i = (-\infty, R_i]$ and $\mathcal{F}'_i = (-\infty, R_i']$, $1 \leq i \leq n$, for some random variables $R_i$s and $R_i'$s. Then $\mathcal{F} \geq_{\text{hit}} (\geq_{\text{miss}}, \geq_{\text{hit-miss}}) \mathcal{F}'$ if and only if $(R_1, \ldots, R_n) \geq_{\text{pod}} (\geq_{\text{pod}}) (R'_1, \ldots, R'_n)$. A multi-dimensional example will be given later in Example 3.4.

In general, the hit and miss dependence orders are two different notions, except for the case when $n = 2$, where $(\mathcal{F}_1, \mathcal{F}_2) \geq_{\text{hit}} (\mathcal{F}'_1, \mathcal{F}'_2)$ if and only if $(\mathcal{F}_1, \mathcal{F}_2) \geq_{\text{miss}} (\mathcal{F}'_1, \mathcal{F}'_2)$, due to the fact that $\Pr\{\mathcal{F}_1 \cap U_1 \neq \emptyset, \mathcal{F}_2 \cap U_2 \neq \emptyset\} = 1 - \Pr\{\mathcal{F}_1 \cap U_1 = \emptyset\} - \Pr\{\mathcal{F}_2 \cap U_2 = \emptyset\} + \Pr\{\mathcal{F}_1 \cap U_1 = \emptyset, \mathcal{F}_2 \cap U_2 = \emptyset\}$ for any two compact subsets $\{U_1, U_2\}$.

Denote $\mathcal{F}_J := \{\mathcal{F}_j, j \in J\}$ and $\mathcal{F}'_J := \{\mathcal{F}'_j, j \in J\}$, for any $J \subseteq \{1, \ldots, n\}$. Obviously, if $\mathcal{F} \geq_{\text{hit}} (\geq_{\text{miss}}, \geq_{\text{hit-miss}}) \mathcal{F}'$, then $\mathcal{F}_J \geq_{\text{hit}} (\geq_{\text{miss}}, \geq_{\text{hit-miss}}) \mathcal{F}'_J$, for any $J \subseteq \{1, \ldots, n\}$. Thus all three set dependence orders introduced in Definition 3.1 are preserved under marginalization. It is also easy to see that those orders are transitive and reflective. The equivalence property of those partial orders is established in the following proposition.

**Proposition 3.2.** If $\mathcal{F} \geq_{\text{hit}} (\geq_{\text{miss}}, \geq_{\text{hit-miss}}) \mathcal{F}'$ and $\mathcal{F} \geq_{\text{hit}} (\geq_{\text{miss}}, \geq_{\text{hit-miss}}) \mathcal{F}$, then $\cup_{i=1}^n \mathcal{F}_i$ and $\cup_{i=1}^n \mathcal{F}'_i$ have the same distribution.
Note that if $\mathcal{F} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{F}')$, then for any $z_j \in \mathbb{R}^d$, $j = 1, \ldots, n$,
\[
\Pr\{z_j \in \mathcal{F}_j, 1 \leq j \leq n\} \geq \Pr\{z_j \in \mathcal{F}_j', 1 \leq j \leq n\} \quad (\Pr\{z_j \notin \mathcal{F}_j, 1 \leq j \leq n\} \geq \Pr\{z_j \notin \mathcal{F}_j', 1 \leq j \leq n\}).
\]
Since $\mathcal{F} \geq_{\text{hit}} \mathcal{F}'$ is equivalent to $\mathcal{F} \geq_{\text{miss}} \mathcal{F}'$ for $n = 2$,
\[
\mathcal{F} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{F}') \iff \Pr\{z_i \in \mathcal{F}_i, z_j \in \mathcal{F}_j\} \geq \Pr\{z_i \in \mathcal{F}_i', z_j \in \mathcal{F}_j\}, \text{ for any } i \neq j.
\]
That is, the spatial pairwise covariance function of $\mathcal{F}$ is larger than that of $\mathcal{F}'$.

Various set-theoretic operations on random closed sets yield composite closed sets. For any two random closed sets $\mathcal{Y}$ and $\mathcal{Y}'$, the union $\mathcal{Y} \cup \mathcal{Y}'$ and the intersection $\mathcal{Y} \cap \mathcal{Y}'$ are also random closed sets. It is also clear that scaling of $\mathcal{Y}$ with respect to a scaling parameter $r \neq 0$, i.e., $r\mathcal{Y} = \{ry : y \in \mathcal{Y}\}$, and translation of $\mathcal{Y}$ with respect to a location point $y'$, i.e., $\mathcal{Y} + y' = \{y + y' : y \in \mathcal{Y}\}$, are all random closed sets. The Minkowski-sum of two sets $\mathcal{Y}$ and $\mathcal{Y}'$ is defined as
\[
\mathcal{Y} \oplus \mathcal{Y}' := \{y + y' : y \in \mathcal{Y}, y' \in \mathcal{Y}'\} = \cup_{y' \in \mathcal{Y}'}(\mathcal{Y} + y') = \cup_{y \in \mathcal{Y}}(y + \mathcal{Y}').
\]
(3.1)

The Minkowski-sum is a closed set if both $\mathcal{Y}$ and $\mathcal{Y}'$ are closed sets. For example, if $\mathcal{Y}' = \mathcal{B}[0, R]$, then $\mathcal{Y} \oplus \mathcal{B}[0, R]$ is the union of all location points that are of distance no more than $R$ from $\mathcal{Y}$. The Minkowski subtraction is defined as
\[
\mathcal{Y} \ominus \mathcal{Y}' := \{x : x + \mathcal{Y}' \subseteq \mathcal{Y}\} = \cap_{y' \in \mathcal{Y}'}(\mathcal{Y} - y') = (\mathcal{Y}^c \oplus \mathcal{Y}')^c.
\]
(3.2)
The Minkowski subtraction is closed if $\mathcal{Y}$ is closed and $\mathcal{Y}'$ is open. For example, if $\mathcal{Y}' = \mathcal{B}(0, R)$, then $\mathcal{Y} \ominus \mathcal{B}(0, R)$ is the union of all location points within $\mathcal{Y}$ that are of distance more than $R$ from $\mathcal{Y}^c$. The union (intersection) of random closed sets generalizes the notion of the maximum (minimum) of random variables, whereas the Minkowski sum generalizes the notion of the sum of random variables. Note, however, that the Minkowski subtraction is not an inverse of Minkowski addition in the sense that $(\mathcal{Y} \oplus \mathcal{Y}') \ominus \mathcal{Y}' \neq \mathcal{Y}$.

The preservation of the hit-or-miss dependence orders under these monotone set operations is summarized below.

