Vine copulas with asymmetric tail dependence and applications to financial return data

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Abstract

It has been shown that vine copulas constructed from bivariate t copulas can provide good fits to multivariate financial asset return data. However, there might be stronger tail dependence of returns in the joint lower tail of assets than the upper tail. To this end, vine copula models with appropriate choices of bivariate reflection asymmetric linking copulas will be used to assess such tail asymmetries. Comparisons of various vine copulas are made in terms of likelihood fit and forecasting of extreme quantiles.

Keywords: copula-GARCH; inference functions for margins; reflection asymmetry; Value-at-Risk

1. Introduction

One goal in the theory of dependence modeling and multivariate copulas is to develop parametric families that are appropriate as models for multivariate data with different dependence structures including features such as tail dependence. In this paper, we focus on copula models with flexible tail dependence for applications to financial return data. The desired properties of multivariate copula families for modeling multivariate continuous data are given below.

P1: Wide range of dependence, allowing both positive and negative dependence.
P2: Flexible range of upper and lower tail dependence.
P3: Computationally feasible density for (likelihood) estimation.
P4: Closure property under marginalization, meaning that lower-order marginals belong to the same parametric family.

None of the existing parametric families of multivariate copulas satisfy all these conditions. Multivariate Archimedean copulas allow only exchangeable structure with a narrower range of negative dependence as the dimension increases, see for example, Joe (1997) and McNeil and Nešlehová (2009). Partially symmetric copulas extend Archimedean to a class with a non-exchangeable structure, but they don’t provide flexible dependence; see Joe (1993). Mixtures of max-id copulas in Joe and Hu (1996) provide flexible positive dependence by construction. Their upper tail dependence is flexible, but not lower tail dependence. The multivariate normal copula family, studied by Abdous et al. (2005) and Fang et al. (2002), satisfies properties P1, P3 and P4, but does not have tail dependence. The multivariate t copula family, see for example, Demarta and McNeil (2005) and Nikoloulopoulos et al. (2009), satisfies all properties but only P2 in part; reflection symmetry implies that the tail dependence is the same in the upper and lower tails for any bivariate margin. There are also various skewed t copulas, see Demarta and McNeil (2005) and Kotz and Nadarajah (2004), but these are computationally more involved. Vine copulas, see Bedford and Cooke (2002), can...
satisfy all the properties by suitable choices of bivariate linking copulas, but P4 is not satisfied so that applications of vine copulas require a decision on the indexing of variables.

Multivariate t copulas have been used extensively in the context of modeling multivariate financial return data, and have been shown to be generally superior to the normal copula, see for example, Breyermann et al. (2003). The reason is that the t copula has tail dependence, i.e., dependence in extreme values. In the literature, it is reported that

asymmetries are reported to be generally superior to the normal copula, see for example, Longin and Solnik (2001), Ang and Chen (2002) and Hong et al. (2007) among others, meaning that lower tail dependence can be larger than upper tail dependence or vice versa. Some authors say that there is more extremal dependence in downturns/crashes, see for example, Jondeau et al. (2007), Chollete et al. (2009), Giacomini et al. (2009) and the references therein. Hu (2006) applied mixture copula models to capture asymmetrical tail dependence. Caillault and Guegan (2005), providing ‘naive’ estimation for the tail dependence parameters, show a bivariate dependence structure which is symmetric for the Thai / Indonesian markets and asymmetric for the Thai / Malaysian markets and for the Malaysian / Indonesian markets. Hong et al. (2007) use conditional correlations of values below and above thresholds. Rather than two-parameter copula families they use some two-component mixture models with three parameters to capture reflection asymmetry and tail dependence.

Vine copulas (also called the pair-copula construction) have been applied recently for finance asset return and other data; see Schirmacher and Schirmacher (2008), Aas et al. (2009), Aas and Berg (2009), Heinen and Valdesogo (2009), Fischer et al. (2009) and Min and Czado (2010). d-dimensional vine copulas can cover flexible dependence structures through the specification of \( d - 1 \) bivariate marginal copulas at level 1 and \( (d - 1)(d - 2)/2 \) bivariate conditional copulas at higher levels; at level \( \ell \) for \( \ell = 2, \ldots, d - 1 \), there are \( d - \ell \) bivariate conditional copulas that condition on \( \ell - 1 \) variables. Vine copulas include multivariate normal and t copulas as special cases, but can also cover reflection asymmetry and have upper/lower tail dependence parameters being different for each bivariate margin \((i, j)\). Joe et al. (2010) have a main theorem that says that all bivariate margins of the vine copula have upper/lower tail dependence if the bivariate copulas at level 1 have upper/lower tail dependence.

Aas et al. (2009), Aas and Berg (2009) and Fischer et al. (2009) compared fits and inference for vine copulas, when the bivariate copulas are all (i) t copulas, (ii) Gumbel copulas, (iii) Clayton or MTCJ copulas, (iv) Frank/normal copulas. These bivariate copulas are either reflection symmetric or have one-directional (one of upper or lower) tail dependence. In this paper, we make the first use of vine copulas with two-parameter bivariate linking copulas that can have upper tail dependence different from lower tail dependence. This provides a means to check if there is some reflection asymmetry in the joint tails of financial asset returns.

Joe and Hu (1996) and Joe (1997) have three families of bivariate copulas, called BB1, BB4 and BB7 in the latter, that interpolate independence and the Fréchet upper bound copulas and have upper and lower tail dependence that can range independently from 0 to 1. Relevant properties, including some new results, of BB1, BB4, BB7 and their comparisons to bivariate t copulas are summarized in the Appendix. We will compare the use of BB1, BB4 and BB7 versus t in the two boundary cases of C-vines and D-vines. For ease of coding, we take all bivariate copulas to be in the same family (e.g., BB1) for all levels of the vines, although the overall fit might be better, see for example, Chollete et al. (2009), by allowing bivariate linking copulas to come from different parametric families. Our simpler analysis does allow us to assess whether we can improve on vine copulas built from bivariate t. For the comparisons, we use a data set with five European market indexes. However, we had done similar analyses on other data sets of similar size and conclusions are similar.

The remainder of the paper is as follows. Section 2 introduces the vine copulas, and Section 3 discusses dependence measures. In Section 4, an inferential procedure for the copula-GARCH models is summarized, while in Section 5 a goodness-of-fit procedure based on the forecasting is proposed. Section 6 has practical issues related to vine-copula modeling, such as indexing of variables. In the context of a financial returns data set, Section 7 covers diagnostics for dependence and reflection asymmetry, univariate and bivariate analyses, goodness-of-fit diagnostics, estimation of dependence parameters in vines, and forecasting of extreme quantiles of a portfolio. Section 8 concludes with some discussion and directions for further research.

2. Introduction to vine copulas

A \( d \)-variate copula \( C(u_1, \ldots, u_d) \) is a cumulative distribution function (cdf) with uniform marginals on the unit interval; see for example, Joe (1997) and Nelsen (2006). According to the theorem of Sklar (1959), if \( F_j(y_j) \) is
the cdf of a univariate continuous random variable \( Y_j \), then \( C(F_1(y_1), \ldots, F_d(y_d)) \) is a \( d \)-variate distribution for \( Y = (Y_1, \ldots, Y_d) \) with marginal distributions \( F_j, j = 1, \ldots, d \). Conversely, if \( H \) is a continuous \( d \)-variate cdf with univariate marginal cdfs \( F_1, \ldots, F_d \), then there exists a unique \( d \)-variate copula \( C \) such that
\[
F(y) = C(F_1(y_1), \ldots, F_d(y_d)), \quad \forall y = (y_1, \ldots, y_d).
\]
The corresponding density is
\[
f(y) = \frac{\partial^d F(y)}{\partial y_1 \cdots \partial y_d} = c(F_1(y_1), \ldots, F_d(y_d)) \prod_{j=1}^d f_j(y_j),
\]
where \( c(u_1, \ldots, u_d) \) is the \( d \)-variate copula density and \( f_j, j = 1, \ldots, d \), are the corresponding marginal densities.

A copula \( C \) has reflection symmetry if \( (U_1, \ldots, U_d) \sim C \) implies that \( (1-U_1, \ldots, 1-U_d) \) has the same distribution \( C \). When it is necessary to have copula models with reflection asymmetry and flexible lower/upper tail dependence, then vine copulas (see Bedford and Cooke (2002, 2001), Kurowicka and Cooke (2006), Section 4.5 of Joe (1997), and Joe (1996)) may be the best choice. The \( d \)-dimensional vine copulas are built via successive mixing from \( d-1/2 \) bivariate linking copulas on trees and their cdfs involve lower-dimensional integrals. Since the densities of multivariate vine copulas can be factorized in terms of bivariate linking copulas and lower-dimensional margins, they are computationally tractable. Depending on the types of trees, various vine copulas can be constructed. Two boundary cases are D-vines and C-vines. In this paper, we make use of these boundary cases, but for higher-dimensional data, the methods presented here do extend to other regular vines.

