Assumptions in Non-Cooperative Games

Prisoners’ Dilemma, Again

1. Cooperation does not occur in prisoners’ dilemma, because players cannot make binding agreements.
2. But what if binding agreements are possible?
3. This is exactly the class of scenarios studied in cooperative game theory.
Cooperative Games

- Players form coalitions with some binding agreements.
- Each coalition chooses its action (collective strategy), and players can benefit by cooperating within each coalition.
- **Non-transferable utility (NTU) games**: the choice of coalitional actions (by all coalitions) determines each player’s payoff.
- **Transferable utility (TU) games**: the choice of coalitional actions (by all coalitions) determines the payoff of each coalition. The members of the coalition then need to divide this joint payoff.
NTU Games: Joint Publication

Example

- $n$ researchers working at $n$ different universities can form groups to write papers on probability theory.
- Each group of researchers can work together; the composition of a group determines the quality of the paper they produce.
- Each author receives a payoff from his own university (promotion, bonus, teaching load reduction, etc).
- Payoffs are non-transferable.
TU Games: Ice-Cream Example

Three types of ice-cream tubs are for sale:

1. Type 1 (500g) costs $7.
2. Type 2 (750g) costs $9.
3. Type 3 (1000g) costs $11.

- \( n \) children, each has some amount of money (the \( i \)-th child has \( b_i \) dollars).
- Children have utility for ice-cream, and do not care about money.
- The payoff of each group: the maximum quantity of ice-cream the members of the group can buy by pooling their money.
- The ice-cream can be shared arbitrarily within the group.
How Is a Cooperative Game Played?

- Even though players work together they are still selfish.
- The partition into coalitions and payoff distribution should be such that no player (or group of players) has an incentive to deviate.
- We may also want to ensure that the outcome is fair: the payoff of each player is proportional to his contribution.
- We will now see how to formalize these ideas.
Cooperative TU Games

- \( N = \{1, \ldots, n\} \) is a finite set of \( n \geq 2 \) players.
- A subset \( S \subseteq N \) (or \( S \in 2^N \)) of players is called a coalition.
- \( N \) itself is called the grand coalition.
- \( \nu(\cdot) : 2^N \to \mathbb{R} \) is a set function called the characteristic function.
- For each subset of players \( S \), \( \nu(S) \) is the collective payoff that the members in \( S \) can earn by working together.

**Definition**

A cooperative TU game is a pair \((N, \nu)\).
Land Development Game

1 Player 1 owns a piece of land with value of $10,000.
2 Player 2 is interested in buying and can develop the land increase its value to $20,000.
3 Player 3 is interested in buying and can develop the land increase its value to $30,000.

Find the characteristic function $\nu(\cdot)$ of the game.

- Any coalition that does not contain player 1 has a worth of $0$.
- $\nu(S) = 0$ if $1 \notin S$, and

$$\nu\{1\} = 10,000; \quad \nu\{1, 2\} = 20,000;$$

$$\nu\{1, 3\} = 30,000; \quad \nu\{1, 2, 3\} = 30,000.$$
Three types of ice-cream tubs are for sale:

1. Type 1 (500g) costs $7.
2. Type 2 (750g) costs $9.
3. Type 3 (1000g) costs $11.

Three children 1, 2, 3 have $4, $3, $3 respectively. Find the characteristic function \( \nu(\cdot) \) of the game.

- \( \nu(\emptyset) = \nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = 0 \), and \( \nu(\{2, 3\}) = 0 \)
- \( \nu(\{1, 2\}) = \nu(\{1, 3\}) = 500 \), and

\[
\nu(\{1, 2, 3\}) = 750
\]
Garbage Game

1. Each of 4 property owners has one bag of garbage and must dump it on somebody’s property.
2. Owners can form at most two coalitions.
3. If $b$ bags of garbage are dumped on a coalition of property owners, then the coalition receives a reward of $4 - b$.

Find the characteristic function $\nu(\cdot)$ of the game.

- The best strategy for a coalition $S$ is to dump all of their garbage on the property of owners who are not in $S$.
- $\nu(\emptyset) = 0$, and $\nu(\{1, 2, 3, 4\}) = 0$, and

$$\nu(S) = 4 - |S| \text{ for } 1 \leq |S| \leq 3,$$

where $|S|$ denotes the number of players in $S$. 