**Proposition 3.3.** Let $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_n)$ and $\mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_n)$ be two independent vectors of random closed sets, and $\mathcal{F}' = (\mathcal{F}_1', \ldots, \mathcal{F}_n')$ and $\mathcal{G}' = (\mathcal{G}_1', \ldots, \mathcal{G}_n')$ be two independent vectors of random closed sets.

1. $\mathcal{F} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{F}')$ if and only if, for any non-zero real numbers $a_1, \ldots, a_n$ and points $x_1, \ldots, x_n$ in $\mathbb{R}^d$,
\[
(a_1\mathcal{F}_1 + x_1, \ldots, a_n\mathcal{F}_n + x_n) \geq_{\text{hit}}^{\text{hit}} (\mathcal{F}_1' + x_1, \ldots, a_n\mathcal{F}_n' + x_n).
\]
In particular, $\mathcal{F} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{F}')$ if and only if $\mathcal{F} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{F}')$.

2. If $\mathcal{F} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{F}')$ and $\mathcal{G} \geq_{\text{hit}}^{\text{hit}} (\geq_{\text{miss}} \mathcal{G}')$, then
Proposition 3.3 (1) that

where \(d\) for any \(\{\text{closed sets with flexible dependence structures}\}\). Let

The hit-or-miss dependence notions can be applied to germ-grain models to construct random smaller than the aggregate defect \(\cup\) is less likely than the latter to damage the critical region \(U\) is less likely than the latter to damage the critical region.

3. Let \(R\) and \(R'\) be two vectors of nonnegative random variables independent of \(F\) and \(F'\). If \(F \geq \overset{\text{hit}}{\geq} (\overset{\text{hit}}{\geq}_{\text{miss}}, \overset{\text{hit}}{\geq}_{\text{miss}}) F'\), and \(R \geq \overset{\text{prod}}{\geq} (\overset{\text{prod}}{\geq}_{\text{pod}}, \overset{\text{prod}}{\geq}_{\text{pod}}) R'\), then

\[ (F_1 \ominus B(0, R_1), \ldots, F_n \ominus B(0, R_n)) \overset{\text{hit}}{\geq} (\overset{\text{hit}}{\geq}_{\text{miss}}, \overset{\text{hit}}{\geq}_{\text{miss}}) (F'_1 \ominus B(0, R'_1), \ldots, F'_n \ominus B(0, R'_n)). \]

If spatial damages are modeled by random sets, then by Proposition 3.3 (1), (2b), and (2c), more correlated damage growths lead to more correlated cumulative damages, and by Proposition 3.3 (1), (2a) and (3), more correlated recoveries result in more correlated, reduced damages, all in the sense of hit-or-miss dependence orders.

Example 3.4. Let \(c = \{c_j\} \subseteq \mathbb{R}^d\), and \(B(c, R) = \{B[c_j, R_j]\}\) and \(B(c, R') = \{B[c_j, R'_j]\}\) be two vectors of random closed balls. For any compact subsets \(U_j\) of \(\mathbb{R}^d\), we have

\[
\Pr\{B[c_j, R_j] \cap U_j \neq \emptyset, 1 \leq j \leq n\} = \Pr\{R_j \geq d_S(c_j, U_j), 1 \leq j \leq n\},
\]

\[
\Pr\{B[c_j, R_j] \cap U_j = \emptyset, 1 \leq j \leq n\} = \Pr\{R_j < d_S(c_j, U_j), 1 \leq j \leq n\},
\]

where \(d_S(x, D)\) denotes the shortest distance between a point \(x\) and a set \(D\). Thus it follows from Proposition 3.3 (1) that

\[
R \geq \overset{\text{prod}}{\geq} (\overset{\text{prod}}{\geq}_{\text{pod}}, \overset{\text{prod}}{\geq}_{\text{pod}}) R' \iff B(c, R) \geq \overset{\text{hit}}{\geq} (\overset{\text{hit}}{\geq}_{\text{miss}}, \overset{\text{hit}}{\geq}_{\text{miss}}) B(c, R')
\]

\[ \iff B(0, R) \geq \overset{\text{hit}}{\geq} (\overset{\text{hit}}{\geq}_{\text{miss}}, \overset{\text{hit}}{\geq}_{\text{miss}}) B(0, R'). \quad (3.3) \]

Consider the implication of (3.3) to the miss-order. Setting \(U_j = U, 1 \leq j \leq n\), we have

\[ B(c, R) \overset{\text{miss}}{\geq} B(c, R') \implies \Pr\{\cup_{j=1}^n B(c_j, R_j) \cap U = \emptyset\} \geq \Pr\{\cup_{j=1}^n B(c_j, R'_j) \cap U = \emptyset\}, \]

for any \(\{c_j\}\) in \(\mathbb{R}^d\). If the set union \(\cup_{j=1}^n B(c_j, R_j)\) is viewed as the aggregate defect of sub-defects \(\{B(c_j, R_j)\}\) and \(U\) as the critical region whose intactness assures an underlying operating system to function properly, then \(R \geq \overset{\text{prod}}{\geq} R'\) implies that the aggregate defect \(\cup_{j=1}^n B(c_j, R_j)\) is “stochastically smaller” than the aggregate defect \(\cup_{j=1}^n B(c_j, R'_j)\) for any \(\{c_j\}\) in \(\mathbb{R}^d\), in the sense that the former is less likely than the latter to damage the critical region \(U\).

\[ \square \]

4 Dependence Comparison of Boolean Germ-Grain Models

The hit-or-miss dependence notions can be applied to germ-grain models to construct random closed sets with flexible dependence structures. Let \(\{(x_i, Y_i), i = 1, 2, \ldots\}\) be a marked spatial point process, where \(x_i \in \mathbb{R}^d\) is called a germ and the random subset \(Y_i\) is called the grain associated
with the germ \( x_i \). We assume that \( \{ Y_i, i = 1, 2, \ldots \} \) is a sequence of i.i.d. copies of a random closed subset \( Y \subseteq \mathbb{R}^d \), independent of the point process \( \Phi = \{ x_1, x_2, \ldots \} \). The random closed set \( x_i + Y_i \) is the spatial translation of grain \( Y_i \) via germ shift \( x_i, i = 1, 2, \ldots \).

We assume that the germ process \( \Phi \) follows a spatial (non-homogeneous) Poisson process with intensity measure \( \Lambda(\cdot) \). The Poisson germ process \( \Phi \), coupled with a grain process \( \{ Y_i, i = 1, 2, \ldots \} \), results in a Boolean model \([9]\). Although the Boolean germ-grain model lacks interactions among conditionally independently scattered germs, it does allow local dependence within each grain. In a more general setting, \( \Phi \) can be any spatial point process (e.g., Markov point process \([10]\)) that satisfies some regularity conditions. The Boolean model, however, allows us to obtain explicit expressions for the capacity functional and other spatial reliability performance indexes.