For the \( d \)-dimensional D-vine, the pairs at level 1 are \( i, i+1 \), for \( i = 1, \ldots, d-1 \), and for level \( \ell \) (\( 2 \leq \ell < d \)), the (conditional) pairs are \( i, i+\ell \), \( i+1, \ldots, i+\ell-1 \) for \( i = 1, \ldots, d-\ell \). For the \( d \)-dimensional C-vine, the pairs at level 1 are 1, \( i \), for \( i = 2, \ldots, d \), and for level \( \ell \) (\( 2 \leq \ell < d \)), the (conditional) pairs are \( i, i+1, \ldots, i+\ell-1 \) for \( i = 1, \ldots, d-\ell \). That is, for the D-vine, conditional copulas are specified for variables \( i \) and \( i+\ell \) given the variables indexed in between; and for the C-vine, conditional copulas are specified for variables \( \ell \) and \( i \) given those indexed as 1 to \( \ell-1 \).

For C-vines and D-vines the densities are respectively (Aas et al. (2009)),
\[
f(y) = \prod_{k=1}^d f_k(y_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,i+j,i+1,\ldots,i+j-1}(F_{i+1-i}(y_i|y_{i+1},y_{i+j-1}), F_{i+j-1+i-1}(y_{i+j-1}|y_{i+1},y_{i+j-1})),
\]
where \( y_{k;1:k} = (y_{k1}, \ldots, y_{kk}) \) and
\[
f(y) = \prod_{k=1}^d f_k(y_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{j,j+i,i+1,\ldots,j+i-1}(F_{j+i-j}(y_j|y_1,\ldots,y_{j-i}), F_{j+i-1+j-i}(y_{j+i-1}|y_1,\ldots,y_{j-1})),
\]
index \( j \) denotes the tree/level, while \( i \) runs over the edges in each tree.

In Figures 1 and 2, a C-vine and a D-vine with 5 variables and 4 trees/levels are depicted. The densities are decomposed in a simple manner by multiplying the nodes of the nested set of trees, as indicated below for the D-vine,
\[
f = f_1 f_2 f_3 f_4 f_5 \times c_{12}(F_1, F_2) c_{23}(F_2, F_3) c_{34}(F_3, F_4) c_{45}(F_4, F_5) \times c_{132}(F_{12}, F_{32}) c_{243}(F_{23}, F_{43}) c_{354}(F_{34}, F_{54}) \times c_{1423}(F_{123}, F_{423}) c_{2534}(F_{234}, F_{534}) \times c_{15234}(F_{1234}, F_{5234}).
\]

For more general regular vines in dimension \( d \), there are \( d-1 \) pairs at level 1 (and they satisfy a condition that the graph formed with edges based on the pairs has no cycles), \( d-2 \) pairs in level 2 where each pair has one element in common, and for \( \ell = 2, \ldots, d-1 \), there are \( d-\ell \) pairs in level \( \ell \) where each pair has \( \ell-1 \) elements in common. Other conditions for regular vines are given in Bedford and Cooke (2001, 2002).

3. Dependence measures

Since the dependence structure among random variables is represented by copulas, they provide a natural way to study and measure the dependence among random variables. Bivariate concordance measures such as Kendall’s \( \tau \),
Spearman’s $\rho$, and Blomqvist’s $\beta$ are copula-based measures of dependence and invariant under strictly increasing transformations; see Chapter 2 of Joe (1997) and Chapter 5 of Nelsen (2006). We will use only the Blomqvist’s $\beta$, because $\beta$ can be calculated easily for all the parametric families of copulas in contrast with $\tau$ and $\rho$. The copula-based formula for Blomqvist’s $\beta$ is, $\beta = 4C(u, v) - 1$.

Tail dependence is another useful copula-based measure, indicating dependence in extreme values. Moreover, tail dependence is one of the properties that discriminate between the different families of copulas since, while it exists for some of them, there are families that cannot allow tail dependence (e.g., the normal copula). Let $C(u, v) = 1 - u - v + C(u, v)$ be the joint survival function for $C$. If $C$ is such that $\lim_{u \to 1} C(u, u)/(1 - u) = \lambda_U$ exists, then $C$ has upper tail dependence if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. Similarly, if $\lim_{u \to 0} C(u, u)/u = \lambda_L$ exists, then $C$ has lower tail dependence if $\lambda_L \in (0, 1]$ and no upper tail dependence if $\lambda_L = 0$. In the financial data context upper tail dependence means dependence in extremal gains, while lower tail dependence means dependence in extremal losses.

Recently, Nikoloulopoulos et al. (2009) and Joe et al. (2010) generalized the tail dependence concept by introducing the multivariate tail dependence functions as a unified tool for extremal dependence analysis. Joe et al. (2010) studied the extremal dependence of vine copulas and show that vine copulas can have a flexible range of bivariate lower and upper tail dependence parameters, and different upper/lower tail dependence for each bivariate margin when asymmetric bivariate copulas with upper/lower tail dependence are used in level 1 of the vine. That is, in order for a vine copula to have tail dependence for all bivariate margins, it is only necessary for the bivariate copulas in level 1 to have tail dependence and it is not necessary for the conditional bivariate copulas in levels 2, . . . , $d - 1$ to have tail.
dependence.

4. Copula-GARCH model and inferential procedure

In this section, we summarize the copula-GARCH model (Jondeau and Rockinger (2006), Fantazzini (2009), Liu and Luger (2009), Aas and Berg (2009), Ausin and Lopes (2010), Hafner and Reznikova (2010)) and our inference procedures.

Let \( P_t \) (\( t = 0, 1, \ldots, T \)) be a time series of the price on a financial asset such as a market index. The return \( R_t \), for \( t \geq 1 \), is defined as \( \log(P_t/P_{t-1}) \). Suppose that there are \( d \) assets with returns \( R_{1,t}, \ldots, R_{d,t} \). Copula modeling can proceed in two steps. In the first step, we select the models for the individual variables, that is the univariate marginal distributions. For financial return data, a common choice is the GARCH(1,1) time series filter with innovation distribution being the symmetric Student-t distribution with variance 1 (\( \nu > 2 \)), see Jondeau et al. (2007) (Section 4.3.6, page 93).

The model is:

\[
R_{ij} = \mu_j + \sigma_{r,j}Z_{ij}, \quad \sigma_{r,j}^2 = \alpha_0j + \alpha_1R_{t-1,j}^2 + \beta_j\sigma_{r-1,j}^2, \quad j = 1, \ldots, d, t = 1, \ldots, T, \tag{3}
\]

where \( Z_{ij} \) are assumed to be innovations that are independent and identically distributed (Student-t). The vectors \( Z_t = (Z_{1,t}, \ldots, Z_{dt}) \) for \( t = 1, \ldots, T \) are assumed to be independent and identically distributed (iid) with distribution:

\[
F_Z(z_1, \ldots, z_d, \omega) = C(F_1(z_1; \nu_1), \ldots, F_d(z_d; \nu_d); \omega),
\]

where \( \omega \) is a dependence parameter (vector) of a \( d \)-dimensional vine copula \( C \) and \( F_1, \ldots, F_d \) are the scaled Student-t distributions with parameters vectors \( \nu_1, \ldots, \nu_d \). For reference below, we define \( \eta_j = (\nu_j, \mu_j, \alpha_{0j}, \alpha_{1j}, \beta_j) \).

The joint log-likelihood of the model is:

\[
L(\eta_1, \ldots, \eta_d, \omega) = \sum_{t=1}^T \log f_R(r_{1,t}, \ldots, r_{dt}; \eta_1, \ldots, \eta_d, \omega) \tag{4}
\]

\[
= \sum_{t=1}^T \left[ \log c(F_1([r_{1,t} - \mu_1]/s_{1,t}; \nu_1), \ldots, F_d([r_{dt} - \mu_d]/s_{dt}; \nu_d); \omega) + \sum_{j=1}^d \log \left[ s_{ij}^{-1} f_j([r_{ij} - \mu_j]/s_{ij}; \nu_j) \right] \right]
\]

with \( s_{ij} = (\alpha_0j + \alpha_1r_{t-1,j}^2 + \beta_j\sigma_{r-1,j}^2)^{1/2} \), and \( c(\cdot; \omega) \) is the copula density for \( C(\cdot; \omega) \).

Consider the joint log-likelihood where \( f \) is the joint density in (1) for a C-vine or in (2) for a D-vine. Maximum likelihood of (4) is possible, either directly with a quasi-Newton routine or through the modified maximization by parts algorithm in Liu and Luger (2009), but is time-consuming for large \( d \) because the total number of model parameters is of the order \( 4d + d(d - 1) \). When the dependence is not too strong, the Inference Function of Margins (IFM) method (Joe (2005)), which consists of a two-step approach, can efficiently (in sense of computing time and asymptotic variance) estimate the model parameters. In the first step, the GARCH(1,1) filter is applied to the return data to get the filtered data and univariate parameter estimates \( (\hat{\nu}_j, \hat{\mu}_j, \hat{\alpha}_{0j}, \hat{\alpha}_{1j}, \hat{\beta}_j) \) are derived, and in the second step the joint log-likelihood (4) is maximized over the copula parameter vector \( \omega \) with the \( \nu_j \) and other univariate parameters fixed at the estimated values from the first step. The joint log-likelihood at the second step of estimation could be reduced to the copula log-likelihood \( L_C \) (the part of the likelihood related to the copula function) since the part of the log-likelihood \( L_M \) related only to the marginal distributions (parameters) is fixed as estimated at the first step. In the sequel, likelihood evaluation for vine copulas performed using the algorithms in Aas et al. (2009) combined with a numerical quasi-Newton routine, and the GARCH(1,1) models are fitted using the R package \( \text{fGarch} \) — this has a default way of initializing \( s_{0,j} \) in (4).