Definition
A game \((N, \nu)\) is said to be monotone if

1. \(\nu(\cdot)\) is normalized; that is, \(\nu(\emptyset) = 0\).
2. \(\nu(S_1) \leq \nu(S_2)\) for all \(S_1 \subseteq S_2 \subseteq N\). That is, larger coalitions gain more.

Remark
- \(\nu(S) \geq 0\) for all \(S \subseteq N\).
- The land development and ice-cream games are monotone.
- The garbage game is not monotone.
Superadditive Games

Definition
A game \((N, \nu)\) is said to be superadditive if it is monotone and

\[ \nu(S_1 \cup S_2) \geq \nu(S_1) + \nu(S_2) \]

for any two disjoint coalitions \(S_1\) and \(S_2\).

Remark
- A large coalition as a whole is greater than the sum of its parts.
- The land development and ice-cream games are superadditive.
- The garbage game is not superadditive.
Any Measure is a Game

The characteristic function \( \nu(\cdot) \) of a game is known as a non-additive measure (Gustave Choquet, 1953/54).

Definition
A set function \( \nu(\cdot) : 2^N \rightarrow \mathbb{R} \) is called a measure on \( N \) if
- (non-negativity) \( \nu(S) \geq 0 \) for all \( S \subseteq N \).
- (additivity) \( \nu(S_1 \cup S_2) = \nu(S_1) + \nu(S_2) \) for any two disjoint subsets \( S_1 \) and \( S_2 \).

Remark
If \( \nu(\cdot) \) is a measure on \( N \), then
- (monotonicity) \( \nu(S_1) \leq \nu(S_2) \) for all \( S_1 \subseteq S_2 \subseteq N \);
- \( \nu(\emptyset) = 0 \);
- \( \nu(S_1 \cup S_2) + \nu(S_1 \cap S_2) = \nu(S_1) + \nu(S_2) \) for any two subsets \( S_1 \) and \( S_2 \).
Any Probability Measure is a Game

Definition
A set function $\nu(\cdot) : 2^N \rightarrow \mathbb{R}$ is called a probability measure on $N$ if $\nu(\cdot)$ is a measure with $\nu(N) = 1$.

Example
Let $(x_1, \ldots, x_n)$ denote a probability distribution on $N$; that is,

$$x_1 + \cdots + x_n = 1, \quad 0 \leq x_i \leq 1, \quad \forall i \in N.$$

Define $P(S) := \sum_{i \in S} x_i$ for any $S \subseteq N$.

- $P(\cdot)$ is a probability measure on $N$.
- Probability space $(N, P)$ is an additive game.
Transferable Utility Games: Outcome

Let $C = (C_1, \ldots, C_k)$ be a coalition structure (a partition); that is
\[ \bigcup_{i=1}^{k} C_i = N, \quad C_i \cap C_j = \emptyset, \quad \forall i \neq j. \]

Let $x = (x_1, \ldots, x_n)$ be a reward (payoff) vector with
\[ x_i \geq 0, \quad \forall i \in N \]

Definition
An outcome of a TU game $(N, \nu)$ is a pair $(C, x)$ with
\[ \sum_{i \in C_j} x_i = \nu(C_j), \quad \forall 1 \leq j \leq k. \]

That is, $x$ distributes the value of each coalition in $C$. 
Example

- \( N = \{1, 2, 3, 4, 5\} \) with coalitions \( \{1, 2, 3\} \) and \( \{4, 5\} \).
- \( \nu(\{1, 2, 3\}) = 9 \) and \( \nu(\{4, 5\}) = 4 \).
- \( (((\{1, 2, 3\}, \{4, 5\}), (3, 3, 3, 3, 1)) \) is an outcome.
- \( (((\{1, 2, 3\}, \{4, 5\}), (3, 1, 5, 2, 2)) \) is an outcome.
- \( (((\{1, 2, 3\}, \{4, 5\}), (3, 1, 3, 5, 2)) \) is not an outcome.
- Reward transfers between coalitions are not allowed.
Rationality

Let $x = (x_1, x_2, \ldots, x_n)$ be a reward vector such that player $i$ receives a reward $x_i$, $i = 1, \ldots, n$, in a cooperative environment.

