Math 576: Quantitative Risk Management

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Week 1
Outline

1. Flash Crashes and Algorithmic Trading
2. Hidden Risk: The secret financial market only robots can see
3. This Course
4. Risk Factors
2010 US Stock Market Flash Crash

At 2:45pm, May 6, 2010, the Dow Jones Industrial Average plunged about 1000 points (about 9%) for no apparent reason, but quickly recovered most of these losses within the next few minutes.
2013 AP Twitter Feed Flash Crash

At 1:18pm, April 23, 2013, AP’s Twitter feed was hacked to announce, inaccurately, that a bomb had gone off in the White House, and the S&P Index lose $136 billion in a matter of four minutes.
What’s behind the spooked stock market?

Efficient Market Hypothesis (EMH)

Financial markets are “informationally efficient”; that is, one cannot consistently make profit in excess of average market returns, given the information available at the time the investment is made.

- The efficient-market hypothesis is rooted in French mathematician Louis Bachelier’s work, “Theory of Speculation” (1900).
- The extreme movements are unlikely under EMH.
Adaptive Market Hypothesis (AMH, Andrew Lo, 2004)

Financial markets are ecological systems in which various agents (“species”) compete for resources in stochastically changing environments.

Agents could be humans, automatic trading algorithms, or super species-supercomputers.

- The system evolves under financial interactions of competition and natural selection among agents adapting to a changing environment using limited, partial information.
- The extreme events can happen under AMH, due to agents’ herding, overreaction, and other behavioral biases.
- Contributing factors in the flash crashes: Automatic trading algorithms and high speed trading (including high frequency trading).
According to AMH, profit opportunities (e.g., statistical or event arbitrage) generally exist in financial markets.

Interacting agents compete for profit opportunities (= limited resources) by learning and developing trading strategies (sometimes predatory).

Learning and competition result in the gradual erosion of these profit opportunities, previously successful strategies become less profitable (e.g., high frequency trading now becomes less profitable).

But new profit opportunities arise from the changing market environment. Life of learning, competition, and adaptation continues ...

And risk is part of this interacting agent ecology.
In the big data era, these guys don’t stand a chance!
Hidden Risk at Nano Time Scale

“Abrupt rise of new machine ecology beyond human response time”

Key Findings:

- Analysis of millisecond-scale stock trading data uncovers large numbers of subsecond price crashes/spikes (so called ultrafast extreme events, or UEEs).
- The proliferation of these UEEs due to high speed trading shows an intriguing correlation with the onset of the system-wide global financial collapse in 2008.
- High speed trading generates a new behavioral regime at nano scale as humans lose the ability to intervene in real time.
- The emerging ecology of competitive machines featuring ‘crowds’ of predatory algorithms highlights the need for a new scientific theory of financial phenomena at nano time scale.
High speed computers can prepares trades in 740 nanoseconds (1 nanosecond is $10^{-9}$ seconds).

The quickest that people can notice potential danger, is approximately 1 second. Even a chess grandmaster requires approximately 650 milliseconds ($1\text{ millisecond} = 10^{-3} \text{ second}$) just to realize that she is in trouble.

Data Source from Nanex: High-throughput millisecond-resolution price stream across multiple stocks and exchanges starting in 2006.

The dataset comprises 18,520 ultrafast price crashes and spikes (January 3rd 2006 to February 3rd 2011) with durations less than 1500 milliseconds.

Both ultrafast crashes and spikes are typically more than 30 standard deviations larger than the average price movement either side of an event.
A = Allbanc Split Corp; B = Super Micro Computer

Figure: A: Crash (11/04/2009), duration = 25 ms, percentage price change downwards = -14%. B: Spike (10/01/2010), duration = 25 ms, percentage price change upwards = 26%. 
Figure: Ultrafast crashes (red) and spikes (blue) as compared to S&P 500 in black. Green horizontal lines show periods of escalation of UEEs. Dashed green = Non-financials; solid green = financials.
Power Laws

Let $S$ denote the size of a randomly select ultrafast crash/spike. The random variable $S$ is said to follow the power law if

$$P(S > x) \sim x^{-\alpha}, \text{ for sufficiently large } x,$$

where $\alpha > 0$ is known as the heavy-tail exponent.

- The smaller $\alpha$ the heavier right tail $S$ has.
- Unlike the normal distribution or gamma distribution, the probability that $S$ is extremely large cannot be overlooked.

