Washington State University
Department of Mathematics

Cascade Knapsack Problems

Bala Krishnamoorthy
Washington State University

joint work with

Gábor Pataki, UNC Chapel Hill

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- gain computational insights

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  ▶ B&C using lifted cover inequalities takes at least $2^{n/30}$ nodes

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- Krishnamoorthy and Pataki (06) - Column basis reduction and decomposable knapsack problems (preprint available in Optimization Online)

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  – RSRef solves in root node

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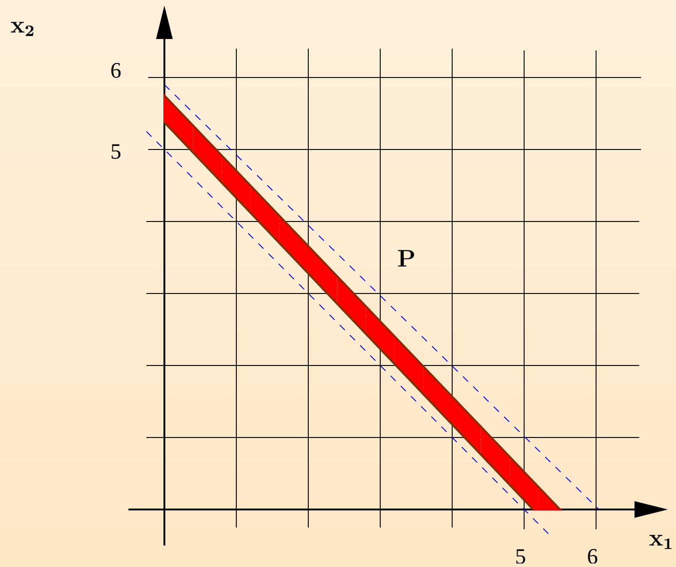

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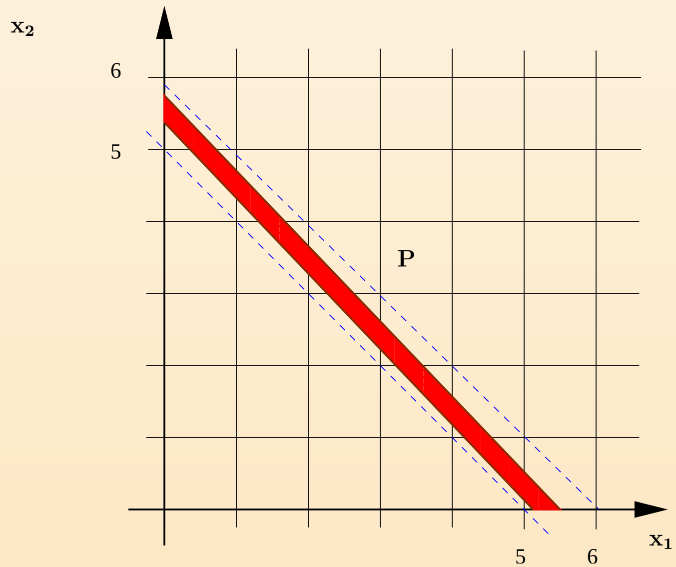

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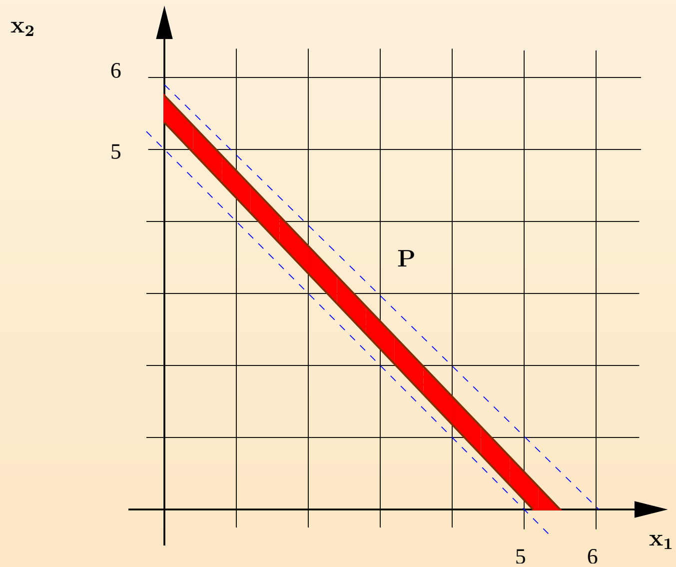