Imputation

A reward vector $x = (x_1, x_2, \ldots, x_n)$ is called an imputation of game $\nu(\cdot)$ if it satisfies

1. Group Rationality: $\nu(N) = \sum_{i=1}^{n} x_i$;
2. Individual Rationality: $\nu(\{x_i\}) \leq x_i$, $i = 1, 2, \ldots, n$.

Land Development Game

Consider $N = \{1, 2, 3\}$. Then $x = (20,000, 5,000, 5,000)$ is an imputation, but $y = (5,000, 10,000, 15,000)$ is not.
Comparison of Imputations

Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be two imputations of a game \( \nu(\cdot) \).

**Domination**

Let \( S \subseteq N \) be a coalition. The imputation \( y \) is said to dominate \( x \) through the coalition \( S \), written as \( y >^S x \), if

1. \( y_i \geq x_i \) for all \( i \in S \) (that is, each member of \( S \) prefers \( y \) to \( x \));
2. \( \sum_{i \in S} y_i \leq \nu(S) \) (that is, the members of \( S \) can attain the rewards given by \( y \)).

**Land Development Game**

Consider \( N = \{1, 2, 3\} \) and \( S = \{1, 3\} \). Let \( x = (19,000, 1,000, 10,000) \), and \( y = (19,800, 100, 10,100) \) be two imputations. Then \( y >^S x \).
What Is a Good Outcome?

Ice-Cream Game

Three children 1, 2, 3 have $4, $3, $3 respectively.

1. Type 1 (500g) costs $7; Type 2 (750g) costs $9; Type 3 (1000g) costs $11.

2. $\nu(\emptyset) = \nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{2, 3\}) = 0$.

3. $\nu(\{1, 2\}) = \nu(\{1, 3\}) = 500$, and $\nu(\{1, 2, 3\}) = 750$.

Analysis

- Three children can buy Type 2 and share as $(200, 200, 350)$.
- But 1 and 2 can get more ice-cream by buying a 500g tub on their own, and splitting it equally.

- $(200, 200, 350)$ is not stable; e.g., $(250, 250, 250) > \{1, 2\} (200, 200, 350)$.
The Core: A Stable Set

Definition (Gillies, 1953)
The core of an $n$-person game $(N, \nu)$ is the set of all undominated imputations.

Theorem

- $I = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \nu(N), \ x_i \geq \nu(\{x_i\}), \ \forall i = 1, \ldots, n\}$
- $\text{Core}(N, \nu) = \{x \in I : \sum_{i \in S} x_i \geq \nu(S), \ \forall S \subset N\}$.

Remark

- The core is an intersection of halfspaces, and hence a convex set.
- The core may be empty.
Example: Ice-Cream

Ice-Cream Game

Three children 1, 2, 3 have $4, $3, $3 respectively.

1. Type 1 (500g) costs $7; Type 2 (750g) costs $9; Type 3 (1000g) costs $11.
2. \( \nu(\emptyset) = \nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = \nu(\{2, 3\}) = 0. \)
3. \( \nu(\{1, 2\}) = \nu(\{1, 3\}) = 500, \) and \( \nu(\{1, 2, 3\}) = 750. \)

- (200, 200, 350) is not in the core.
- (250, 250, 250) is in the core because no subgroup of players can deviate so that each member of the subgroup gets more.
- (750, 0, 0) is in the core because 2 and 3 cannot get more on their own.
Example: Land Development

- $N = \{1, 2, 3\}$.  
- $\nu(S) = 0$ if $1 \notin S$, and 
  
  $$
  \nu(\{1\}) = 10,000; \quad \nu(\{1, 2\}) = 20,000; \\
  \nu(\{1, 3\}) = 30,000; \quad \nu(\{1, 2, 3\}) = 30,000.
  $$

Imputations and Core

- The set of all imputations is given by
  
  $$
  I = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 30,000, \quad x_1 \geq 10,000, \quad x_2 \geq 0, \quad x_3 \geq 0\}
  $$

- $\text{Core}(N, \nu) = \{x : x = (x_1, 0, 30,000 - x_1), \quad x_1 \geq 20,000\}$. 
Empty Core

Consider the game:

- $N = \{1, 2, 3\}$.
- $\nu(S) = 0$ if $|S| = 1$, and
  
  $\nu(\{1, 2\}) = \nu(\{1, 3\}) = \nu(\{2, 3\}) = 1$, $\nu(\{1, 2, 3\}) = 1$.

Imputations and Core

- The set of all imputations is given by
  
  $I = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, \ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0\}$.