To diagnose power laws, one can use the probability log-log plot:

$$\log P(S > x) \sim -\alpha \log x, \text{ for sufficiently large } x.$$
Emergent Power Laws of Sizes of Ultrafast Spikes

Figure: For durations more than 1 second, there is strong evidence for a power-law (p-value is 0.912).
Emergent Power Laws of Sizes of Ultrafast Spikes

Figure: For durations less than 1 second, a power-law can be rejected.
Definition
Risk is the potential of loss resulting from an uncertain event, that can occur at any scale in a complex system.

This course focuses on risk analysis with the following topics:
1. empirical facts of risk factors;
2. risk models, measures and estimation;
3. volatility and dependence modeling;
4. catastrophe modeling and extreme value analysis.

Data-Driven Approach
We will proceed in the context of finance, because datasets are freely available from the global financial market (the world’s largest and most complex techno-social system).
Cautionary Remark

- There are many types of risks. The methods we discuss in this class can be applied to other areas but must be used with care.
- Since 2005, World Economic Forum in Geneva publishes “Global Risks” each year to discuss economic, environmental, geopolitical, societal, and technological risks.
- The annual report was coauthored with some leading institutions such as Wharton Risk Center, University of Pennsylvania, as well as well-known insurance companies such as Swiss Re.
Global Risks Landscape 2015 (WEF)

- Weapons of mass destruction
- Biodiversity loss and ecosystem collapse
- Failure of critical infrastructure
- Unmanageable inflation
- Critical infrastructure breakdown
- Energy price shock
- Failure of financial mechanism or institution
- Spread of infectious diseases
- Fiscal crises
- Food crises
- Terrorist attacks
- Asset bubble
- Profound social instability
- Large-scale involuntary migration
- Data fraud or theft
- Natural catastrophes
- Man-made environmental catastrophes
- State collapse or crisis
- Failure of national governance
- Extreme weather events
- Interstate conflict
- Unemployment or underemployment
Top 5 Global Risks in Terms of Likelihood (WEF)

<table>
<thead>
<tr>
<th>2011</th>
<th>2012</th>
<th>2013</th>
<th>2014</th>
<th>2015</th>
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</thead>
<tbody>
<tr>
<td>Storms and cyclones</td>
<td>Severe income disparity</td>
<td>Severe income disparity</td>
<td>Income disparity</td>
<td>Interstate conflict with regional consequences</td>
</tr>
<tr>
<td>Flooding</td>
<td>Chronic fiscal imbalances</td>
<td>Chronic fiscal imbalances</td>
<td>Extreme weather events</td>
<td>Extreme weather events</td>
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<tr>
<td>Corruption</td>
<td>Rising greenhouse gas emissions</td>
<td>Rising greenhouse gas emissions</td>
<td>Unemployment and underemployment</td>
<td>Failure of national governance</td>
</tr>
<tr>
<td>Biodiversity loss</td>
<td>Cyber attacks</td>
<td>Water supply crises</td>
<td>Climate change</td>
<td>State collapse or crisis</td>
</tr>
<tr>
<td>Climate change</td>
<td>Water supply crises</td>
<td>Mismanagement of population ageing</td>
<td>Cyber attacks</td>
<td>High structural unemployment or underemployment</td>
</tr>
</tbody>
</table>
Top 5 Global Risks in Terms of Impact (WEF)
Societal Risks 2014 → 2015 (WEF)
Geopolitical Risks 2014 → 2015 (WEF)

- Weapons of mass destruction
- Interstate conflict
- Terrorist attacks
- State collapse or crisis

Impact

4.0

5.0
Economic Risks 2014 → 2015 (WEF)

- Energy price shock
- Fiscal crises
- Unemployment or underemployment
- Failure of financial mechanism or institution
- Failure of critical infrastructure

Impact

4.0  5.0

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Environmental Risks 2014 → 2015 (WEF)

- Biodiversity loss and ecosystem collapse
- Failure of climate change adaptation
- Extreme weather events
- Natural catastrophes
- Man-made environmental catastrophes
Technological Risks 2014 → 2015 (WEF)
Stochastic Model: \((\Omega, \mathcal{F}, P)\)

- State space \(\Omega := \{\text{all future states } \omega \text{ of the world}\}\).
- Any \(A \subseteq \Omega\) (subset of \(\Omega\)) is called an event.
- The empty set is denoted by \(\emptyset\), which is also called the impossible event.
- The sample space \(\Omega\) is also called the sure event.
- If \(A, B\) are events, then union \(A \cup B\), intersection \(A \cap B\), and complement \(A^c\) are also events.
- \(\sigma\)-field \(\mathcal{F}\) (\(=\) operational class of observable events): A non-empty class of subsets of \(\Omega\) that are closed under countable union, countable intersection and complements.
- Probability measure \(P(\cdot)\): \(\mathcal{F} \rightarrow [0, 1]\) is a (measurable) function of events. That is, \(P(A) := \text{likelihood of event } A\).
Risk Factors and Loss Distributions