The estimated GARCH parameters \( \alpha_{0j}, \alpha_{1j}, \beta_j \) are not sensitive to the innovation distribution being normal, Student-t, or skewed t, e.t.c., so following Section 5 of Aas et al. (2009), as a sensitivity analysis, we also estimate the dependence parameters of the vine using the semi-parametric (S-P) estimation proposed by Genest et al. (1995). In a recent publication, Fantazzini (2010) studies semi-parametric estimation for t-copulas. With the copula-GARCH
model, S-P estimation is based on maximizing the pseudo-likelihood of the copula likelihood $L_c$ using the re-scaled empirical distributions of the Student-t iid innovations,

$$
\hat{F}_{Z_{ij},T}(z_{ij}) = \sum_{t=1}^{T} \mathbf{1}(Z_{ij} \leq z_{ij})/(T + 1), \quad j = 1, \ldots, d,
$$

instead of the parametric distributions.

5. Goodness-of-fit based on forecasts

Copula model selection is often based on the AIC or formal goodness-of-fit tests, see for example, Genest et al. (2006) and Genest et al. (2009). Aas et al. (2009) and Aas and Berg (2009) adopt these goodness-of-fit procedures for vine copulas by applying them to the bivariate blocks. However as commented in Aas and Berg (2009) this selection mechanism does not guarantee a globally optimal fit.

Because tail risks are important for portfolios, we instead compare and select the best model based on comparing observed and parametric-bootstrapped extreme quantiles for equally-weighted portfolio returns of the form $r_{p,t} = d^{-1} \sum_{j=1}^{d} r_{ij}$. The parametric distributions of portfolio returns are obtained via Monte Carlo simulations using the following algorithm:

Let $K$ be a number like 10000.

1. For a fitted vine copula-based model, do the following for $k = 1, \ldots, K$:
   (a) Simulate a sample $u_{(k)}^{(i)}, \ldots, u_{(k)}^{(d)}, t = 1, \ldots, T$, using the algorithms in Aas et al. (2009).
   (b) For a fixed $j$ ($j = 1, \ldots, d$), convert $u_{(k)}^{(j)}$ to $z_{ij}^{(k)}$, $t = 1, \ldots, T$, using the inverse of the symmetric Student-t distribution with $v_j$ as estimated at the first step of the IFM method on the observed data.
   (c) Convert $z_{ij}^{(k)}$ to the simulated time-series $r_{ij}^{(k)} = \hat{\mu}_j + \sigma_j \epsilon_{ij}^{(k)}$, $j = 1, \ldots, d$, where $\sigma_j$ and $\hat{\mu}_j$ are the conditional standard deviation (volatility) predictions and mean values as derived in the first step of the IFM method on the observed data.
   (d) Compute the return of the portfolio as $r_{p,t}^{(k)} = \sum_{j=1}^{d} r_{ij}^{(k)} / d$.

2. For $q \in (0.01, 0.99)$ calculate the one-day $100q$-th percentile $r_p(q)$ of $r_{p,t}^{(k)}, k = 1, \ldots, K$. VaR$_{0.01} = -r_p(0.01)$ is the Value-at-Risk value for losses.

3. If $r_p(0.99)$ is greater than or $r_p(0.01)$ is less than the observed value of $r_{p,t}$ for day $t$, then a violation is said to occur.

Consider the indicator sequence of violations $I_t, t = 1, \ldots, T$, constructed using the preceding algorithm. To validate the interval forecast we use the likelihood ratio tests of unconditional coverage $LR_{uc}$, independence $LR_{ind}$, and the combination of coverage and independence $LR_{cc}$ proposed by Christoffersen (1998). $LR_{uc}$ is the log-likelihood ratio statistic of the observed violation rate to the (null) theoretical violation rate $q$:

$$
LR_{uc} = -2 \log \left( \frac{L(q; I_1, \ldots, I_T)}{L(\hat{\pi}; I_1, \ldots, I_T)} \right) = -2 \log \left( \frac{q^T}{x^T} \right) \sim \chi_1^2,
$$

where $x$ is the number of violations and $\hat{\pi} = x/T$. Unconditional coverage tests are known to have low power since they account for only the number of violations and not the order of violations/non-violations in the indicator sequence.

Next, $LR_{ind}$ is the log-likelihood ratio statistic for Markov dependence versus time independence ($H_0$),

$$
LR_{ind} = -2 \log \left( \frac{L(\hat{\pi}_2; I_1, \ldots, I_T)}{L(\hat{\pi}_0; \hat{\pi}_1; I_1, \ldots, I_T)} \right) = -2 \log \left( \frac{\hat{\pi}_2^{n_0+n_1}}{\hat{\pi}_1^{n_0}\hat{\pi}_0^{n_1}(1-\hat{\pi}_1)^{n_0+n_1}} \right) \sim \chi_1^2,
$$

where $\pi_{ij} = Pr(I_t = j| I_{t-1} = i)$, $n_{ij} = T \hat{\pi}_{ij}$ is the corresponding frequency and $\hat{\pi}_2 = (n_0 + n_1)/(n_0 + n_1 + n_{11})$. Furthermore, $LR_{cc}$ is the log-likelihood ratio statistic that combines the previous two tests with $H_0$ being serial independence and a violate rate of $q$:

$$
LR_{cc} = -2 \log \left( \frac{L(q; I_1, \ldots, I_T)}{L(\hat{\pi}_0, \hat{\pi}_1; I_1, \ldots, I_T)} \right) \sim \chi_2^2.
$$
6. Construction of vine copulas

A multivariate copula can be decomposed into a sequence of bivariate conditional copulas in many ways if the conditional copulas depend on the values of the conditioning variables. Vine copulas involve a sequence of conditional bivariate copulas, but assume the bivariate conditional copulas are constant over the conditioning variables. For example, multivariate normal copulas satisfy this property with conditional correlations being equal to partial correlations. In general, a given multivariate copula might be well-approximated by several different vine copulas, and likelihood inference could be used to find such vines.

For fitting multivariate data with possible tail dependence (in which case, multivariate normal copulas are inappropriate), we would like to compare the fits of different vine copulas with parametric families for the bivariate building blocks. One reason for using the faster IFM method (described in Section 4) over full maximum likelihood is in order to more quickly compare different vines and different bivariate families as building blocks.

In this section, we discuss the following practical issues for vine-copula modeling:

1. type of the vine,
2. type of bivariate copula families as building blocks,
3. given the vine, the matching of variables to labels/indexes, since for a $d$-dimensional C-vine or D-vine copula there are $d!/2$ possible different choices (distinct permutations).

6.1. Building blocks

Partly as a diagnostic, we suggest to choose the building blocks by fitting various bivariate copulas (with a range of tail behavior) to all bivariate margins and select the most appropriate bivariate copula for each margin, or a bivariate copula family for all margins at level 1 of the vine, based on the AIC. By using the IFM method, the AIC is $-2\times \log$-likelihood $+2\times (#dependence \ parameters)$ and a smaller AIC indicates a better fitting model. The discussion below could also apply to other information criteria.

In our candidate set for vine copula modeling, families that have different strengths of tail behavior (see Heffernan (2000)) are included:

- (a) Copulas with tail independence satisfying $C(u, u; \delta) = O(u^2)$ and $\overline{C}(1 - u, 1 - u; \delta) = O(u^2)$ as $u \to 0$, such as the Frank copula (used as a possible baseline for comparison for strength of tail dependence).
- (b) Copulas with intermediate tail independence (term as used in Hua and Joe (2009)) such as the bivariate normal copula, which satisfies $C(u, u, \rho) = O(u^{2/(1+\rho)}(-\log u)^{-\rho/(1+\rho)})$ as $u \to 0$.
- (c) Copulas with upper tail dependence only such as the Gumbel copula.
- (d) Copulas with lower tail dependence only such as the Mardia-Takahasi-Cook-Johnson (MTCJ) copula.
- (e) Copulas with reflection symmetric upper and lower tail dependence such as the t copula in the elliptical family.
- (f) Copulas with different upper and lower tail dependence such as BB1, BB4 and BB7 and their reflected forms.

If (e) and (f) have the lowest AIC values (for all bivariate margins), then upper and lower tail dependence would be suggested. If (c) or (d) have the lowest AIC values, then one-sided tail dependence would be suggested. If (a) has the lowest AIC values, then no tail dependence is suggested. This method to assess tail dependence is in addition to the non-parametric estimators based on the empirical copula and the tail copula, see Dobrić and Schmid (2005) and Schmidt and Stadtmüller (2006).