- $\text{Core}(N, \nu) = \emptyset$.

- The game is superadditive but has no core.
The core is a very attractive solution concept.
However, some games have empty cores.
People have developed the concepts of $\epsilon$-core and least core to describe approximately stable outcomes.
Another issue about cores is lack of fairness.
Core and Superadditivity

Consider the game:

- \( N = \{1, 2, 3, 4\} \).
- \( \nu(S) = 0 \) if \( |S| \leq 1 \), and \( \nu(S) = 1 \) if \( |S| > 1 \).

- The set of all imputations is given by
  \[
  I = \{ x : x_1 + x_2 + x_3 + x_4 = 1, \ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0 \}.
  \]
- \( \text{Core}(N, \nu) = \emptyset \).
- The game is not superadditive:
  \[
  \nu(\{1, 2\}) + \nu(\{3, 4\}) = 2 > 1 = \nu(\{1, 2, 3, 4\}).
  \]
Revise the game by adding cost for stability:

- \( N = \{1, 2, 3, 4\} \).
- \( \nu(S) = 0 \) if \(|S| \leq 1\), and \( \nu(S) = 1 \) if \( 1 < |S| < 4 \), and \( \nu(\{1, 2, 3, 4\}) = 1 + 1 \) (cost).

- The set of all imputations is given by

\[
I = \{ x : x_1 + x_2 + x_3 + x_4 = 2, \ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0 \}.
\]

- \( \text{Core}(N, \nu) = \{(0.5, 0.5, 0.5, 0.5)\} \).
- The game becomes superadditive.
Stability vs. Fairness

Outcomes in the core may be unfair.

Example

- \( N = \{1, 2\} \).
- \( \nu(\emptyset) = 0, \nu(\{1\}) = \nu(\{2\}) = 5, \) and \( \nu(\{1, 2\}) = 20 \).
- The imputation \((15, 5)\) is in the core: player 2 cannot get more by deviating.
- However, this is unfair since 1 and 2 make the same contribution.
- How do we divide payoffs in a fair way?
An optimal solution of a game $\nu(\cdot)$ on players $N = \{1, 2, \ldots, n\}$ should give more rewards to the game’s most important coalitions, and the core provides such a solution concept.

Instead of maximizing rewards for the most important coalition in a game, Lloyd Shapley proposed the idea to specify the desirable properties that an optimal solution should satisfy and then derive the tractable expression of an optimal solution using duality.

**Notation:**
For a game $\nu(\cdot)$, a reward vector, depending on $\nu(\cdot)$, is written as follows:

$$x = (x_1, \ldots, x_n) := (\phi_1(\nu), \ldots, \phi_n(\nu)).$$
Axiomatic Approach

1 Symmetry: If two players $i$ and $j$ are equivalent in the sense that
\[ \nu(S \cup \{i\}) = \nu(S \cup \{j\}) \]
for every coalition $S$ of $N$ which contains neither $i$ nor $j$, then
\[ \phi_i(\nu) = \phi_j(\nu). \]

2 Efficiency (or Group Rationality): \[ \sum_{i \in N} \phi_i(\nu) = \nu(N). \]

3 Null Player: If \[ \nu(S \cup \{i\}) = \nu(S) \] for all coalitions $S$, then
\[ \phi_i(\nu) = 0. \]

4 Additivity: For any two games $\nu(\cdot)$ and $\mu(\cdot)$,
\[ \phi_i(\nu + \mu) = \phi_i(\nu) + \phi_i(\mu), \quad i = 1, 2, \ldots, n. \]

The validity of these axioms has often been questioned, especially during the recent global financial crisis.
The Shapley Value

Theorem (Shapley, 1953)

Given any \( n \)-person game \( \nu(\cdot) \), there exists a unique reward vector \( x = (\phi_1(\nu), \ldots, \phi_n(\nu)) \) satisfying Axioms 1-4 and having the following expression:

\[
\phi_i(\nu) = \sum_{\text{all } S \text{ for which } i \notin S} \left[ \nu(S \cup \{i\}) - \nu(S) \right] p(S), \quad i = 1, 2, \ldots, n
\]

where \( p(S) = \frac{|S|!(n-|S|-1)!}{n!}, \quad S \subseteq N. \)

**Notation:** \( n! = n(n-1) \cdots 2(1) \) denotes the factorial of \( n \). Note that \( 0! = 1 \).
Imagine the coalition being formed one player at a time.