- Risk factors are random variables $X, Y, Z, ...$: Variables associated with "unfavorable" random events.
- Risk factor $X : \Omega \rightarrow \mathbb{R}$ is a real-valued (measurable) function defined on $\Omega$.
- "Measurable" means that these risk events are observable:
  
  $$X^{-1}(a, b] := \{ a < X \leq b \} \in \mathcal{F} \text{ for all } a, b \in \mathbb{R};$$

- A loss distribution is the CDF $F(x) := P(X \leq x), x \in \mathbb{R}$.

\[ \lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1. \]
Portfolio Loss

- A portfolio = a collection of stocks or bonds, a book of derivatives, risky loans, and other risky assets.
- $V(s) =$ value of this portfolio at time $s$, $s \geq 0$.
- Given a time horizon $\Delta$ (e.g., 1 or 10 days), the loss of the portfolio over $[s, s + \Delta]$ is given by

$$L_{[s, s+\Delta]} := -(V(s + \Delta) - V(s)).$$

- The loss distribution is the CDF of $L_{[s, s+\Delta]}$.
- For simplicity, take $\Delta = 1$ (time unit, e.g., one hour, one day, one week or one month), and

$$L_{t+1} := -(V(t + 1) - V(t)), \quad t \geq 0.$$
Mapping of Risks

Let $Z_t = (Z_{t,1}, \ldots, Z_{t,d})$ denote the vector of observable risk factors at time $t$.

Mapping of risks:

$$V_t = f(t, Z_{t,1}, \ldots, Z_{t,d}) = f(t, Z_t), \quad t \geq 0,$$

for some measurable function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$.

Risk-factor change $X_{t+1} = (X_{t+1,1}, \ldots, X_{t+1,d}) = Z_{t+1} - Z_t, \quad t \geq 0$.

The portfolio loss:

$$L_{t+1} = -(f(t + 1, Z_t + X_{t+1}) - f(t, Z_t))$$

depends on risk factors at time $t$ and factor changes from $t$ to $t + 1$. 

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Let \( Z_t = (Z_{t,1}, \ldots, Z_{t,d}) \) denote the vector of observable risk factors at time \( t \).

Risk-factor change \( X_{t+1} = (X_{t+1,1}, \ldots, X_{t+1,d}) = Z_{t+1} - Z_t, \ t \geq 0 \).

Using the Taylor Expansion, the linearized portfolio loss:

\[
L_{t+1} = -(f(t + 1, Z_t + X_{t+1}) - f(t, Z_t)) \\
\approx - \left( f_t(t, Z_t) + \sum_{i=1}^{d} f_{Z_i}(t, Z_t) X_{t+1,i} \right)
\]

where the subscripts of \( f \) denote partial derivatives.

The approximation is useful if the risk-factor changes are likely to be small (i.e., if we are measuring risk over a short horizon time unit \( \Delta \)) and if the portfolio value is almost linear in the risk factors.
Example: A Stock Portfolio

- A portfolio consists of $d$ stocks, with $\lambda_i$ shares of stock $i$.
- The price of stock $i$ at time $t$ is denoted by $S_{t,i}$.
- The risk factor $Z_{t,i} := \log S_{t,i}$, $i = 1, \ldots, d$.
- The risk factor change $X_{t+1,i} := \log S_{t+1,i} - \log S_{t,i} = \log \frac{S_{t+1,i}}{S_{t,i}}$ (known as log-returns).
- Risk mapping:
  \[
  V_t = \sum_{i=1}^{d} \lambda_i S_{t,i} = \sum_{i=1}^{d} \lambda_i e^{Z_{t,i}}.
  \]
- The portfolio loss:
  \[
  L_{t+1} = -(V_{t+1} - V_t) = \sum_{i=1}^{d} \lambda_i S_{t,i}(e^{X_{t+1,i}} - 1).
  \]
- The linearized loss:
  \[
  L_{t+1} \approx -V_t \sum_{i=1}^{d} w_{t,i} X_{t+1,i}, \quad w_{t,i} = \frac{\lambda_i S_{t,i}}{V_t}.
  \]