To describe local interactions within each grain, we assume the generic grain \( Y \) consists of a random number of sub-grains \( Y = (Z_1, \ldots, Z_N) \) where \( 0 \leq N \leq n \) is an integer-valued random variable, independent of random closed subsets \( (Z_1, Z_2, \ldots, Z_n) \). To avoid ambiguity of spatial labeling, we assume that the distribution of \( (Z_1, Z_2, \ldots, Z_n) \) is permutation-symmetric. For example, let \( Z_j = B[c_j, R_j], j = 1, 2, \ldots, n \), where the ball centers \( \{ c_j \} \) are i.i.d. and uniformly scattered within open ball \( B(0, r) \) and radii \( \{ R_j \} \) have a symmetric joint distribution and are independent of \( \{ c_j \} \) and \( N \). Note that uniform scattering of ball centers \( \{ c_j \} \) within a close neighborhood of 0 resembles the idea of the Matérn process (see, e.g., \([10]\)).

Let \( Y_i = (Z_{1i}, \ldots, Z_{Ni}), i \geq 1 \), be i.i.d. having the same distribution as that of the generic grain \( Y \). The union of all random closed sets in the Boolean germ-grain model

\[
D = \bigcup_{x_i \in \Phi} (x_i + Y_i) = \bigcup_{x_i \in \Phi} (x_i + \bigcup_{j=1}^{N_i} Z_{ij})
\]  

(4.1)

can be used to represent aggregated defects in spatial reliability modeling. We are interested in the following probabilities that spatial defects damage a deterministic compact set \( A \subset \mathbb{R}^d \), which may represent, for example, some network structure.

1. **Spatial Damages**: The probability that set \( A \) is not infected by the cumulative damage \( D \) is defined by

\[
R_D(A) = \Pr\{ A \cap D = \emptyset \} = 1 - T_D(A),
\]  

(4.2)

where \( T_D(A) \), the capacity functional for \( A \), is given by

\[
T_D(A) = \Pr\left\{ \bigcup_{x_i \in \Phi} \left( \bigcup_{j=1}^{N_i} (A \cap (x_i + Z_{ij})) \neq \emptyset \right) \right\}.
\]  

(4.3)

Recall that \( d_S(x, D) \) be the shortest distance between \( x \) and \( D \), then the probability that there is no damage within the distance of \( r \) from a location \( x \) is given by

\[
\bar{H}_x(r) = \Pr\{ d_S(x, D) > r \} = 1 - H_x(r), \quad r \geq 0,
\]  

(4.4)

where

\[
H_x(r) = \Pr\{ D \cap B[x, r] \neq \emptyset \} = T_D(B[x, r])
\]  

(4.5)

is called the (spherical) contact distribution function \([9]\).
2. Simultaneous Damages: The probability that set $A$ is infected simultaneously by every sub-grain from a grain is given by

$$T_D^S(A) = \Pr \left\{ \bigcup_{x_i \in \Phi} \{ \bigcap_{j=1}^N [A \cap (x_i + \mathcal{Z}_{ij})] \neq \emptyset \} \right\}.$$  

(4.6)

Since the point pattern $\Phi$ follows a spatial Poisson process with intensity measure $\Lambda$, the indexes (4.2)-(4.6) can be expressed explicitly in terms of the generic grain $\mathcal{Y}$ and set $A$. It follows from the capacity functional formula for Boolean models (see, e.g., [9]) that

$$R_D(A) = \exp(-E\Lambda(\mathcal{Y} \oplus A)) = \exp(-E\Lambda((\bigcup_{j=1}^N \mathcal{Z}_j) \oplus A)).$$  

(4.7)

The expression of $\bar{H}_x(r)$ can also be obtained from (4.7), with $A$ being replaced by $\mathcal{B}[x,r]$. The expression of $T_D^S(A)$ can be obtained by thinning as follows.

**Proposition 4.1.** Let $(\Phi, \mathcal{Y})$ be a Boolean model as defined above, where $\Phi$ is a homogeneous Poisson process with intensity $\lambda$. For any compact set $A$, we have,

$$T_D^S(A) = 1 - \exp \left( -\lambda E\nu(\bigcap_{j=1}^N (\mathcal{Z}_j \oplus A)) \right)$$

$$= 1 - \exp \left( E \sum_{\emptyset \neq J \subseteq \{1, \ldots, N\}} (-1)^{|J|-1} \lambda E\nu(\mathcal{Y}_J \oplus A) \right),$$

(4.8)

where $\mathcal{Y}_J = \{ \mathcal{Z}_j : j \in J \}$ for $J \subseteq \{1, \ldots, n\}$, and $\nu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Notice from (4.7)-(4.8) that those spatial reliability measures depend on $E\nu(\mathcal{Y}_J \oplus A)$, $J \subseteq \{1, \ldots, n\}$. In general, $E\nu(\mathcal{Y}_J \oplus A)$ is analytically intractable, especially when $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ are dependent random sets. Computationally, it is known that if $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ and $A$ are polyconvex sets (finite unions of convex sets), then $E\nu(\mathcal{Y}_J \oplus A)$ can be evaluated via the *Steiner formula* (see, e.g., [9]). If, in particular, $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ are random balls, then the evaluation of $E\nu(\mathcal{Y}_J \oplus A)$ boils down to determining the expectations of Minkowski functionals on finite unions of random balls, which, in the case of small $n$, reduces to numerically tractable integrations. Unfortunately, numerical evaluations of (4.7)-(4.8) shed little insight on how dependence among sub-grains would affect the system performance. For this, dependence analysis on random closed sets is needed.

Consider two Boolean models $(\Phi, \mathcal{Y})$ and $(\Phi', \mathcal{Y}')$ with the same germ pattern $\Phi$ and generic defects, $\mathcal{Y} = (\mathcal{Z}_1, \ldots, \mathcal{Z}_N)$ and $\mathcal{Y}' = (\mathcal{Z}'_1, \ldots, \mathcal{Z}'_N)$, respectively. Let $R_D(A)$, $\bar{H}_x(s)$ and $T_D^S(A)$ ($R_D'(A)$, $\bar{H}'_x(s)$ and $T_D^S'(A)$) denote the spatial reliability indexes of $(\Phi, \mathcal{Y})$ ($(\Phi', \mathcal{Y}')$) given by (4.2), (4.4) and (4.6), respectively. The following theorem states that, if the sub-grains within each grain are more miss-dependent, then the probability that set $A$ is not infected by the aggregated damage $D$ is higher, whereas if the sub-grains within each grain are more hit-dependent, then the probability that set $A$ is simultaneously infected by all sub-grains of each grain is higher.

