Asymptotic Analysis of Multivariate Coherent Risks

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- Univariate Coherent Risk Measures
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Univariate Coherent Risk Measures

Let $\mathcal{L}$ be the convex cone consisting of all the variables $X$ which may represent losses of portfolios at the end of a given period.

Risk $\varrho(X)$ for loss $X$ corresponds to the amount of extra capital requirement that has to be invested in some secure instrument so that the resulting position $\varrho(X) - X$ is acceptable to regulator/supervisor.
Univariate Coherent Risk Measures

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A mapping $\varrho : \mathcal{L} \rightarrow \mathbb{R}$ is called a coherent risk measure if $\varrho$ satisfies the four coherent axioms (Artzner et al. Math Fin 1999):

- (monotonicity) For $X_1, X_2 \in \mathcal{L}$ with $X_1 \leq X_2$ almost surely, $\varrho(X_1) \leq \varrho(X_2)$.
- (subadditivity) For all $X_1, X_2 \in \mathcal{L}$, $\varrho(X_1 + X_2) \leq \varrho(X_1) + \varrho(X_2)$.
- (positive homogeneity) For all $X \in \mathcal{L}$ and every $\lambda > 0$, $\varrho(\lambda X) = \lambda \varrho(X)$.
- (translation invariance) For all $X \in \mathcal{L}$ and every $l \in \mathbb{R}$, $\varrho(X + l) = \varrho(X) + l$.

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Duality

Under some regularity conditions (Fatou property, ...), a coherent risk measure \( \varrho(X) \) arises as the supremum of expected values of loss \( X \) under various scenarios:

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\varrho(X) = \sup_{Q \in S} E_Q(X)
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where \( S \) is a convex set of probability measures on states, that are absolutely continuous with respect to the underlying measure \( \mathbb{P} \).
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If the scenario set \( S = \{ \mathbb{P}(\cdot | A) : \mathbb{P}(A) \geq 1 - p \} \), then \( \varrho(X) \) is known as the *worst conditional expectation*.

In the case of continuous losses, \( \varrho(X) \) equals to the *tail conditional expectation* (TCE):

\[
TCE_p(X) := E(X | X > \text{VaR}_p(X))
\]

where \( \text{VaR}_p(X) := \inf \{ x \in \mathbb{R} : \Pr\{X > x\} \leq 1 - p \} \) is known as the *Value-at-Risk* (VaR) with confidence level \( p \). Note that VaR is not coherent.
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Note that VaR is not coherent.
What Happens If $p \to 1$?

For light-tailed loss distributions, such as the normal distribution and any phase-type distribution, $\text{TCE}_p(X) \approx \text{VaR}_p(X)$ as $p \to 1$.

A Heavy-Tail Example: For the Pareto distributed loss with survival function $F(r) := 1 - F(r) = (1 + r)^{-\alpha}$, $r \geq 0$, if $\alpha > 1$, then $\text{TCE}_p(X) = \frac{\alpha}{\alpha - 1} \text{VaR}_p(X)$, $0 < p < 1$.

In general, if $X$ has a regularly varying survival function $F(r) = r^{-\alpha} L(r)$, $r > 0$, $\alpha > 0$, where $L > 0$ is a slowly varying function (i.e., $\lim_{r \to \infty} L(cr)/L(r) = 1$ for any $c > 0$), then the straightforward calculations lead to $\text{TCE}_p(X) = \text{VaR}_p(X) + \int_{\infty}^{\text{VaR}_p(X)} \frac{\text{Pr}\{X > x\}}{\text{Pr}\{X > \text{VaR}_p(X)\}} \, dx$. 