In Nikoloulopoulos and Karlis (2008), Monte Carlo simulations were performed to examine whether data simulated from a given one-parameter copula family can be recognized and attributed, based on the likelihood principle, to the true generating copula for different sample sizes and levels of dependence. It is shown that in practice it is not easy to distinguish one-parameter copula families with similar tail dependence properties. To this end, we perform a simulation study to examine (a) whether a mixture of max-id copula in (f) with different upper and lower tail dependence can be distinguished from the t copula with reflection symmetric upper and lower tail dependence, and vice-versa, and (b) whether the model-based estimates from the “true” model match with the non-parametric (N-P) estimates (average of non-parametric estimates in Dobrić and Schmid (2005)) of the tail dependence parameters. We also do not restrict ourselves to the bivariate case by using a trivariate vine copula; this is both a C-vine and a D-vine. The candidate copula families are the t, BB1, BB7 copulas and the BB1 copula in survival or reflected form (denoted as s.BB1) in the trivariate vine copula; see the Appendix for why we also use the reflected or survival form of BB1.
but not BB4 or BB7. Monte Carlo simulations are performed using the first step, (a) to (c), of the algorithm in Section 5 and the dependence parameters are estimated with IFM and the S-P approach in Section 4. Conditional standard deviation (volatility) predictions and mean values are derived in the first step of the IFM method on the observed data in Section 7. We report here typical results from these experiments where the true parameters at the second tree/level are close to the independence copula.

Table 1 contains the Bias, standard deviation (SD) and root mean square error (RMSE) of the N-P, S-P and IFM estimates when the data are generated for a trivariate vine copula with BB1. The lower/upper tail dependence parameters for the (1,2) and (1,3) margin are \( \lambda_L = 0.6, \lambda_U = 0.2 \) and \( \lambda_L = 0.7, \lambda_U = 0.3 \), respectively, while \( C_{23|1} \) is conditional independent. However the unconditional lower/upper tail dependence parameters for the (2,3) margin are \( \lambda_L = 0.51, \lambda_U = 0.09 \) and obtained using the methods in Section 4 of Joe et al. (2010) where the lower and upper tail dependence functions of the baseline BB1 copulas are:

\[
b(w_1, w_2; \theta, \delta) = \frac{w_1 - \theta \delta + w_2 - \theta \delta}{\theta \delta},
\]

and

\[
b^*(w_1, w_2; \delta) = w_1 + w_2 - (w_1^\delta + w_2^\delta)^{1/\delta},
\]

respectively.

Table 1: Bias, standard deviation (SD) and root mean square error (RMSE) of N-P, S-P and IFM estimates when the data are generated for a trivariate vine copula with BB1. The two last rows of the table represent the times of selection as a better fitting model of BB1, s. BB1, BB7 and t; the latter is not listed since it is not selected at all.

<table>
<thead>
<tr>
<th>Method</th>
<th>Bias</th>
<th>SD</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\lambda}_L )</td>
<td>( \hat{\lambda}_U )</td>
<td>( \hat{\lambda}_L )</td>
<td>( \hat{\lambda}_U )</td>
</tr>
<tr>
<td>N-P</td>
<td>-0.04</td>
<td>0.04</td>
<td>0.07</td>
</tr>
<tr>
<td>S-P</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>IFM</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>Bias</th>
<th>SD</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\lambda}_L )</td>
<td>( \hat{\lambda}_U )</td>
<td>( \hat{\lambda}_L )</td>
<td>( \hat{\lambda}_U )</td>
</tr>
<tr>
<td>N-P</td>
<td>-0.05</td>
<td>0.03</td>
<td>0.06</td>
</tr>
<tr>
<td>S-P</td>
<td>-0.02</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>IFM</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>Bias</th>
<th>SD</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\lambda}_L )</td>
<td>( \hat{\lambda}_U )</td>
<td>( \hat{\lambda}_L )</td>
<td>( \hat{\lambda}_U )</td>
</tr>
<tr>
<td>N-P</td>
<td>-0.05</td>
<td>0.05</td>
<td>0.08</td>
</tr>
<tr>
<td>S-P</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>IFM</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
</tbody>
</table>

| Times of selection | | | |
| BB1 | s.BB1 | BB7 |
| S-P | 7114 | 1660 | 1226 |
| IFM | 6997 | 2400 | 603 |

Conclusions from the values in the Table 1 and other simulations that we have done are: (a) the mixture of max-id copulas BB1, BB4, BB7 cannot distinguish data generated each other, (b) mixture of max-id copulas can be easily distinguished from t, and (c) the N-P estimates are similar with the model-based estimates of tail dependence when the model is correctly specified. These conclusions are based on sample sizes of the order 1000 or less. Note here that as reported in Dobrič and Schmid (2005) when the normal copula (or t copula with large df) is the “true” model enormous bias can occur in resulting estimates.

6.2. Type of vine and indexing of the variables

The next step would be to decide on the vine, and the bivariate pairs to put at level 1 of the vine. For some data sets of financial returns where we fitted all possible permutations of the C-vine and D-vine and compared AIC values, we propose some empirical rules of selecting the \( d - 1 \) pairs in the first tree. Note that these rules would need further evaluation, especially for larger values of the dimension \( d \).
A rule for selecting the “best” permutation for D-vines consists in choosing and connecting the most dependent pairs in the first tree; this could lead to a few candidate permutations. For C-vines, we have developed empirically three different rules. First select a pilot variable 1 that has strong dependence with all other variables. Then

1. list the most dependent variables with the pilot as decreasing in dependence order;
2. list the least dependent variables with the pilot as increasing in dependence order;
3. sequentially list the least dependent variable with the previous selected.

Another rule is to try to get a C-vine with conditional independence after level m. If strengths of conditional dependence match strengths of unconditional dependence, then this would be closer to the first of the above cases. These rules are in line with Aas et al. (2009).

7. Diagnostics, model fitting and forecasting for financial return data

In this section, there are subsections for diagnostics for dependence and reflection asymmetry, univariate and bivariate analyses, goodness-of-fit diagnostics, estimation of dependence parameters in vines, and forecasting of extreme quantiles of a portfolio.

7.1. Diagnostics for dependence and reflection asymmetry

Two types of asymmetric dependence have been mentioned (Longin and Solnik (2001), Tsafack (2009)): one is stronger dependence in returns of different assets during bear markets than bull markets — this is easily seen in scatterplots, the second one is stronger tail dependence in the lower tail (losses) compared with the upper tail (gains) for pairs of assets — this is harder to see in scatterplots, but is suggested by some diagnostics such as those below.

Before proceeding to the vine copulas dependence modeling we performed some non-parametric analysis through simple descriptive statistics and diagnostics to motivate the use of vine copulas with asymmetrical dependence for multivariate financial return data. We considered a few European market indexes: CAC40 France, DAX Germany, OSEAX Norway, SMI Switzerland and FTSE England. In terms of time periods, we considered downturns (bear markets) and upturns (bull markets):

- The period 2003–2006 considered as a period where markets did well.
- The period Fall 2007 to April 2009 considered a period where markets were in a downward trend.

As an initial analysis, we ignore serial dependence and convert the return data to normal scores using the normal quantiles of their empirical distributions: \( z_{ij}^* = \Phi^{-1}(Rank_{ij}/(T + 1)) \), where \( \Phi \) denotes the univariate normal cdf, \( Rank_{ij} \) the rank of the return \( R_{ij} \) over \( t \), for \( j = 1, \ldots, d \), and \( T \) is the number of observations. Note that bivariate normal scores plots are better than uniform scores for assessing tail asymmetry and dependence. Some copula families have infinite density at \((0,0)\) and \((1,1)\) making the assessment of tail dependence unclear from bivariate uniform scores plots. With a bivariate normal scores plot one can check for deviations from the elliptical shape that would be expected with the normal copula. For an assessment of upper and lower tail dependence, we computed empirical versions of:

\[
\begin{align*}
\Pr(Z_1 > z|Z_2 > z) &= \Pr(Z_1 > z, Z_2 > z) / \Pr(Z_2 > z) & \text{upper tail,} \\
\Pr(Z_1 \leq -z|Z_2 \leq -z) &= \Pr(Z_1 \leq -z, Z_2 \leq -z) / \Pr(Z_2 \leq -z) & \text{lower tail,}
\end{align*}
\]

where \( Z_1, Z_2 \) are standard normal, and \( z = \Phi^{-1}(u) \) where \( u \in \{.9,.95,.98,.99\} \). We also computed the conditional versions of Spearman’s \( \rho \) to assess the dependence structure in the tails, see Dobrić et al. (2010).

The summary of findings for the non-parametric tail dependence analysis of the European stock market index data for 2003–2006, and 2007–2009 separately are the following:

- There does seem to be some tail dependence; this is also suggested from the sharper than elliptical shape of the scatterplot cloud.
- There is a slight tendency to more tail dependence in the lower tail than upper tail, but this might not be significant (true for both periods). The Spearman rho values in the conditional tails suggest the same.

Some representative results are presented in Table 2.

The descriptive statistics suggest some skewness to lower tail for many pairs of market index returns. Models such as vine copulas with asymmetrical dependence can be used to check whether the two tails are significantly different. This pattern was also seen for other data sets of financial returns.
Table 2: Tail measures and conditional Spearman’s ρ

<table>
<thead>
<tr>
<th></th>
<th>CAC40, FTSE</th>
<th>DAX, SMI</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>lower tail</td>
<td>upper tail</td>
</tr>
<tr>
<td>1.282</td>
<td>0.76</td>
<td>0.65</td>
</tr>
<tr>
<td>1.645</td>
<td>0.74</td>
<td>0.66</td>
</tr>
<tr>
<td>2.054</td>
<td>0.68</td>
<td>0.53</td>
</tr>
<tr>
<td>2.326</td>
<td>0.78</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Spearman’s ρ on [0, 0.3]²: 0.71
Spearman’s ρ on [0.7, 1]²: 0.63
Spearman’s ρ on [0, 0.5]²: 0.65
Spearman’s ρ on (0, 0.5, 1)²: 0.54

7.2. Univariate and bivariate analyses

In this subsection, some results of univariate and bivariate analyses for the period 2003–2006 are presented. Particularly we study time series of daily log-return data for CAC40, DAX, FTSE, OSEAX, and SMI, denoted by $R_1$ to $R_5$.