**Theorem 4.2.** 1. If $\mathcal{Y} \geq_{\text{miss}} \mathcal{Y}'$, then $R_D(A) \geq R_D'(A)$ and $\bar{H}_x(r) \geq \bar{H}'_x(r)$.
2. If \( \mathcal{V} \geq \text{hit} \mathcal{V} \), then \( T_D^S(A) \geq T_D^S(A) \).

3. If \( \mathcal{V} \geq \text{miss} \mathcal{V} \), then \( R_D(A) \geq R_D(A) \) and \( T_D^S(A) \geq T_D^S(A) \).

The above theorem and Proposition 3.3 lead to the following corollary, which shows how dependent growth of spatial defect via Minkowski-addition and dependent recovery of spatial defect via Minkowski-subtraction would impact spatial reliability measures.

**Corollary 4.3.** 1. Let \( \mathcal{V} = (Z_1 \oplus B[0, R_1], \ldots, Z_n \oplus B[0, R_n]) \) and \( \mathcal{V}' = (Z_1 \oplus B[0, R_1'], \ldots, Z_n \oplus B[0, R_n']) \). If \( R \geq \text{plod} (\geq \text{plod}) R' \), then \( R_D(A) \geq R_D(A) \) and \( H_x(s) \geq H_x(s) \) \( (T_D^S(A) \geq T_D^S(A)) \).

2. Let \( \mathcal{V} = (Z_1 \oplus B(0, R_1), \ldots, Z_n \oplus B(0, R_n)) \) and \( \mathcal{V}' = (Z_1 \oplus B(0, R_1'), \ldots, Z_n \oplus B(0, R_n')) \). If \( R \geq \text{plod} (\geq \text{plod}) R' \), then \( R_D(A) \geq R_D(A) \) and \( H_x(s) \geq H_x(s) \) \( (T_D^S(A) \geq T_D^S(A)) \).

Some easily computable bounds for spatial reliability indexes can be also derived. For expositional simplicity, let \( N_i = n, 1 \leq i \leq n \). Let \( R_D(A) \) and \( T_D^S(A) \) denote the reliability indexes of \( (\Phi, \mathcal{V}) \) described by (4.2) and (4.6), respectively, with \( N = n \). Recall that random sets \( \mathcal{V}' = (Z'_1, \ldots, Z'_n) \) are independent if and only if \( \Pr \{ Z'_j \cap U_j \neq \emptyset, 1 \leq j \leq n \} = \prod_{j=1}^{n} \Pr \{ Z'_j \cap U_j \neq \emptyset \} \), for all compact sets \( U_1, \ldots, U_n \) (see, e.g., [9]).

**Theorem 4.4.** Consider a Boolean model \((\Phi, \mathcal{V})\) as defined above, where \( \Phi \) is a homogeneous Poisson process with intensity \( \lambda \). Let \( \mathcal{V}' = (Z'_1, \ldots, Z'_n) \) be independent, and \( Z'_j \) and \( Z_j \) be identically distributed for all \( j \).

1. Denote \( \nu(Z_{[n]}):= \max_{1 \leq j \leq n} \{ \nu(Z_j) \} \). If \( \mathcal{V} \geq \text{miss} \mathcal{V} \), then
   \[
   \exp \{ -\lambda E\nu \big( \check{Z}_{[n]} + A \big) \} \geq R_D(A) \geq \exp \{ -\lambda E\nu \big( \cup_{j=1}^{n} \check{Z}'_j + A \big) \} \geq \prod_{j=1}^{n} \exp \{ -\lambda E\nu \big( \check{Z}_j + A \big) \}. \tag{4.9}
   \]

2. Denote \( \nu(Z_{[1]}):= \min_{1 \leq j \leq n} \{ \nu(Z_j) \} \). If \( \mathcal{V} \geq \text{hit} \mathcal{V} \), then
   \[
   1 - \exp \{ -\lambda E\nu \big( \check{Z}_{[1]} + A \big) \} \geq T_D^S(A) \geq 1 - \exp \{ -\lambda E\nu \big( \cap_{j=1}^{n} \check{Z}'_j + A \big) \}. \tag{4.10}
   \]

The quality of the bounds developed in Theorem 4.4 is examined in the following example.

**Example 4.5.** Suppose \( Z_j = B[c_j, R_j], Z'_j = B[c_j, R'_j] \), where \( c_j \in B(0, r) \). Note that \( \lim_{r \to 0} \cup_{j=1}^{n} Z_j = B(0, R_{[n]}) \), where \( R_{[n]} = \max_{1 \leq j \leq n} \{ R_j \} \). This indicates that the upper bound of \( R_D(A) \) in (4.9) is tight when the radius \( r \) of ball \( B(0, r) \) is sufficiently small. The product-form lower bound given in (4.9) is easy to compute, since it requires less effort to evaluate \( E\nu \big( B[c_j, R'_j] + A \big) \) than \( E\nu \big( \cup_{j=1}^{n} B[c_j, R_j'] \big) + A \). Yet the product-form bound is rather coarse if the radius \( r \) of \( B(0, r) \) is small relative to the values of \( R'_j \). To ease computational effort in evaluating the tighter lower bound \( (4.9) \), let \( B[0, L_{[n]}'] \) be the smallest ball centered at \( 0 \) that contains \( \cup_{j=1}^{n} Z'_j \). Then we have
   \[
   \exp \{ -\lambda E\nu \big( B[0, L_{[n]}'] + A \big) \} \leq R_D(A) \leq \exp \{ -\lambda E\nu \big( B[0, R_{[n]}] + A \big) \}. \tag{4.11}
   \]
where $L'_{[n]} = \max_{1 \leq j \leq n}\{d_S(0, c_j) + R'_j\}$. Since $\{R'_j\}$ are independent and also independent of $\{c_j\}$, the distribution of $L'_{[n]}$, and subsequently, $E\nu(\mathcal{B}[0, L'_{[n]}] \oplus A)$, can be obtained easily (see the next example). Observe that $\lim_{r \to 0} \mathcal{B}[0, L'_{[n]}] = \mathcal{B}[0, R'_{[n]}]$, where $R'_{[n]} = \max_{1 \leq j \leq n}\{R_j\}$. Thus when $r$ is sufficiently small and $\{R_j\}$ are not highly dependent, the lower and upper bounds in (4.11) are approximately equal.