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$$\text{TCE}_p(X) = \text{VaR}_p(X) + \int_{\text{VaR}_p(X)}^{\infty} \Pr\{X > x\} dx / \Pr\{X > \text{VaR}_p(X)\}.$$
Karamata’s Theorem

What is the limit of

\[
\int_{\text{VaR}_p(X)}^\infty \frac{\text{Pr}\{X > x\} dx}{\text{Pr}\{X > \text{VaR}_p(X)\}} \rightarrow ? \text{ as } p \rightarrow 1.
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If \( X \) has a regularly varying survival function \( \bar{F}(r) = r^{-\alpha}L(r) \), \( r > 0, \alpha > 1 \), then

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Tail Estimate of TCE via the Karamata’s Theorem

If \( \alpha > 1 \), then \( \text{TCE}_p(X) \approx \frac{\alpha}{\alpha-1} \text{VaR}_p(X) \), \( \text{as} \ p \rightarrow 1. \)
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$$\int_{\text{VaR}_p(X)}^{\infty} \frac{\Pr\{X > x\}}{\Pr\{X > \text{VaR}_p(X)\}} dx \rightarrow ? \text{ as } p \rightarrow 1.$$ 

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If $X$ has a regularly varying survival function $\bar{F}(r) = r^{-\alpha}L(r), r > 0, \alpha > 1$, then

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If $\alpha > 1$, then $\text{TCE}_p(X) \approx \frac{\alpha}{\alpha-1} \text{VaR}_p(X), \text{ as } p \rightarrow 1.$

The Karamata’s Theorem plays a crucial role in estimating the tail integrals of heavy-tailed distributions, in the univariate case as well as in the multivariate situation.
Consider $\mathbf{X} = (X_1, \ldots, X_d)$ from a multi-asset portfolio at the end of a given period, where $X_i = \text{loss of the position on the } i\text{-th market.}$
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**Motivation:**

- Investors are sometimes not able to aggregate their multivariate portfolios on various security markets because of liquidity problems and/or transaction costs .... (Jouini et al. Fin. & Stoch. 2004).

- A risk measure $R(X)$ for loss vector $X$ corresponds to an upper closed subset of $\mathbb{R}^d$ consisting of all the deterministic portfolios $x$ such that the modified positions $x - X$ is acceptable to regulator/supervisor. That is, any deterministic portfolio in set $R(X)$ ‘cancels’ the risk imposed by losses $X$ on various markets.
Coherent Risks (Jouini et al. Fin. & Stoch. 2004)

A multivariate risk measure $R(X)$ with $0 \in R(0) \neq \mathbb{R}^d$ is called coherent if

- (Monotonicity) For any $X$ and $Y$, $X \leq Y$ component-wise implies that $R(X) \supseteq R(Y)$,
- (Subadditivity) For any $X$ and $Y$, $R(X + Y) \supseteq R(X) + R(Y)$,
- (Positive Homogeneity) For any $X$ and positive $s$, $R(sX) = sR(X)$,
- (Translation Invariance) For any $X$ and any deterministic vector $l$, $R(X + l) = R(X) + l$.

Here we use the component-wise ordering to simplify the discussion. In general, a partial ordering induced by a closed convex cone $K \supseteq \mathbb{R}^d$ can be used to account for some frictions on the financial market such as transaction costs, liquidity problems, irreversible transfers, etc...
A multivariate risk measure $R(\mathbf{X})$ with $\mathbf{0} \in R(\mathbf{0}) \neq \mathbb{R}^d$ is called **coherent** if

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Scenario-Based Representation

Vector-Valued Coherent Risk Measure with the Fatou Property

\[ R(X) = \{ x \in \mathbb{R}^d : E_Q(x - X) \geq 0, \ \forall \ Q \in S \}, \]

where \( S \) is a closed convex set of probability measures that are absolutely continuous with respect to the underlying probability measure \( \mathbb{P} \).
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If \( S \) is taken to be the set of conditional probability measures given various tail events, \( R(X) \) is the worst conditional expectation:

\[
WCE_p(X) := \{ x \in \mathbb{R}^d : E(x - X | B) \geq 0, \forall B \in \mathcal{F}, \mathbb{P}(B) \geq 1 - p \}. 
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For any continuous random loss vector \( X \), \( WCE_p(X) \) equals the vector-valued tail conditional expectation:

\[ TCE_p(X) = \bigcap_{A \in Q_p(X)} (E(X \mid X \in A) + \mathbb{R}_+^d), \ 0 < p < 1, \]

where \( Q_p(X) = \{ A \subseteq \mathbb{R}^d : A \text{ is upper, } \Pr\{X \in A\} \geq 1 - p \}. \]
Multivariate Regular Variation