The GARCH(1,1) filter is applied to the individual time series with symmetric Student-t distributed iid residuals $z_{jt}, j = 1, \ldots, 5$. For our data, preliminary checks did not suggest the need for an asymmetric Student-t distribution because some skewness measures (such as third central moment and difference of 100(1 − $p$) and 100$p$ percentiles for $p = 0.2, 0.1, 0.05$) were slightly positive and others were slightly negative. Table 3 has the estimates from the univariate GARCH fits. If the GARCH(1,1) models are successful at modeling the serial correlation in the conditional mean and the conditional variance, there should be no autocorrelation left in the standardised residuals and squared standardised residuals. We use the tests of Box and Pierce (1970) and Ljung and Box (1978) to check this. For all series and both tests, the null hypothesis that there is no autocorrelation left cannot be rejected at the 5% level; see Table 3.

Table 3: Estimates from the univariate GARCH models, along with the P-values of autocorrelation tests of Box and Pierce (1970) (BP) and Ljung and Box (1978) (LB) in the standardised residuals $z_j$ and squared standardised residuals $z_j^2$.

<table>
<thead>
<tr>
<th>j (market)</th>
<th>$\mu_j$</th>
<th>$\alpha_{0j}$</th>
<th>$\alpha_{1j}$</th>
<th>$\beta_j$</th>
<th>$\gamma_j$</th>
<th>BP: $z_j$</th>
<th>LB: $z_j$</th>
<th>BP: $z_j^2$</th>
<th>LB: $z_j^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 CAC40</td>
<td>0.089</td>
<td>0.019</td>
<td>0.072</td>
<td>0.91</td>
<td>10.60</td>
<td>0.22</td>
<td>0.21</td>
<td>0.53</td>
<td>0.53</td>
</tr>
<tr>
<td>2 DAX</td>
<td>0.120</td>
<td>0.015</td>
<td>0.077</td>
<td>0.91</td>
<td>8.90</td>
<td>0.62</td>
<td>0.62</td>
<td>0.79</td>
<td>0.79</td>
</tr>
<tr>
<td>3 FTSE</td>
<td>0.063</td>
<td>0.017</td>
<td>0.089</td>
<td>0.88</td>
<td>14.80</td>
<td>0.05+</td>
<td>0.05+</td>
<td>0.92</td>
<td>0.92</td>
</tr>
<tr>
<td>4 OSEAX</td>
<td>0.206</td>
<td>0.081</td>
<td>0.092</td>
<td>0.84</td>
<td>8.06</td>
<td>0.55</td>
<td>0.55</td>
<td>0.14</td>
<td>0.14</td>
</tr>
<tr>
<td>5 SMI</td>
<td>0.088</td>
<td>0.014</td>
<td>0.081</td>
<td>0.90</td>
<td>11.80</td>
<td>0.32</td>
<td>0.32</td>
<td>0.37</td>
<td>0.37</td>
</tr>
</tbody>
</table>

After the non-parametric tail dependence analysis in Section 7.1, we think that bivariate copulas which provide both upper and lower tail dependence are the most adequate building blocks, especially for tree 1 of the vine. Hence we next fit the t, BB1, s.BB1, and BB7 copulas to pairwise GARCH-filtered returns (with univariate parameters fixed as estimated in equation (3)). For comparisons, we also fit the Frank and bivariate normal copulas.

Table 4 has the resulting IFM log-likelihoods ($L_C$) for each bivariate margin. The results are quite similar using S-P estimation. From Table 4, the t copula is overall the best and the BB1/s.BB1 copulas are second best and the Frank copula (tail independence) is worst; the bivariate normal copula which has intermediate tail dependence has larger likelihood values than the Frank copula. We conclude that upper and lower tail dependence of the GARCH-filtered data are suggested. Also for the GARCH-filtered data, the tail dependence measures, like those in Table 2, show a tendency to more lower than upper tail dependence. In the most of the cases the t copula leads to the largest log-likelihood. Therefore we explored the reason of the probable superiority of the t copula through diagnostics on the pairs of random variables.
Table 4: Bivariate log-likelihoods $L_C$ for the period 2003–2006. Based on likelihood, overall $t$ is best and BB1/s.BB1 are second best, and the Frank copula is worst, and upper and lower tail dependence are suggested.

<table>
<thead>
<tr>
<th>Pair</th>
<th>BB1</th>
<th>s.BB1</th>
<th>BB7</th>
<th>$t$</th>
<th>Frank</th>
<th>Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC40 DAX</td>
<td>840.9</td>
<td>839.4</td>
<td>810.4</td>
<td>845.6</td>
<td>769.0</td>
<td>820.2</td>
</tr>
<tr>
<td>CAC40 FTSE</td>
<td>598.3</td>
<td>598.1</td>
<td>582.7</td>
<td>601.6</td>
<td>538.4</td>
<td>595.8</td>
</tr>
<tr>
<td>CAC40 OSEAX</td>
<td>167.2</td>
<td>163.9</td>
<td>167.3</td>
<td>162.5</td>
<td>130.5</td>
<td>160.3</td>
</tr>
<tr>
<td>CAC40 SMI</td>
<td>561.3</td>
<td>562.0</td>
<td>544.2</td>
<td>565.3</td>
<td>511.6</td>
<td>548.1</td>
</tr>
<tr>
<td>DAX FTSE</td>
<td>449.4</td>
<td>447.2</td>
<td>436.4</td>
<td>451.6</td>
<td>415.0</td>
<td>443.5</td>
</tr>
<tr>
<td>DAX OSEAX</td>
<td>123.5</td>
<td>121.2</td>
<td>123.7</td>
<td>120.8</td>
<td>96.5</td>
<td>120.2</td>
</tr>
<tr>
<td>DAX SMI</td>
<td>453.1</td>
<td>451.5</td>
<td>437.6</td>
<td>456.8</td>
<td>425.6</td>
<td>452.6</td>
</tr>
<tr>
<td>FTSE OSEAX</td>
<td>177.4</td>
<td>178.1</td>
<td>175.9</td>
<td>177.2</td>
<td>151.5</td>
<td>170.5</td>
</tr>
<tr>
<td>FTSE SMI</td>
<td>444.6</td>
<td>446.6</td>
<td>434.1</td>
<td>447.8</td>
<td>401.8</td>
<td>446.5</td>
</tr>
<tr>
<td>OSEAX SMI</td>
<td>108.8</td>
<td>108.4</td>
<td>109.4</td>
<td>106.5</td>
<td>82.4</td>
<td>101.4</td>
</tr>
</tbody>
</table>

7.3. Goodness-of-fit diagnostics based on discretized normal scores

In this subsection, we do some comparisons to understand why the $t$ copula seems to be the best fit to bivariate margins.

Let us consider here the $(j,k)$ margin. The GARCH-filtered data are transformed to uniform scores $U_j$ and $U_k$ in $(0,1)$ and then grouped into 6 intervals $[a_0,a_1),\ldots,[a_5,a_6)$ and $[b_0,b_1),\ldots,[b_5,b_6)$ respectively, where $a_i = b_i = \Phi(q_i)$, $\Phi$ denotes the univariate standard normal cdf, and $q_i \in \{-4,-1.4,-0.7,0,0.7,1.4,4\}$. For the rectangle $[a_i,a_{i+1}) \times [b_j,b_{j+1})$, one can calculate the observed relative frequency $O_{ij}$, as well as the expected or model-estimated frequency of the transformed data using any fitted copula model $C(\cdot; \hat{\omega})$:

$$E_{ij} = C(a_{i+1},b_{j+1}; \hat{\omega}) - C(a_i,b_{j+1}; \hat{\omega}) - C(a_{i+1},b_j; \hat{\omega}) + C(a_i,b_j; \hat{\omega}).$$

(5)

Using these frequencies a $\chi^2$ goodness-of-fit measure can be defined

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}};$$

we use this for diagnostics and not for formal hypothesis tests.

Table 5: $\chi^2$ goodness-of-fit measures for the period 2003–2006. Overall $t$ is best and BB7 is worst.