Since $\lim_{r \to 0} \cap_{j=1}^n \tilde{Z}_j = \lim_{r \to 0} \mathcal{Z}_{[1]} = \mathcal{B}[0, R_{[1]}], \text{ where } R_{[1]} = \min_{1 \leq j \leq n}\{R_j\}$, both the upper and lower bounds of $T_D(A)$ are of good quality when the radius $r$ of $\mathcal{B}(0, r)$ is sufficiently small. Similar to (4.11), we can ease the calculation of (4.10) by introducing ball $\mathcal{B}[c, L'_{[1]}]$, which is the largest ball that is contained in $\cap_{j=1}^n \tilde{Z}'_j$, and we have

$$1 - \exp\left\{-\lambda E\nu \left(\mathcal{B}[c, L'_{[1]}] \oplus A\right)\right\} \leq T_D^{\mathcal{B}}(A) \leq 1 - \exp\left\{-\lambda E\nu \left(\mathcal{B}[0, R_{[1]}] \oplus A\right)\right\}. \tag{4.12}$$

The lower and upper bounds of (4.12) approach to the same value when $r \to 0$. □

The above example shows that when $(Z_1, \ldots, Z_n)$ is a vector of random balls, then all the bounds in (4.11) and (4.12) only involve $E\nu(\mathcal{B} \oplus A)$, where $\mathcal{B}$ is a random ball, that can be calculated explicitly for any polyconvex set $A$ via the Steiner formula

$$\nu(\mathcal{B}[0, r] \oplus A) = \sum_{k=0}^d \binom{d}{k} W_k(A)r^k,$$

where $W_k$s are the Minkowski functionals (see page 27 in [9]). In the two-dimensional case, $\nu(\mathcal{B}[0, r] \oplus A) = \text{AREA}(A) + \text{BL}(A)r + \pi r^2$, where $\text{AREA}(A)$ denotes the area of $A$, and $\text{BL}(A)$ the boundary length of $A$.

**Example 4.5 (continued).** We only discuss the case that $A$ is a polyconvex subset of $\mathbb{R}^2$.

1. Let $F_r$ be the distribution of $c$ on $\mathcal{B}(0, r)$. Using the Steiner formula, we obtain that,

$$\exp(-\lambda E\nu(\mathcal{B}(c, R) \oplus A)) = \int \exp(-\lambda E\nu(-\mathcal{B}[c, R] \oplus A))dF_r(c)$$

$$= \int \exp(-\lambda E\nu(-\mathcal{B}[0, R] \oplus (A + c)))dF_r(c)$$

$$= \int \exp(-\lambda (\text{AREA}(A + c) + 2E[R]\text{BL}(A + c) + 2\pi E[R^2]))dF_r(c),$$

$$= \exp(-\lambda (\text{AREA}(A) + 2E[R]\text{BL}(A) + 2\pi E[R^2])), \tag{4.13}$$

where the last equality follows from the invariance properties of the area and boundary. If $A = (V, E)$ is a finite graph representing a network, where $V$ is a set of nodes and $E$ consists of the edges connecting distinct nodes, then $\text{AREA}(A) = 0$, and (4.13) is reduced to $\exp(-\lambda E\nu(\mathcal{B}(c, R) \oplus A)) = \exp(-\lambda (4E[R]\text{L}(A) + 2\pi E[R^2]))$, where $\text{L}(A)$ is the total length of the edges in $E$.

2. Let $Z_j = \mathcal{B}[c_j, R_j]$, where $c_j$ is uniformly distributed within $\mathcal{B}(0, r)$, and let $(R_1, \ldots, R_n)$ have a symmetric distribution with finite marginal moments of any order. The explicit expressions
of the bounds in (4.11)-(4.12) follow immediately from the formulas obtained in the above. For example, the upper bound in (4.11) is given by

$$\exp\{-\lambda E\nu(\tilde{B}(0, R_{[n]} \oplus A))\} = \exp\left\{-\lambda \left(\text{AREA}(A) + 2E[R_{[n]}]\text{BL}(A) + 2\pi E[R_{[n]}^2]\right)\right\},$$

where \(R_{[n]} = \max_{1 \leq j \leq n}\{R_j\}\).

5 Concluding Remarks

We introduce in this note the hit, miss and hit-or-miss dependence notions of random closed sets, and study their properties in the context of the Boolean model. There may be several ways to describe dependence of random closed sets, but any natural dependence property of random sets should emerge from set operations, similar to the fact that the most useful notions of the expectation of a random closed set are only those that are associated with certain types of strong laws of large numbers for random sets [5]. It is well-known (see [3]) that orthant dependence naturally emerges as a limiting dependence property of component-wise maximums of vector-valued samples, and whether or not the hit-or-miss dependence would emerge as the dependence structure of certain limiting union-scheme of random closed sets is worth further investigation.

Appendix: Proofs

Proof of Proposition 3.2

We need to show that the capacity functionals of \(\bigcup_{i=1}^{n} F_i\) and \(\bigcup_{i=1}^{n} F'_i\) are the same under a given ordering. Consider first the miss dependence. If \(\mathcal{F} \geq_{\text{miss}} \mathcal{F}'\) and \(\mathcal{F}' \geq_{\text{miss}} \mathcal{F}\), we have, for any compact set \(U\),

$$\Pr\{\bigcup_{i=1}^{n} F_i \cap U \neq \emptyset\} = 1 - \Pr\{F_j \cap U = \emptyset, 1 \leq j \leq n\} = 1 - \Pr\{F'_j \cap U = \emptyset, 1 \leq j \leq n\} = \Pr\{\bigcup_{i=1}^{n} F'_i \cap U \neq \emptyset\}. \tag*{□}$$

If \(\mathcal{F} \geq_{\text{hit}} \mathcal{F}'\) and \(\mathcal{F}' \geq_{\text{hit}} \mathcal{F}\), then \(\mathcal{F}_J \geq_{\text{hit}} \mathcal{F}'_J\) and \(\mathcal{F}'_J \geq_{\text{hit}} \mathcal{F}_J\) for any \(J \subseteq \{1, \ldots, n\}\). Thus for any compact set \(U\),

$$\Pr\{\bigcup_{j=1}^{n} F_j \cap U \neq \emptyset\} = \sum_{\emptyset \neq J \subseteq \{1, \ldots, n\}} (-1)^{|J|-1} \Pr\{F_j \cap U \neq \emptyset, j \in J\}$$

$$= \sum_{\emptyset \neq J \subseteq \{1, \ldots, n\}} (-1)^{|J|-1} \Pr\{F'_j \cap U \neq \emptyset, j \in J\} = \Pr\{\bigcup_{j=1}^{n} F'_j \cap U \neq \emptyset\}. \tag*{□}$$

The distributional equivalence under the assumption that \(\mathcal{F} \geq_{\text{hit miss}} \mathcal{F}'\) and \(\mathcal{F}' \geq_{\text{hit miss}} \mathcal{F}\) follows from the fact that the order \(\geq_{\text{hit miss}}\) is stronger than both the orders \(\geq_{\text{hit}}\) and \(\geq_{\text{miss}}\). \(\tag*{□}\)
Proof of Proposition 3.3

Proof. (1) This part follows from the observations that \((a_j F_j + x_j) \cap U_j \neq \emptyset\) if and only if \(F_j \cap (U_j - x_j)/a_j \neq \emptyset\), and \((U_j - x_j)/a_j\) is a compact set for any compact subset \(U_j\).