- There are $|S|!$ possible ways to form an coalition $S$. There are $(n - |S| - 1)!$ possible ways to form an coalition $N \setminus S \cup \{i\}$.
- $p(S) = \frac{|S|!(n-|S|-1)!}{n!}$ is the probability that player $i$ joins the coalition $S$.
- Player $i$, upon joining $S$, demands his fair compensation $\nu(S \cup \{i\}) - \nu(S)$, and then the Shapley value

$$\phi_i(\nu) = \sum_{\text{all } S \text{ for which } i \notin S} [\nu(S \cup \{i\}) - \nu(S)]p(S)$$

is the average over all the possible different permutations in which the coalition $S$ can be formed.
Let $\Pi$ denote the set of all $n!$ permutations on $N$. For $\pi \in \Pi$, define:

$$\pi^i = \{ j \in N : \pi(j) \leq \pi(i) \}$$

as the set of all players with rank not exceeding the rank of player $i$.

**Definition**

The marginal contribution vector for a game $(N, \nu)$ is defined as

$$m^\pi(\nu) = (m_1^\pi(\nu), \ldots, m_n^\pi(\nu)),$$

where

$$m_i^\pi(\nu) = \nu(\pi^i) - \nu(\pi^i \setminus \{i\}), \quad i \in N.$$

The Shapley value is the average of marginal contributions:

$$\phi_i(\nu) = \frac{1}{n!} \sum_{\pi \in \Pi} m_i^\pi(\nu), \quad i \in N.$$
Illustrative Example

- $N = \{1, 2, 3\}$.
- $\nu(\{1, 3\}) = \nu(\{2, 3\}) = \nu(\{1, 2, 3\}) = 1$; $\nu(S) = 0$ otherwise.

**Solution:** All permutations:

1. $(1, 2, 3)$: $\nu(\{1\}) - \nu(\emptyset) = 0$; $\nu(\{1, 2, 3\}) - \nu(\{1, 2\}) = 1$.
2. $(1, 3, 2)$: $\nu(\{1\}) - \nu(\emptyset) = 0$; $\nu(\{1, 3\}) - \nu(\{1\}) = 1$.
3. $(2, 1, 3)$: $\nu(\{2, 1\}) - \nu(\{2\}) = 0$; $\nu(\{2, 1, 3\}) - \nu(\{2, 1\}) = 1$.
4. $(2, 3, 1)$: $\nu(\{2, 3, 1\}) - \nu(\{2, 3\}) = 0$; $\nu(\{2, 3\}) - \nu(\{2\}) = 1$.
5. $(3, 1, 2)$: $\nu(\{3, 1\}) - \nu(\{3\}) = 1$; $\nu(\{3\}) - \nu(\emptyset) = 0$.
6. $(3, 2, 1)$: $\nu(\{3, 2, 1\}) - \nu(\{3, 2\}) = 0$; $\nu(\{3\}) - \nu(\emptyset) = 0$.

The core consists of $(0, 0, 1)$.

The Shapley Value: $\phi_1(\nu) = \phi_2(\nu) = 1/6$, $\phi_3(\nu) = 4/6$. 
The Drug Game

- Company 1 has invented a new drug, but cannot manufacture the drug itself and has to sell the drug’s formula to company 2, or company 3.
- The lucky company will split a $1 million profit with company 1.
- Find the core and the Shapley value for this game.

Solution: Let $N = \{1, 2, 3\}$.

1. If $1 \notin S \subseteq N$, then $\nu(S) = 0$ and $\nu(\{1\}) = 0$.
   $\nu(\{1, 2\}) = \nu(\{1, 3\}) = \nu(\{1, 2, 3\}) = \$1,000,000$.
2. The core consists of $(\$1,000,000, \$0, \$0)$.
3. The Shapley value $\phi_1(\nu) = \frac{\$4,000,000}{6}$, $\phi_2(\nu) = \frac{\$1,000,000}{6}$, and $\phi_3(\nu) = \frac{\$1,000,000}{6}$. 
Convex Games

Definition (Shapley, 1971)
A game \((N, \nu)\) is called a convex game if for all \(i \in N\) and all \(S \subseteq T \subseteq N\),

\[
\nu(S \cup \{i\}) - \nu(S) \leq \nu(T \cup \{i\}) - \nu(T).
\]