Consider random loss vectors on $\mathbb{R}^d = [0, \infty]^d$. The extreme value analysis of TCE $TCE_p(X)$ as $p \to 1$ boils down to analyzing asymptotic behaviors of $E(X \mid X \in rB)$ as $r \to \infty$ for various upper set $B$. 

A random vector $X$ is said to have a multivariate regularly varying (MRV, Resnick 2007) distribution $F$ if there exists a Radon measure $\mu$, called the intensity measure, on $\mathbb{R}^d_+ \{0\}$ such that

$$\lim_{r \to \infty} \frac{\Pr\{X \in rB\}}{\Pr\{|X| > r\}} = \mu(B),$$

for any relatively compact set $B \subset \mathbb{R}^d_+ \{0\}$ with $\mu(\partial B) = 0$, where $|\cdot|$ denote a norm on $\mathbb{R}^d$. Note that $\mu(rB) = r^{\alpha} \mu(B)$ for any $r > 0$ and any subset $B$ that is bounded away from the origin. We assume that tail index $\alpha > 1$.

Example: Multivariate $t$, heavy-tailed scale mixtures or any distribution with heavy-tailed margins and copula whose tail dependence function exists.
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Note that $\mu(rB) = r^{-\alpha} \mu(B)$ for any $r > 0$ and any subset $B$ that is bounded away from the origin. We assume that tail index $\alpha > 1$.

**Example:** Multivariate $t$, heavy-tailed scale mixtures or any distribution with heavy-tailed margins and copula whose tail dependence function exists.
Tail Estimates of Multivariate Regular Variation

**MRV Rewrite:** For any subset $B$ bounded away from the origin,

$$\Pr\{X \in rB\} \approx \frac{\mu(B)}{\mu((1, \infty] \times \mathbb{R}^{d-1}_+)} \Pr\{X_i > r\}, \text{ as } r \to \infty.$$
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That is, multivariate heavy-tails can be approximated proportionally by univariate heavy-tails where the proportionality constant $\frac{\mu(B)}{\mu((1, \infty] \times \mathbb{R}^{d-1})}$ encodes the extremal dependence information of multivariate extremes.
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\[
\Pr\{ \|\mathbf{X}\| > r \} \approx \frac{\mu(\{ \mathbf{x} : \|\mathbf{x}\| > 1 \})}{\mu((1, \infty] \times \mathbb{R}^{d-1}_+)} \Pr\{ X_i > r \}, \text{ as } r \rightarrow \infty.
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Tail Estimates of Multivariate Regular Variation

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Thus, VaR for the tail risk of portfolio aggregations can be approximated proportionally by marginal VaRs. For example,

$$\text{VaR}_p\left(\sum_{i=1}^{d} X_i\right) \approx \frac{\mu\{|x: \sum_{i=1}^{d} x_i > 1\})}{\mu((1, \infty] \times \mathbb{R}^{d-1}_+)} \text{VaR}_p(X_i), \text{ as } p \to 1.$$
Tail Risk of Multivariate Regular Variation

Let $\mathbf{X}$ be a non-negative loss vector that has an MRV df with intensity measure $\mu$ and tail index $\alpha > 1$. Define $u_j(B; \mu) := \int_0^\infty \frac{\mu(A_j(w) \cap B)}{\mu(B)} dw$, where $A_j(w) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_j > w\}, 1 \leq j \leq d$.