(2a) Note that \(F_j \cap G_j\) and \(F'_j \cap G'_j\) have the identical marginals, \(1 \leq j \leq n\). The inequalities follow by conditioning on \(G = G = (G_1, \ldots, G_n)\), where \(\{G_j\}\) are compact sets.

(2b) It can be easily derived, by the independence of \(F\) and \(G\) and of \(F'\) and \(G'\), that

\[
\Pr \{ (F_j \cup G_j) \cap U_j = \emptyset, \forall j \} = \Pr \{ F_j \cap U_j = \emptyset, \forall j \} \Pr \{ G_j \cap U_j = \emptyset, \forall j \} \\
\geq \Pr \{ F'_j \cap U_j = \emptyset, \forall j \} \Pr \{ G'_j \cap U_j = \emptyset, \forall j \} = \Pr \{ (F'_j \cup G'_j) \cap U_j = \emptyset, \forall j \},
\]

which, together with identical margins, imply that the miss order is closed under the union of independent random sets. To establish the closure property for the hit order, it is sufficient to show that \(\Pr \{ (F_j \cup G_j) \cap U_j \neq \emptyset, \forall j \} \geq \Pr \{ (F'_j \cup G'_j) \cap U_j \neq \emptyset, \forall j \}\). For this, we condition on \(G = G = (G_1, \ldots, G_n)\), where \(\{G_j\}\) are compact sets. Let \(J_G = \{ j : G_j \cap U_j = \emptyset \}\), and then

\[
\Pr \{ (F_j \cup G_j) \cap U_j \neq \emptyset, \forall j \} = \Pr \{ (F_j \cap U_j \neq \emptyset \text{ or } G_j \cap U_j \neq \emptyset), \forall j \} \\
= \Pr \{ F_j \cap U_j \neq \emptyset, \forall j \in J_G \} \geq \Pr \{ F'_j \cap U_j \neq \emptyset, \forall j \in J_G \} \\
= \Pr \{ (F'_j \cup G_j) \cap U_j = \emptyset, \forall j \} \tag{5.1}
\]

where the inequality follows since the hit order is preserved under marginalization. Unconditioning on \(G\) leads to the desired result.

(2c) Without loss of generality, let \(F'\) and \(G\) be independent. It suffices to show that

\[
(F_1 \oplus G_1, \ldots, F_n \oplus G_n) \geq_{hit} (\geq_{miss}, \geq_{hit}) (F'_1 \oplus G_1, \ldots, F'_n \oplus G_n). \tag{5.2}
\]

First, since \(F_j\) and \(F'_j\) have the same distribution, and both are independent of \(G_j\), we can easily verify that \(F_j \oplus G_j\) and \(F'_j \oplus G_j\) have the same marginal distribution. Conditioning on \(G = (G_1, \ldots, G_n)\), where \(\{G_j\}\) are compact sets, it is easy to verify that for any compact subset \(U_j\),

\[
(F_j \oplus G_j) \cap U_j \neq \emptyset \text{ if and only if } F_j \cap (G_j \oplus U_j) \neq \emptyset. \tag{5.3}
\]

Note that \(G_j \oplus U_j\) is compact for any compacts \(G_j\) and \(U_j\), and it then follows from \(5.3\) that

\[
\Pr \{ (F_j \oplus G_j) \cap U_j \neq \emptyset, 1 \leq j \leq n \} \geq \Pr \{ (F'_j \cap (G_j \oplus U_j) \neq \emptyset, 1 \leq j \leq n \}.
\]

Unconditioning on \(G = (G_1, \ldots, G_n)\) yields \(5.2\) for the hit-order. The proofs for the other two orders are similar.

(3) The proof of this part depends on the following equivalence relation: for any compact set \(U_j\) and open ball \(B(0, r_j)\),

\[
(F_j \oplus B(0, r_j)) \cap U_j = \emptyset \text{ if and only if } F_j \cap (U_j \oplus B(0, r_j)) = \emptyset. \tag{5.4}
\]
To see this, recall that $\mathcal{F}_j \ominus \mathcal{B}(0, r_j) = (\mathcal{F}_j^c \ominus \mathcal{B}(0, r_j))^c = \cap_{b \in \mathcal{B}(0, r_j)}(\mathcal{F}_j - b)$, is the union of all location points within $\mathcal{F}_j$ that are of distance at least $r_j$ from $\mathcal{F}_j^c$. Note that

$$\{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U_j = \emptyset\}$$

(5.5)

$$= \{\mathcal{F}_j \cap U_j = \emptyset\} \cup \{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U_j = \emptyset, [\mathcal{F}_j \cap (\mathcal{F}_j \ominus \mathcal{B}(0, r_j))^c] \cap U_j \neq \emptyset\}.$$

Obviously, the first subset of (5.5), $\{\mathcal{F}_j \cap U_j = \emptyset\}$, is contained in $\{\mathcal{F}_j \cap (U \ominus \mathcal{B}(0, r_j)) = \emptyset\}$. To see the second subset of (5.5) is also contained in $\{\mathcal{F}_j \cap (U \ominus \mathcal{B}(0, r_j)) = \emptyset\}$, consider any $x \in [\mathcal{F}_j \cap (\mathcal{F}_j \ominus \mathcal{B}(0, r_j))^c] \cap U_j$ under the condition that $(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U = \emptyset$. Since the shortest distance $d_S(x, \mathcal{F}_j \ominus \mathcal{B}(0, r_j))$ between $x$ and $\mathcal{F}_j \ominus \mathcal{B}(0, r_j)$ is less than $r_j$, the shortest distance $d_S(x, U_j^c)$ between $x$ and $U_j^c$ is also less than $r_j$. This implies that $x \notin U_j \ominus \mathcal{B}(0, r_j)$, and thus $[\mathcal{F}_j \cap (\mathcal{F}_j \ominus \mathcal{B}(0, r_j))^c] \cap [U_j \ominus \mathcal{B}(0, r_j)] = \emptyset$. We then have in this case that under $(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U_j = \emptyset$,

$$\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) = \{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cup [\mathcal{F}_j \cap (\mathcal{F}_j \ominus \mathcal{B}(0, r_j))^c] \cap [U_j \ominus \mathcal{B}(0, r_j)]\} \cap (U_j \ominus \mathcal{B}(0, r_j))$$

$$= \{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap (U_j \ominus \mathcal{B}(0, r_j))\} \cup \{(\mathcal{F}_j \cap (\mathcal{F}_j \ominus \mathcal{B}(0, r_j))^c) \cap (U_j \ominus \mathcal{B}(0, r_j))\}$$

$$\subseteq \{\mathcal{F}_j \ominus \mathcal{B}(0, r_j)\} \cup \{\mathcal{F}_j \cap (\mathcal{F}_j \ominus \mathcal{B}(0, r_j))^c\} \cap (U_j \ominus \mathcal{B}(0, r_j)) = \emptyset.$$