<table>
<thead>
<tr>
<th>margin</th>
<th>BB1</th>
<th>s.BB1</th>
<th>BB7</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC40 DAX</td>
<td>0.044</td>
<td>0.045</td>
<td>0.073</td>
<td>0.034</td>
</tr>
<tr>
<td>CAC40 FTSE</td>
<td>0.030</td>
<td>0.028</td>
<td>0.042</td>
<td>0.029</td>
</tr>
<tr>
<td>CAC40 OSEAX</td>
<td>0.026</td>
<td>0.029</td>
<td>0.027</td>
<td>0.032</td>
</tr>
<tr>
<td>CAC40 SMI</td>
<td>0.034</td>
<td>0.034</td>
<td>0.049</td>
<td>0.027</td>
</tr>
<tr>
<td>DAX FTSE</td>
<td>0.040</td>
<td>0.039</td>
<td>0.054</td>
<td>0.032</td>
</tr>
<tr>
<td>DAX OSEAX</td>
<td>0.041</td>
<td>0.043</td>
<td>0.043</td>
<td>0.044</td>
</tr>
<tr>
<td>DAX SMI</td>
<td>0.040</td>
<td>0.041</td>
<td>0.060</td>
<td>0.032</td>
</tr>
<tr>
<td>FTSE OSEAX</td>
<td>0.029</td>
<td>0.027</td>
<td>0.028</td>
<td>0.030</td>
</tr>
<tr>
<td>FTSE SMI</td>
<td>0.026</td>
<td>0.023</td>
<td>0.038</td>
<td>0.025</td>
</tr>
<tr>
<td>OSEAX SMI</td>
<td>0.037</td>
<td>0.035</td>
<td>0.036</td>
<td>0.039</td>
</tr>
</tbody>
</table>

From the measures based on discretized normal scores, the conclusions from applying the method to all bivariate margins for the period 2003–2006 are the following

1. The $t$ copula fits a little better in the middle than the other two-parameter copulas.
2. The two-parameter copulas provide similar but not optimum fit in the bivariate upper and lower tails.
3. The $\chi^2$ goodness-of-fit measures (Table 5) provide similar information with likelihood (Table 4).
7.4. Estimation of dependence parameters for vines and extreme quantiles of portfolio

Using the sample Kendall’s taus and Blomqvist’s betas in Table 6 and the rules at the end of Section 6, we decide on D- and C-vines.

<table>
<thead>
<tr>
<th>Pair</th>
<th>$\hat{\tau}$</th>
<th>$\hat{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC40 DAX</td>
<td>0.73</td>
<td>0.71</td>
</tr>
<tr>
<td>CAC40 FTSE</td>
<td>0.64</td>
<td>0.63</td>
</tr>
<tr>
<td>CAC40 OSEAX</td>
<td>0.34</td>
<td>0.30</td>
</tr>
<tr>
<td>CAC40 SMI</td>
<td>0.62</td>
<td>0.63</td>
</tr>
<tr>
<td>DAX FTSE</td>
<td>0.57</td>
<td>0.55</td>
</tr>
<tr>
<td>DAX OSEAX</td>
<td>0.29</td>
<td>0.25</td>
</tr>
<tr>
<td>DAX SMI</td>
<td>0.58</td>
<td>0.58</td>
</tr>
<tr>
<td>FTSE OSEAX</td>
<td>0.36</td>
<td>0.33</td>
</tr>
<tr>
<td>FTSE SMI</td>
<td>0.57</td>
<td>0.55</td>
</tr>
<tr>
<td>OSEAX SMI</td>
<td>0.27</td>
<td>0.23</td>
</tr>
</tbody>
</table>

The D-vine rule for choice of permutations leads to $(4,3,1,2,5) = (\text{OSEAX, FTSE, CAC40, DAX, SMI})$ for the D-vine as this has large dependence for all consecutive pairs. The C-vine rule leads to variable 1 (CAC40) as the pilot variable.

1. The most dependent variables with the pilot in decreasing in dependence order leads to $(1,2,5,3,4) = (\text{CAC40, DAX, SMI, FTSE, OSEAX})$.
2. The least dependent variables with the pilot in increasing in dependence order leads to $(1,4,3,5,2) = (\text{CAC40, OSEAX, FTSE, SMI, DAX})$.
3. Sequentially the least dependent variable with the previous selected leads to $(1,4,5,3,2) = (\text{CAC40, OSEAX, SMI, FTSE, DAX})$.

Table 7 and Table 8 give the estimated parameters, and joint log-likelihoods $L_C$ for the best permutations of vine copulas builded by BB1, s.BB1, BB7 and $t$ bivariate blocks using the IFM and the S-P method of estimation, respectively. For these data, for the C-vine and D-vine, the fits at second to fourth levels were mostly close to the independence copula. Therefore the three different best permutations for C-vines are similar after fixing the first pivot as variable 1. In the table we present results only for the C-vine permutation 14532. The vine copulas with $t$ are the best models based on $L_C$, followed by vines with BB1/s.BB1. These have strongest dependence in the baseline copulas and weak conditional dependence.

Table 9 and Table 10 present the model-based tail dependence parameters and Blomqvist’s $\beta$’s at the level-1 copulas of the C-vine and D-vine composed by BB1, s.BB1, BB7, and $t$ copulas estimated using IFM and the S-P method of estimation along with N-P estimates. For our data example, the estimates of tail dependence based on S-P are clearly better than the IFM-based parametric estimates for tail inferences since they more closely match the N-P estimates of tail dependence. The non-parametric results suggest more lower than upper tail dependence and the vine copula with $t$ is inadequate for modeling such tail inferences.

Next, the S-P estimates are used for forecasting of extreme quantiles of the observed equally-weighted portfolio. In Table 11, the likelihood ratio tests of unconditional coverage $LR_{uc}$, independence $LR_{ind}$, and the combination of coverage and independence $LR_{cc}$ are presented for the estimated vine copula models composed by BB1, s.BB1 and $t$ (the best fits according to the Table 8). One can see from Table 11 that the C-vine copula composed by bivariate s.BB1 copulas performs best in terms of forecasting. All the other models with asymmetrical BB1/s.BB1 copulas provide similar prediction of extreme quantiles. However, the C- and D-vine copula with $t$ overestimate the 1st percentile (lower tail) and hence underestimate the VaR of losses.
Table 7: Estimated dependence parameters, and joint log-likelihoods $L_C$ (step 2 of IFM) for the best permutations of vine copulas built by BB1, s.BB1, BB7, and t bivariate blocks. For BB1/s.BB1: $\theta > 0, \delta \geq 1$; for BB7, $\theta \geq 1, \delta > 0$.

<table>
<thead>
<tr>
<th>block</th>
<th>$L_C$</th>
<th>Estimated C-vine copula parameters $\omega$</th>
<th>level-1</th>
<th>level-2</th>
<th>level-3</th>
<th>level-4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>margin:</td>
<td>14</td>
<td>15</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>BB1</td>
<td>2242.5</td>
<td>$\hat{\theta}$'s:</td>
<td>0.40</td>
<td>0.53</td>
<td>0.59</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\delta$'s:</td>
<td>1.25</td>
<td>2.03</td>
<td>2.06</td>
<td>2.81</td>
</tr>
<tr>
<td>s.BB1</td>
<td>2233.2</td>
<td>$\hat{\theta}$'s:</td>
<td>0.26</td>
<td>0.58</td>
<td>0.54</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\delta$'s:</td>
<td>1.33</td>
<td>2.00</td>
<td>2.11</td>
<td>2.55</td>
</tr>
<tr>
<td>BB7</td>
<td>2178.5</td>
<td>$\hat{\theta}$'s:</td>
<td>1.33</td>
<td>2.38</td>
<td>2.42</td>
<td>3.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\delta$'s:</td>
<td>0.60</td>
<td>1.46</td>
<td>1.63</td>
<td>2.17</td>
</tr>
<tr>
<td>t</td>
<td>2257.5</td>
<td>$\hat{\rho}$'s:</td>
<td>0.53</td>
<td>0.83</td>
<td>0.85</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\hat{\nu}$'s:</td>
<td>13.4</td>
<td>6.94</td>
<td>13.3</td>
<td>5.72</td>
</tr>
</tbody>
</table>

Table 8: Estimated dependence parameters, and joint log-likelihoods $L_C$ (S-P estimation) for the best permutations of vine copulas built by BB1, s.BB1, BB7, and t bivariate blocks. For BB1/s.BB1: $\theta > 0, \delta \geq 1$; for BB7, $\theta \geq 1, \delta > 0$.

<table>
<thead>
<tr>
<th>block</th>
<th>$L_C$</th>
<th>Estimated D-vine copula parameters $\omega$</th>
<th>level-1</th>
<th>level-2</th>
<th>level-3</th>
<th>level-4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>margin:</td>
<td>14</td>
<td>15</td>
<td>12</td>
<td>25</td>
</tr>
<tr>
<td>BB1</td>
<td>2233.6</td>
<td>$\hat{\theta}$'s:</td>
<td>0.28</td>
<td>0.65</td>
<td>0.57</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\delta$'s:</td>
<td>1.38</td>
<td>2.03</td>
<td>2.73</td>
<td>1.91</td>
</tr>
<tr>
<td>s.BB1</td>
<td>2233.2</td>
<td>$\hat{\theta}$'s:</td>
<td>0.39</td>
<td>0.52</td>
<td>0.73</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\delta$'s:</td>
<td>1.32</td>
<td>2.14</td>
<td>2.55</td>
<td>1.76</td>
</tr>
<tr>
<td>BB7</td>
<td>2176.8</td>
<td>$\hat{\theta}$'s:</td>
<td>1.51</td>
<td>2.40</td>
<td>3.34</td>
<td>2.19</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\delta$'s:</td>
<td>0.54</td>
<td>1.74</td>
<td>2.19</td>
<td>1.09</td>
</tr>
<tr>
<td>t</td>
<td>2256.9</td>
<td>$\hat{\rho}$'s:</td>
<td>0.55</td>
<td>0.85</td>
<td>0.91</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\hat{\nu}$'s:</td>
<td>9.10</td>
<td>14.5</td>
<td>5.79</td>
<td>13.0</td>
</tr>
</tbody>
</table>

*aVariable 1=CAC40, 2=DAX, 3=FTSE, 4=OSEAX, 5=SMI.*
8. Discussion and further research

We have compared vine copulas with BB1/BB7 and vine copulas with t for multivariate financial return data, and have used for the first time BB1/BB7 copulas with vines. For the data set in Section 7 and other similar data sets, we found that vine copulas with bivariate t linking copulas tend to be best based on a likelihood or AIC comparison because bivariate t copulas provide better fit in the middle for the first level of the vine. However, for inferences involving tails, our recommendation is that the “best-fitting” copula should not just be likelihood-based but also depend on matching to N-P tail dependence measures and extreme quantiles. One needs to do both IFM and S-P estimation as a sensitivity analysis and compare with the fully non-parametric approach of Dobrić and Schmid (2005) to assess goodness-of-fit for tail inference. Empirically with other financial return data, sometimes IFM and S-P agree with the N-P tail dependence, sometimes just one of them does, and sometimes neither does.