Tail Estimates of TCE (Joe and Li, MCAP 2010)

- Let $B$ be an upper set bounded away from $0$, then
  \[
  \lim_{r \to \infty} r^{-1} E(X_j \mid \mathbf{X} \in rB) = u_j(B; \mu).
  \]

- Let $Q_{\|\cdot\|} := \{B \subseteq \mathbb{R}^d : B \text{ is upper}, B \cap S_{d-1}^+ \neq \emptyset, B \subseteq B_1^d(0)^c\}$, where $S_{d-1}^+$ is the unit sphere and $B_1^d(0)$ is the unit open ball (w.r.t. the norm $\|\cdot\|$). As $p \to 1$,
  \[
  TCE_p(\mathbf{X}) \approx \bigcap_{B \in Q_{\|\cdot\|}} \text{VaR}_{1-(1-p)/\mu(B)}(\|\mathbf{X}\|) \left( (u_1(B; \mu), \ldots, u_d(B; \mu)) + \mathbb{R}^d \right).
  \]
Remark:

- Take $\|X\| = \sum_{i=1}^{d} X_i$, we have,

$$E\left(X_j \Big| \sum_{i=1}^{d} X_i > \text{VaR}_p\left(\sum_{i=1}^{d} X_i\right)\right) \approx \text{VaR}_p\left(\sum_{i=1}^{d} X_i\right) u_j(B; \mu), \text{ as } p \to 1,$$

where $B = \{x : \sum_{i=1}^{d} x_i > 1\}$.

- The tail estimate of $E\left(X_j \mid \sum_{i=1}^{d} X_i > \text{VaR}_p(\sum_{i=1}^{d} X_i)\right)$ provides the contribution to the total tail risk attributable to risk $j$, as measured by TCEs.

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Question: How to estimate the limiting proportionality constants $u_j(B; \mu) = \int_{0}^{\infty} \frac{\mu(A_j(w) \cap B)}{\mu(B)} dw$? We use the tail dependence functions of copulas.
Assume that df $F$ of random vector $\mathbf{X} = (X_1, \ldots, X_d)$ has continuous margins $F_1, \ldots, F_d$, and then the copula $C$ of $F$ can be uniquely expressed as

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \ (u_1, \ldots, u_n) \in [0, 1]^d,$$

where $F_j^{-1}$, $1 \leq j \leq d$, are the quantile functions of the margins.
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The extremal dependence of a df $F$ can be described the \textit{upper tail dependence function}, defined as follows,

$$b^*(\mathbf{w}) := \lim_{u \downarrow 0} \frac{C(1 - uw_j, 1 \leq j \leq d)}{u}, \ \forall \mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}_+^d.$$
Copulas and Tail Dependence Functions

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Define also the upper exponent function of $C$ as follows

$$a^*(w) := \sum_{J \subseteq \{1, \ldots, d\}, J \neq \emptyset} (-1)^{|J|-1} b^*_J(w_i, i \in J; C_J),$$

where $b^*_J(w_i, i \in J; C_J)$ denotes the upper tail dependence function of the margin $C_J$ of $C$ with component indexes in $J$. 
Tail Dependence Function and Intensity Measure

**Example:** (Joe et al. JMVA 2010) Let \( X \) be a loss vector with survival Archimedean copula \( \widehat{C}(u; \phi) = \phi(\sum_{i=1}^{d} \phi^{-1}(u_i)) \), where the Laplace transform \( \phi \) is regularly varying at \( \infty \) with tail index \( \alpha > 0 \).

\[
b^*(w_1, \ldots, w_d) = \left( \sum_{j=1}^{d} w_j^{-1/\alpha} \right)^{-\alpha}.
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**Example:** (Genest and Rivest, SPL 1989) Let \( X \) be a loss vector with Archimedean copula \( C(u; \phi) = \phi(\sum_{i=1}^{d} \phi^{-1}(u_i)) \), where the inverse Laplace transform \( \phi^{-1} \) is regularly varying at 1 with tail index \( \beta > 0 \).