Remark
- Any player has more incentive to join a larger coalition.
- A game \((N, \nu)\) is convex if and only if

\[
\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B), \quad \forall A, B \subseteq N.
\]
- If \((N, \nu)\) is monotone and convex, then \((N, \nu)\) is superadditive.
Core and Shapley Value

Theorem (Shapley, 1971)
If \((N, \nu)\) is convex, then

- The core exists and is the convex hull of the marginal contribution vectors \(m^\pi(\nu) = (m_1^\pi(\nu), \ldots, m_n^\pi(\nu)), \pi \in \Pi\).
- The Shapley value

\[
(\phi_1(\nu), \ldots, \phi_n(\nu)) = \frac{1}{n!} \sum_{\pi \in \Pi} (m_1^\pi(\nu), \ldots, m_n^\pi(\nu))
\]

is the barycenter (center of gravity) of the core vertices.

Remark
Lloyd Shapley and Robert Aumann (1974) extended the concept of the Shapley value to infinite games (defined with respect to a non-atomic measure).
Illustrative Example

- \( N = \{1, 2\} \).
- \( \nu(\{1\}) = \nu(\{2\}) = 5 \), and \( \nu(\{1, 2\}) = 20 \).

Solution: All permutations:
- 1 (1, 2): \( \nu(\{1\}) - \nu(\emptyset) = 5; \nu(\{1, 2\}) - \nu(\{1\}) = 15 \).
- 2 (2, 1): \( \nu(\{2, 1\}) - \nu(2) = 15; \nu(\{2\}) - \nu(\emptyset) = 5 \).

The core = \( \{(x_1, x_2) : x_1 + x_2 = 20, x_1 \geq 5, x_2 \geq 5\} \).

The Shapley Value: \( \phi_1(\nu) = \phi_2(\nu) = 10 \).
Bankruptcy Game (Aumann and Mashler, 1985)

- \( E > 0 \) is the estate, there are \( n \) claimants;
- \( \mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_+ \) is the claim vector (\( c_i \) is the claim of the \( i \)-th claimant).
- \( \nu(S) = \max\{0, E - \sum_{i \in N \setminus S} c_i\}, S \subseteq N. \)

**Theorem**

Each bankruptcy game is convex.
"In August 2011, a 66 year-old man in Livingston, New Jersey, gets a new kidney. The donor is a complete stranger, a 44 year-old man from Riverside, California, who offers his left kidney as a genuinely altruistic gesture. The recipient’s niece is prepared to donate one of her kidneys to her uncle, but belongs to the wrong blood group. Instead, the niece is asked to donate her kidney to an unknown woman in Wisconsin, whose ex-boyfriend in turn donates one of his kidneys to another anonymous patient in Pittsburgh. The chain does not come to an end until 60 coordinated transplants have taken place across the entire United States."

“Four days before Christmas Eve, a 30th patient with chronic kidney failure, a 45 year-old man in Chicago, resumes a normal life after a difficult year of dialysis...”
2012 Nobel Memorial Prize

“for the theory of stable allocations and the practice of market design”

Figure: Alvin E. Roth  Lloyd S. Shapley
“Professor Shapley: ... You and David Gale are the founders of matching theory, and the deferred-acceptance algorithm you discovered is the cornerstone on which theory and applications rest.”

“Professor Roth: Your innovative work comprises theory, empirical evaluation, laboratory experiments, and design of actual markets where prices cannot be used, for ethical or legal reasons....”

“You have never worked together. But together your contributions constitute one of those unexpected journeys, from basic research motivated by sheer curiosity, to practical use for the benefit of mankind....”
2007 Nobel Memorial Prize

“for having laid the foundations of mechanism design theory”

Figure: Leonid Hurwicz   Eric S. Maskin   Roger B. Myerson
2005 Nobel Memorial Prize

“for having enhanced our understanding of conflict and cooperation through game-theory analysis”

Figure: Thomas C. Schelling  Robert J. Aumann
1994 Nobel Memorial Prize

“for their pioneering analysis of equilibria in the theory of non-cooperative games”

Figure: John C. Harsanyi  John F. Nash Jr.  Reinhard Selten
1975 Nobel Memorial Prize

“for their contributions to the theory of optimum allocation of resources”

Figure: Leonid V. Kantorovich  Tjalling C. Koopmans