In the latter case, as for example in Section 7 where IFM estimates of tail dependence do not match with the N-P estimates, models such as vine copulas with the asymmetric BB1/BB7 can be used for sensitivity analysis for
Table 11: Likelihood ratio tests of unconditional coverage \( LR_{uc} \), independence \( LR_{ind} \), and the combination of coverage and independence \( LR_{cc} \) for the estimated vine copula models composed by BB1, S.BB1 and t.

<table>
<thead>
<tr>
<th>1st</th>
<th>Block</th>
<th>( LR_{uc} )</th>
<th>p-value</th>
<th>( LR_{ind} )</th>
<th>p-value</th>
<th>( LR_{cc} )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>BB1</td>
<td>0.58</td>
<td>0.45</td>
<td>2.21</td>
<td>0.14</td>
<td>2.81</td>
<td>0.24</td>
</tr>
<tr>
<td>C</td>
<td>s.BB1</td>
<td>0.58</td>
<td>0.45</td>
<td>2.21</td>
<td>0.14</td>
<td>2.81</td>
<td>0.24</td>
</tr>
<tr>
<td>C</td>
<td>t</td>
<td>4.73</td>
<td>0.03</td>
<td>4.53</td>
<td>0.03</td>
<td>9.29</td>
<td>0.01</td>
</tr>
<tr>
<td>D</td>
<td>BB1</td>
<td>0.58</td>
<td>0.45</td>
<td>2.21</td>
<td>0.14</td>
<td>2.81</td>
<td>0.24</td>
</tr>
<tr>
<td>D</td>
<td>s.BB1</td>
<td>0.58</td>
<td>0.45</td>
<td>2.21</td>
<td>0.14</td>
<td>2.81</td>
<td>0.24</td>
</tr>
<tr>
<td>D</td>
<td>t</td>
<td>4.73</td>
<td>0.03</td>
<td>4.53</td>
<td>0.03</td>
<td>9.29</td>
<td>0.01</td>
</tr>
</tbody>
</table>

99th

<table>
<thead>
<tr>
<th>99th</th>
<th>Block</th>
<th>( LR_{uc} )</th>
<th>p-value</th>
<th>( LR_{ind} )</th>
<th>p-value</th>
<th>( LR_{cc} )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>BB1</td>
<td>2.67</td>
<td>0.10</td>
<td>0.05</td>
<td>0.82</td>
<td>2.73</td>
<td>0.25</td>
</tr>
<tr>
<td>C</td>
<td>s.BB1</td>
<td>1.55</td>
<td>0.21</td>
<td>0.08</td>
<td>0.78</td>
<td>1.64</td>
<td>0.44</td>
</tr>
<tr>
<td>C</td>
<td>t</td>
<td>2.67</td>
<td>0.10</td>
<td>0.05</td>
<td>0.82</td>
<td>2.73</td>
<td>0.25</td>
</tr>
<tr>
<td>D</td>
<td>BB1</td>
<td>2.67</td>
<td>0.10</td>
<td>0.05</td>
<td>0.82</td>
<td>2.73</td>
<td>0.25</td>
</tr>
<tr>
<td>D</td>
<td>s.BB1</td>
<td>2.67</td>
<td>0.10</td>
<td>0.05</td>
<td>0.82</td>
<td>2.73</td>
<td>0.25</td>
</tr>
<tr>
<td>D</td>
<td>t</td>
<td>2.67</td>
<td>0.10</td>
<td>0.05</td>
<td>0.82</td>
<td>2.73</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 12: Likelihood ratio tests of unconditional coverage \( LR_{uc} \), independence \( LR_{ind} \), and the combination of coverage and independence \( LR_{cc} \) for a C-vine copula with BB1 where lower tail dependence has been increased and upper tail dependence has been decreased compared with the IFM estimates on the observed data (Model 1).

<table>
<thead>
<tr>
<th>1st</th>
<th>Model increase of ( \lambda_L )</th>
<th>( LR_{uc} )</th>
<th>p-value</th>
<th>( LR_{ind} )</th>
<th>p-value</th>
<th>( LR_{cc} )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>1.12</td>
<td>0.29</td>
<td>1.93</td>
<td>0.16</td>
<td>3.07</td>
<td>0.22</td>
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<tr>
<td>2</td>
<td>0.10</td>
<td>0.58</td>
<td>0.45</td>
<td>2.21</td>
<td>0.14</td>
<td>2.81</td>
<td>0.24</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>0.21</td>
<td>0.65</td>
<td>2.53</td>
<td>0.11</td>
<td>2.76</td>
<td>0.25</td>
</tr>
</tbody>
</table>

99th

<table>
<thead>
<tr>
<th>99th</th>
<th>Model decrease of ( \lambda_U )</th>
<th>( LR_{uc} )</th>
<th>p-value</th>
<th>( LR_{ind} )</th>
<th>p-value</th>
<th>( LR_{cc} )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>4.19</td>
<td>0.04</td>
<td>0.03</td>
<td>0.85</td>
<td>4.24</td>
<td>0.12</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>1.55</td>
<td>0.21</td>
<td>0.08</td>
<td>0.78</td>
<td>1.64</td>
<td>0.44</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>0.77</td>
<td>0.38</td>
<td>0.10</td>
<td>0.75</td>
<td>0.89</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Forecasting of extreme quantiles. To this end, in Table 12 the likelihood ratio tests of unconditional coverage \( LR_{uc} \), independence \( LR_{ind} \), and coverage and independence \( LR_{cc} \) are presented when the lower tail dependence has been increased and upper tail dependence has been decreased compared with the IFM estimates in Table 7. Sensitivity analysis, for extreme value inference of a portfolio, that use more lower tail dependence than likelihood-based estimates, can be performed with BB1/BB7 copulas (but not t copulas) in the vine and the forecasting can be substantially improved.

Another conclusion might be that if there is asymmetric tail dependence, then it is not easy to detect it for data over a period of several years. Perhaps the asymmetric tail dependence with stronger lower tails occurs only temporarily for short periods of time. A vine copula model with BB1/BB7 and time-varying dependence can be used to assess whether this holds; Giacomini et al. (2009) use a copula model with only lower tail dependence and a single dependence parameter varying over time, and their methodology might be applied to the flexible copula families considered in this paper.

When we tried all permutations of variables for low-dimensional C-vines and D-vines, the best vine in terms of AIC is usually one that has the highest bivariate dependence at level 1. Also the conditional dependence at levels 2 and higher was relatively weak. Having highly-dependent pairs at level 1 could lead to a vine that is neither the C-vine
nor the D-vine. For example, if matrix of bivariate dependence measures looks like

\[
\begin{pmatrix}
1 & .7 & .6 & .7 & .3 \\
.7 & 1 & .5 & .5 & .5 \\
.6 & .5 & 1 & .4 & .3 \\
.7 & .5 & .4 & 1 & .5 \\
.3 & .5 & .3 & .5 & 1 \\
\end{pmatrix}
\]

then one might consider a vine with pairs 12, 13, 14, 45 at level 1. Joe (2010a) shows that there are four 5-dimensional vines other than the C-vine and D-vine, and three of these have pairs 12, 13, 14, 45 at level 1 (after suitable permutations). In dimensions \( d \geq 6 \), there are many more vines in between the C-vine and D-vine.

To get a vine copula model with tail dependence in all pairs of variables, results of Joe et al. (2010) imply that tail dependence only in bivariate copulas at level 1 of the vine is needed. At levels 2 and higher, independence or Gaussian copulas might be adequate. In this paper we were using a common family at all levels of the vine in order to make a comparison of BB1/BB7 versus t. Future research will include non-C/non-D vines for dimensions \( d \geq 6 \); because the total number of dependence parameters could be increasing quadratically in \( d \), a parsimonious approach is to fit vine structures that have approximately conditional independence after level \( m \) for some \( 1 \leq m \leq d - 1 \). This is more important than insisting on the boundary cases of C-vines or D-vines. If \( m = 1 \), then we have a model has Markov structure, but in general this is not expected. A big advantage of vine copulas is that they can accommodate more general dependence than Markov models.