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**Equivalence** (Li and Sun, JAP 2009)

$$b^*(w) = \frac{\mu((\prod_{i=1}^{d} w_i^{-1/\alpha}, \infty])}{\mu([0,1]^{d-1})}, \text{ and } \frac{\mu((w, \infty])}{\mu([0,1]^{d-1})} = \frac{b^*(w_1^{-\alpha}, \ldots, w_d^{-\alpha})}{a^*(1, \ldots, 1)}.$$
Assume that tail index $\alpha > 1$ and tail dependence function $b^*(w) > 0$. Let $\| \cdot \|_{\text{max}}$ denote the maximum norm.
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For $1 \leq j \leq d$,

$$\lim_{r \to \infty} \frac{1}{r} E(X_j \mid X \in r(x, \infty]) = \int_0^\infty \frac{b^*(x_1^{-\alpha}, \ldots, (w_j \vee x_j)^{-\alpha}, \ldots, x_d^{-\alpha})}{b^*(x_1^{-\alpha}, \ldots, x_d^{-\alpha})} \, dw_j.$$
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Let $S_j(b^*, \alpha) := \int_0^\infty \frac{b^*(1, \ldots, 1, (w_j \vee 1)^{-\alpha}, 1, \ldots, 1)}{b^*(1, \ldots, 1)} \, dw_j$, $1 \leq j \leq d$. For sufficiently small $1 - p$, a superset bound is given by

$$\text{TCE}_p(X) \subseteq \text{VaR}_{1 - (1 - p)\frac{a^*(1, \ldots, 1)}{b^*(1, \ldots, 1)}}(\|X\|_{\text{max}}) ((S_1(b^*, \alpha), \ldots, S_d(b^*, \alpha)) + \mathbb{R}^d_+).$$
Tail Estimates via Tail Dependence Functions

For sufficiently small $1 - \rho$, a subset bound is given by

$$\text{VaR}_{\rho}(\|X\|_{\max}) \left( (s_1(b^*, \alpha), \ldots, s_d(b^*, \alpha)) + \mathbb{R}_+^d \right) \subseteq TCE_{\rho}(X)$$

where, for $1 \leq j \leq d$,

$$s_j(b^*, \alpha) := \frac{\alpha}{\alpha - 1} \frac{1}{b^*(1, \ldots, 1) + \sum_{\emptyset \neq S \subseteq \{i:i \neq j\}} (-1)^{|S|} \int_0^1 w_j d b^*_{\{j\} \cup S}(w_j^{-\alpha}, 1, \ldots, 1; C_{\{j\} \cup S}) b^*(1, \ldots, 1)},$$

and $b^*_{\{j\} \cup S}(w_j^{-\alpha}, 1, \ldots, 1; C_{\{j\} \cup S})$ denotes the upper tail dependence function of the multivariate margin $C_{\{j\} \cup S}$ evaluated with the $j$-th argument being $w_j^{-\alpha}$ and others being one.
Example: Second Order Expansion

Consider the following second order expansion:

\[ \overline{C}(1 - uw_j, 1 \leq j \leq d) \approx u b^*(w_1, \ldots, w_d) + u^{1+\zeta} b^*_2(w_1, \ldots, w_d). \]

It is intuitive that if \( \zeta \) is larger (especially if \( \zeta \geq 1 \)), then the second order term is less important. We show that bounds are better with more tail dependence and a larger \( \zeta \).
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The MTCJ copula (or Mardia-Takahasi-Cook-Johnson copula) in dimension \( d \), with dependence increasing in \( \delta \), is:

\[ C(u_1, \ldots, u_d; \delta) = \left[ u_1^{-\delta} + \cdots + u_d^{-\delta} - (d - 1) \right]^{-1/\delta}, \quad \delta > 0. \]
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As $u \to 0$,

$$C((uw_1, \ldots, uw_d); \delta) = ub^*(w_1, \ldots, w_d; \delta) + u^{1+\delta} b_2^*(w_1, \ldots, w_d; \delta),$$

where $b^*(w_1, \ldots, w_d; \delta) = (w_1^{-\delta} + \cdots + w_d^{-\delta})^{-1/\delta},$