That is, the second set of (5.5) is a subset of $\{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) = \emptyset\}$. Therefore, $\{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U_j = \emptyset\} \subseteq \{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) = \emptyset\}$. Similarly, or by switching $\mathcal{F}_j$ and $U_j$, we also have $\{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) \cap U_j = \emptyset\} \supseteq \{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) = \emptyset\}$, and (5.4) holds.

Observe that $U_j \ominus \mathcal{B}(0, r_j)$ is a compact set. Thus for any $j$,

$$\Pr\{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U_j \neq \emptyset\} = \Pr\{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) \neq \emptyset\}$$

$$= \Pr\{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) \neq \emptyset\} = \Pr\{(\mathcal{F}_j^c \ominus \mathcal{B}(0, r_j)) \cap U_j \neq \emptyset\}.$$

That is, $\mathcal{F}_j \ominus \mathcal{B}(0, r_j)$ and $\mathcal{F}_j^c \ominus \mathcal{B}(0, r_j)$ have the same distribution. Therefore, $\mathcal{F}_j \ominus \mathcal{B}(0, R_j)$ and $\mathcal{F}_j^c \ominus \mathcal{B}(0, R_j')$ have the same distribution, $1 \leq j \leq n$. For the hit-order, it follows from (5.4) that,

$$\Pr\{(\mathcal{F}_j \ominus \mathcal{B}(0, r_j)) \cap U_j \neq \emptyset, 1 \leq j \leq n\} = \Pr\{\mathcal{F}_j \cap (U_j \ominus \mathcal{B}(0, r_j)) \neq \emptyset, 1 \leq j \leq n\}$$

$$\geq \Pr\{\mathcal{F}_j^c \cap (U_j \ominus \mathcal{B}(0, r_j)) \neq \emptyset, 1 \leq j \leq n\} = \Pr\{(\mathcal{F}_j^c \ominus \mathcal{B}(0, r_j)) \cap U_j \neq \emptyset, 1 \leq j \leq n\}.$$

Unconditioning on $\mathbf{R} = (r_1, \ldots, r_n)$ yields

$$((\mathcal{F}_1 \ominus \mathcal{B}(0, R_1)), \ldots, (\mathcal{F}_n \ominus \mathcal{B}(0, R_n))] \geq_{hit} ((\mathcal{F}_1 \ominus \mathcal{B}(0, R_1)), \ldots, (\mathcal{F}_n \ominus \mathcal{B}(0, R_n))).$$

Next, conditioning on $\mathcal{F}' = (\mathcal{F}_1', \ldots, \mathcal{F}_n')$, the indicator function $\mathbf{I}\{(\mathcal{F}_j' \ominus \mathcal{B}(0, r_j)) \cap U_j \neq \emptyset\}$ is decreasing in $r_j$. Thus $\mathbf{R} \geq_{pdist} \mathbf{R}'$ and Lemma 2.1 (2) imply that

$$\Pr\{(\mathcal{F}_j' \ominus \mathcal{B}(0, R_j)) \cap U_j \neq \emptyset, \forall j\} = E \left[ \prod_{j=1}^{n} I\{(\mathcal{F}_j' \ominus \mathcal{B}(0, R_j)) \cap U_j \neq \emptyset\} \right]$$

$$\geq E \left[ \prod_{j=1}^{n} I\{(\mathcal{F}_j' \ominus \mathcal{B}(0, R_j')) \cap U_j \neq \emptyset\} \right] = \Pr\{(\mathcal{F}_j' \ominus \mathcal{B}(0, R_j')) \cap U_j \neq \emptyset, \forall j\}.$$
Unconditioning on $\mathcal{F} = (F_1', \ldots, F_n')$, we have

$$((\mathcal{F}_1' \ominus \mathcal{B}(0, R_1)), \ldots, (\mathcal{F}_n' \ominus \mathcal{B}(0, R_n))) \geq_{hit} ((\mathcal{F}_1' \ominus \mathcal{B}(0, R_1)), \ldots, (\mathcal{F}_n' \ominus \mathcal{B}(0, R_n))).$$

Hence (3) holds for the hit-order. The proofs for the other two orders are similar. \qed

Proof of Proposition 4.1

Proof. Define a thinned process $\Phi_p$ as $\Phi_p = \{x_i \in \Phi : (x_i + Z_{ij}) \cap A \neq \emptyset, j = 1, \ldots, N_i\}$. In words, $\Phi_p$ contains the germs $x_i \in \Phi$ whose sub-grain $x_i + Z_{ij}, 1 \leq j \leq n$, hit set $A$ simultaneously. Because $\Phi$ is a spatial Poisson process, and also because whether or not a germ is deleted by this thinning is independent of thinning happening to other germs, $\Phi_p$ is a non-homogeneous spatial Poisson process with intensity $\Lambda_p(B) = \lambda \int_B p(x)dx$, $B \subseteq \mathbb{R}^d$, where

$$p(x) = \Pr\{(x + Z_j) \cap A \neq \emptyset, j = 1, \ldots, N\} = \Pr\{\cap_{j=1}^{N}((x + Z_j) \cap A \neq \emptyset)\} \quad (5.6)$$

is the probability that every sub-grain of germ $x$ hits $A$. Since $T^\Phi_0(A) = 1 - \Pr\{\Phi_p = 0\} = 1 - \exp(-\Lambda_p(\mathbb{R}^d))$, we need to obtain $\Lambda_p(\mathbb{R}^d)$. Using (5.6) and the fact that $(x + Z_j) \cap A \neq \emptyset$ if and only if $x \in \mathcal{Z}_j + A$, we obtain (4.8) as follows,

$$\Lambda_p(\mathbb{R}^d) = \lambda \int p(x)dx = \lambda \int \Pr\{x \in \cap_{j=1}^{N} (\mathcal{Z}_j \oplus A)\} dx = \lambda \int \mathbb{E}\{1 \{x \in \cap_{j=1}^{N} (\mathcal{Z}_j \oplus A)\}\} dx = \lambda \mathbb{E}\left(\cap_{j=1}^{N} (\mathcal{Z}_j \oplus A)\right). \quad (5.7)$$