**Appendix**

The max-id mixture form for bivariate copula is:

\[
C(u, v; \theta, \delta) = \psi_\theta(e^{-\psi^{-1}(u); \delta}, e^{-\psi^{-1}(v); \delta})
\]

where \( K \) is a bivariate max-id copula \( [K \) is max-id if \( K^\alpha \) is a cdf for all \( \alpha > 0 \) and \( \psi \) is a Laplace transform (LT). If \( K(\cdot; \delta) \) is increasing in concordance as \( \delta \) increases, then \( C(\cdot; \theta, \delta) \) is increasing in concordance in \( \delta \). Increasing on concordance for \( \theta \) cannot be established in general and would have to be checked individually.

Below we summarize the BB1, BB4, BB7 families, where the notation comes from Joe (1997), and then list some properties and make some comparisons.

**Family BB1.** [Example 5.1 in Joe and Hu (1996)]. In (6), let \( K \) be the bivariate Gumbel copula and let \( \psi \) be the gamma LT: \( \psi_\theta = (1 + \delta)^{-1/\theta} \). Then

\[
C(u, v; \theta, \delta) = \left( 1 + \left[ (u^{-\theta} - 1)^\delta + (v^{-\theta} - 1)^\delta \right]^{1/\delta} \right)^{-1/\theta}, \quad \theta > 0, \ \delta \geq 1.
\]

The tail dependence parameters are \( \lambda_L = 2^{1/\delta(\theta)} \) and \( \lambda_U = 2 - 2^{1/\delta} \). Given tail parameters \( \lambda_U, \lambda_L; \delta = \log(2 - \lambda_U) \) and \( \theta = \log(2)/(-\log \lambda_L) = \log(2 - \lambda_U)/(-\log \lambda_L) \), Blomqvist’s \( \beta \) is \( 4C(\frac{1}{2}, \frac{1}{2}; \theta, \delta) - 1 \) and

\[
C(\frac{1}{2}, \frac{1}{2}; \theta, \delta) = \{ 1 + 2^{1/\delta}(2^{\theta} - 1) \}^{-1/\theta} = \{ 1 + (2 - \lambda_U)(2^{\theta} - 1) \}^{-1/\theta} = \{ 2^{\theta+1} - 1 - \lambda_U(2^{\theta} - 1) \}^{-1/\theta}.
\]

**Family BB4.** [Example 5.3 in Joe and Hu (1996)]. In (6), let \( K \) be the bivariate Galambos copula and let \( \psi \) be the gamma LT. Then

\[
C(u, v; \theta, \delta) = \left( u^{\theta} + v^{\theta} - 1 - [(u^{\theta} - 1)^{-\delta} + (v^{\theta} - 1)^{-\delta}]^{-1/\delta} \right)^{-1/\theta}, \quad \theta \geq 0, \ \delta > 0.
\]

The tail dependence parameters are \( \lambda_L = (2 - 2^{-1/\delta})^{-1/\theta} \) and \( \lambda_U = 2^{-1/\delta} \). Given tail parameters \( \lambda_U, \lambda_L; \delta = \log(2)/(-\log \lambda_L) \) and \( \theta = \log(2 - 2^{-1/\delta})/(-\log \lambda_L) = \log(2 - \lambda_U)/(-\log \lambda_L) \) (same \( \theta \) as BB1). Blomqvist’s \( \beta \) is \( 4C(\frac{1}{2}, \frac{1}{2}; \theta, \delta) - 1 \) and

\[
C(\frac{1}{2}, \frac{1}{2}; \theta, \delta) = [2^{\theta+1} - 2^{-\delta}(2^{\theta} - 1)]^{-1/\theta} = [2^{\theta+1} - 1 - \lambda_U(2^{\theta} - 1)]^{-1/\theta}.
\]
With $\lambda_L, \lambda_U$ fixed, this is the same as BB1.

**Family BB7.** [This family was in the first draft of Joe and Hu (1996) but didn’t appear in the published version.]

In (6), let $K$ be the bivatiate MTCJ family and let $\psi$ be the Sibuya LT family: $\psi_\theta(s) = 1 - (1 - e^{-s})^{1/\theta}$.

$$C(u, v; \theta, \delta) = 1 - \left(1 - \left(1 - \frac{(1 - u)^{\delta} + (1 - v)^{\delta}}{1 - \nu} - 1\right)^{1/\theta}\right), \quad \theta \geq 1, \delta > 0.$$ 

The tail dependence parameters are $\lambda_L = 2^{-1/\theta}, \lambda_U = 2 - 2^{1/\theta}$. Given tail parameters $\lambda_U, \lambda_L; \delta = \lfloor \log 2 \rfloor/(\log 2 - \lambda_U)$ and $\theta = \lfloor \log 2 \rfloor/\log(2 - \lambda_U)$. Blomqvist’s $\beta$ is $4C(\frac{1}{2}, \frac{1}{2}; \theta, \delta) - 1$ and

$$C(\frac{1}{2}, \frac{1}{2}; \theta, \delta) = 1 - \left(1 - \left(2(1 - 2^{-\theta})^{\delta} - 1\right)^{1/\delta}\right)^{1/\theta}.$$ 

Some properties of BB1, BB4 and BB7 which are relevant for modeling are the follows.

- BB1 increasing in concordance as $\theta$ increase, for BB4, this property has only been shown numerically, BB7 is not increasing in concordance in $\theta$ over all $\delta$; see Joe (2010b).
- The BB1 and BB4 copulas are similar (very small $L_\infty$ distance of the cdfs), when their $\theta, \delta$ values are chosen to achieve some fixed $\lambda_U, \lambda_L$. But the Gumbel and Galambos copulas are known to be very similar (see Nikoloulopoulos and Karlis (2008)) and hence it is not surprising that BB1 and BB4 are so similar, since they only differ in the Gumbel versus Galambos copula for $K$ in (6).
- Define the reflected copula $C_R(u, v; \theta, \delta) = u + v - 1 + C(1 - u, 1 - v; \theta, \delta)$ for a bivariate copula $C$. With $\lambda_U, \lambda_L$ fixed in order to determine $\theta, \delta$ and $\theta_R, \delta_R$, we compared the $L_\infty$ distance of $C(\cdot; \theta, \delta)$ and $C_R(\cdot; \theta_R, \delta_R)$ for BB1 and BB7. In this comparison, we also found that BB7 had smaller $L_\infty$ distance than BB1. That is, for BB7, $C(\cdot; \theta, \delta)$ and $C_R(\cdot; \theta_R, \delta_R)$ are closer to each other.

Because of these properties, for the modeling with vine copulas in Section 7, we use only BB1, reflected or survival BB1 and BB7 for the bivariate copula families with asymmetric lower and upper tail dependence.

We made further comparisons with t copulas when $\lambda_L = \lambda_U$. Numerically $\beta$ is smaller for BB7 than BB1/BB4, when $\lambda_L, \lambda_U$ are fixed. The difference of the $\beta$ values can be over 0.1 for some $(\lambda_L, \lambda_U)$ pairs. This means for a fixed $\beta$, BB7 has heavier tail dependence. Table 13 has values of $\beta$ for fixed $\lambda_L = \lambda_U$ in $0.1(0.1)0.9$ for BB1/BB4, BB7 and the t copula with 2,5,10 degrees of freedom. For the t copula, with $T_k$ being the t cdf with $k$ degrees of freedom,

$$\lambda = \lambda_L = \lambda_U = 2T_{v+1}\left(\frac{\sqrt{v+1} \sqrt{1-\rho}}{\sqrt{1+\rho}}\right).$$

and given $\lambda_L = \lambda_U$ and $v$, the parameter $\rho$ is determined, and then $\beta$ can be computed. For the t copula, as the degrees of freedom increases, fixed $\lambda_U = \lambda_L$ means that Blomqvist’s $\beta$ increases as $v$ increases. Equivalently for fixed $\beta$, tail dependence decreases as $v$ increases. The table shows that BB1/BB4 are closer to the t copula with 5 df than BB7.

<table>
<thead>
<tr>
<th>$\lambda_L = \lambda_U$</th>
<th>BB1/BB4</th>
<th>BB7</th>
<th>$t_2$</th>
<th>$t_5$</th>
<th>$t_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.181</td>
<td>0.156</td>
<td>-0.192</td>
<td>0.146</td>
<td>0.368</td>
</tr>
<tr>
<td>0.2</td>
<td>0.274</td>
<td>0.226</td>
<td>0.036</td>
<td>0.323</td>
<td>0.503</td>
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<tr>
<td>0.3</td>
<td>0.363</td>
<td>0.294</td>
<td>0.204</td>
<td>0.448</td>
<td>0.596</td>
</tr>
<tr>
<td>0.4</td>
<td>0.450</td>
<td>0.363</td>
<td>0.345</td>
<td>0.549</td>
<td>0.671</td>
</tr>
<tr>
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<td>0.537</td>
<td>0.433</td>
<td>0.470</td>
<td>0.637</td>
<td>0.736</td>
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<tr>
<td>0.6</td>
<td>0.625</td>
<td>0.508</td>
<td>0.586</td>
<td>0.717</td>
<td>0.794</td>
</tr>
<tr>
<td>0.7</td>
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<td>0.588</td>
<td>0.694</td>
<td>0.792</td>
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<tr>
<td>0.8</td>
<td>0.807</td>
<td>0.680</td>
<td>0.798</td>
<td>0.865</td>
<td>0.901</td>
</tr>
<tr>
<td>0.9</td>
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<td>0.807</td>
<td>0.900</td>
<td>0.932</td>
<td>0.951</td>
</tr>
</tbody>
</table>
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References