$b_2^*(w_1, \ldots, w_d; \delta) = (d - 1)^{-1}(w_1^{-\delta} + \cdots + w_d^{-\delta})^{-1/\delta-1}$. The second order term of $C((uw_1, \ldots, uw_d); \delta)$ is $O(u^{1+\zeta})$, where $\zeta = \delta$ increases with more dependence.
Suppose \((X_1, \ldots, X_d)\) has the multivariate Pareto distribution with univariate marginal survival function \(x^{-\alpha}\) for \(x > 1\) for all \(d\) margins and the MTCJ survival copula. That is,

\[
\bar{F}(x_1, \ldots, x_d) = \left[ x_1^{\delta \alpha} + \cdots + x_d^{\delta \alpha} - (d - 1) \right]^{-1/\delta}, \quad x_j > 1, \ j = 1, \ldots, d.
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Multivariate Pareto Distribution

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The exact calculation, first and second order approximations of TCEs can be calculated via one dimensional numerical integrations.

The results show that the first order approximation is worse only when the dependence is weak and the exponent \(\zeta\) of the second order term is much less than 1; in these cases, the second order term of the expansion is useful.
Multivariate Pareto Distribution

**Table:** Values of exact TCE minus $x_1$, together with first/second order approximations for the bivariate MTCJ copula with Pareto survival margins; $r = (1 - p)^{-1/\alpha}$, $x_1 = x_2 = 1$, $p = 0.999$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\alpha = 2$</th>
<th></th>
<th></th>
<th>$\alpha = 5$</th>
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<tr>
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<td>appr1</td>
<td>appr2</td>
<td>exact</td>
<td>appr1</td>
<td>appr2</td>
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<td>1.761</td>
<td>0.3883</td>
<td>0.3892</td>
<td>0.3883</td>
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<tr>
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<td>1.624</td>
<td>1.622</td>
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<td>0.3692</td>
<td>0.3690</td>
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<td>1.526</td>
<td>1.526</td>
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<tr>
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<td>1.326</td>
<td>1.326</td>
<td>0.3200</td>
<td>0.3200</td>
<td>0.3200</td>
</tr>
</tbody>
</table>
**Multivariate Pareto Distribution**

**Table:** Subset bound (LB) and superset bound (UB) for the MTCJ copula \((d = 2, 3)\) with Pareto survival margins; \(p = 0.999, (1 - p)^{-1/\alpha} \alpha / (\alpha - 1) = 63.25\) and 4.98 provides an intermediate value for \(\alpha = 2\) and 5 respectively.

<table>
<thead>
<tr>
<th>(\delta)</th>
<th>(\alpha = 2)</th>
<th>(\alpha = 5)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>(LB_2)</td>
<td>(UB_2)</td>
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<td>4.0</td>
<td>62.74</td>
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<tr>
<td>5.0</td>
<td>62.91</td>
<td>81.66</td>
</tr>
<tr>
<td>8.0</td>
<td>63.11</td>
<td>74.54</td>
</tr>
</tbody>
</table>
Remarks

- Our results illustrate how tail risk is quantitatively affected by extremal dependence and show how the tool of tail dependence functions can be used to estimate such an asymptotic relation.

The lower and upper bounds for multivariate TCEs become approximately equal for highly tail dependent distributions, and thus our method is effective for analyzing extremal risks for loss variables with significant tail dependence. The quality of the bounds presented might be poor for the distributions with weaker tail dependence. In this situation, one may (1) aggregate loss variables with weak tail dependence, or (2) use the higher order expansions to reveal the intermediate tail dependence structure at sub-extreme levels so that more accurate, tractable bounds can be developed.
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