Observe that

$$\Pr\{x \in \cap_{j=1}^{n} (\mathcal{Z}_j \oplus A)\} = 1 - \sum_{\emptyset \neq J \subseteq \{1, \ldots, n\}} (-1)^{|J|-1} \left(1 - \Pr\{x \in \cup_{j \in J} (\mathcal{Z}_j \oplus A)\}\right)$$

$$= \sum_{\emptyset \neq J \subseteq \{1, \ldots, n\}} (-1)^{|J|-1} \Pr\{x \in \mathcal{Y}_j \oplus A\}. \quad (5.8)$$

Using the similar idea as in (5.7), we obtain that $\Lambda_p(\mathbb{R}^d) = E \sum_{\emptyset \neq J \subseteq \{1, \ldots, N\}} (-1)^{|J|-1} \lambda \mathbb{E}\nu[\mathcal{Y}_j \oplus A]$, and (4.8) follows. \qed

Proof of Theorem 4.2

Proof. As a random measure, $\Phi(B)$ denotes the number of points in $B$, where $B$ is a Borel subset of $\mathbb{R}^d$. Let $\mathcal{Y}_i = (Z_{i1}, \ldots, Z_{iN_i}), i \geq 1$, be i.i.d. having the same distribution as that of $\mathcal{Y}$, and $\mathcal{Y}'_i = (Z'_{i1}, \ldots, Z'_{iN_i}), i \geq 1$, be i.i.d. having the same distribution as that of $\mathcal{Y}'$. Since $A$ is compact, then $A \subseteq W$ where $W$ is a ball centered at the origin with finite radius. We prove Parts 1 and 2 by conditioning on the point pattern $\Phi = \varphi = \{x_1, \ldots, x_{\varphi(W)}\}$, where $\varphi(W)$ is a realization of $\Phi(W)$, and $N = \{N_1, \ldots, N_{\varphi(W)}\} = \{n_1, \ldots, n_{\varphi(W)}\} = n$. 

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(1) Conditioning on $\Phi = \varphi, N = n$, $\mathcal{Y}_i \geq_{miss} \mathcal{Y}_i'$ implies that

$$R_D(A | \Phi = \varphi, N = n) := \Pr\{D \cap A = \emptyset \mid \Phi = \varphi, N = n\} = \prod_{i=1}^{\varphi(W)} \Pr\{(x_i + Z_{ij}) \cap A = \emptyset, j = 1, \ldots, n_i\} \geq \prod_{i=1}^{\varphi(W)} \Pr\{(x_i + Z_{ij}') \cap A = \emptyset, j = 1, \ldots, n_i\} = \Pr\{D' \cap A = \emptyset \mid \Phi = \varphi, N = n\} =: R_{D'}(A | \Phi = \varphi, N = n),$$

where the second and third equalities follow since $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_{\varphi(W)}\}$ are i.i.d. and $\{\mathcal{Y}_1', \ldots, \mathcal{Y}_{\varphi(W)'}\}$ are i.i.d.. The inequality $R_D(A) \geq R_{D'}(A)$ follows after unconditioning on $\Phi$ and $N$. In particular, $\bar{H}_x(r) = R_D(\mathcal{B}[x, r]) \geq R_{D'}(\mathcal{B}[x, r]) = \bar{H}_x'(r)$.

(2) Since $\mathcal{Y}_i \geq_{hit} \mathcal{Y}_i'$, we have, given $\Phi = \varphi$ and $N = n$,

$$T_D^S(A | \Phi = \varphi, N = n) := 1 - \Pr\{\cap_{i=1}^{\varphi(W)} \cup_{j=1}^{n_i} [A \cap (x_i + Z_{ij}) = \emptyset] \} \mid \Phi = \varphi, N = n\} = 1 - \prod_{i=1}^{\varphi(W)} (1 - \Pr\{A \cap (x_i + Z_{ij}) \neq \emptyset, j = 1, \ldots, n_i\}) \geq 1 - \prod_{i=1}^{\varphi(W)} (1 - \Pr\{A \cap (x_i + Z_{ij}') \neq \emptyset, j = 1, \ldots, n_i\}) =: T_{D'}^S(A | \Phi = \varphi, N = n).$$

Part (2) follows after unconditioning on $\Phi$ and $N$. \qed

**Proof of Theorem 4.4**

**Proof.** (1) Since $Z_{[n]} \subseteq \cup_{j=1}^n Z_j$, we have $\nu(\tilde{Z}_{[n]} \oplus A) \leq \nu((\cup_{j=1}^n \tilde{Z}_j) \oplus A)$, and then $R_D(A) = \exp\{-\lambda \nu((\cup_{j=1}^n \tilde{Z}_j) \oplus A)\} \leq \exp\{-\lambda \nu(\tilde{Z}_{[n]} \oplus A)\}$. For the lower bounds, we have

$$R_D(A) \geq R_{D'}(A) = \exp\{-\lambda \nu((\cup_{j=1}^n \tilde{Z}_j') \oplus A)\} \geq \prod_{j=1}^n \exp\{-\lambda \nu(\tilde{Z}_j \oplus A)\},$$

where the first inequality follows from Theorem 4.2 (1), the equality uses (4.7), and the second inequality uses the fact that $\nu((\cup_{j=1}^n \tilde{Z}_j') \oplus A) \leq \sum_{j=1}^n \nu(\tilde{Z}_j' \oplus A)$ and $Z_j$ and $Z_j'$ are identically distributed. This proves (4.9).

(2) Since $Z_{[1]} \supseteq (\cap_{j=1}^n Z_j)$, we have $\nu(\tilde{Z}_{[1]} \oplus A) \geq \nu\left((\cap_{j=1}^n (\tilde{Z}_j \oplus A)\right)$, and then $T_D^S(A) = 1 - \exp\{-\lambda \nu((\cap_{j=1}^n (\tilde{Z}_j \oplus A)\right)} \leq 1 - \exp\{-\lambda \nu(\tilde{Z}_{[1]} \oplus A)\}$. On the other hand, from Theorem 4.2 (2), we have

$$T_{D'}^S(A) \geq T_{D'}^S(A) = 1 - \exp\{-\lambda \nu((\cap_{j=1}^n (\tilde{Z}_j' \oplus A)\right)} = 1 - \exp\{-\lambda \nu((\cap_{j=1}^n \tilde{Z}_j') \oplus A)\}$$

where the last equality holds because $\cap_{j=1}^n (\tilde{Z}_j' \oplus A) = (\cap_{j=1}^n \tilde{Z}_j' \oplus A)$. \qed
References