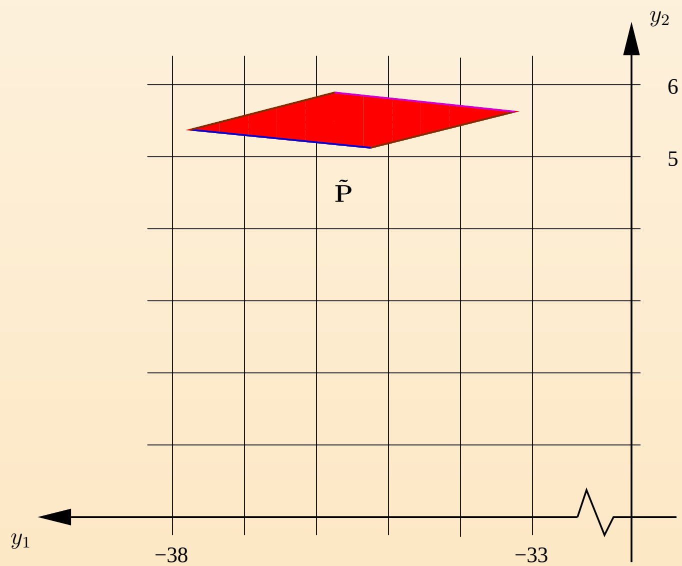

DKPs get harder as t grows

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Two infeasible knapsack problems: Can you tell which one is harder?

$$\begin{aligned}1473x_1 + 1524x_2 + 1569x_3 + 1570x_4 + 1575x_5 + 1624x_6 + 1625x_7\\ +2160x_8 + 2206x_9 + 2207x_{10} + 2211x_{11} + 2211x_{12} + 2257x_{13}\\ +2260x_{14} + 2305x_{15} + 2843x_{16} + 2943x_{17} + 2947x_{18} + 2991x_{19}\\ +2993x_{20} + 2997x_{21} + 3528x_{22} + 3577x_{23} + 3631x_{24} + 3677x_{25}\\ = 28980,\ x_i \in \{0,1\}\end{aligned}$$

$$\begin{aligned}1314x_1 + 1315x_2 + 1317x_3 + 1318x_4 + 1971x_5 + 1972x_6 + 1973x_7\\ +1976x_8 + 1977x_9 + 1977x_{10} + 2629x_{11} + 2630x_{12} + 2631x_{13}\\ +2631x_{14} + 2633x_{15} + 2634x_{16} + 2635x_{17} + 2635x_{18} + 3287x_{19}\\ +3287x_{20} + 3287x_{21} + 3289x_{22} + 3292x_{23} + 3293x_{24} + 3293x_{25}\\ = 28981,\ x_i \in \{0,1\}\end{aligned}$$

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$\text{iwidth}(\boldsymbol{p})$: # branches created by branching on the hyperplane $\boldsymbol{px}$

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- effect of branching on $\boldsymbol{p}_1 \boldsymbol{x}$ *cascades* to the next level $\boldsymbol{p}_2 \boldsymbol{x}$

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  - $\max\{\boldsymbol{p}_2\boldsymbol{x}\,|\mathsf{CKP}_1 \wedge \boldsymbol{p}_1\boldsymbol{x} = 31\} = 21.989,$
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- comparable DKP: $\boldsymbol{a} = \boldsymbol{p}_1 M + \boldsymbol{r}$, with $M = 136$; for $x_j \in \{0, 1\}$

Example 1: CKP_1 – Properties

- $\max\{\boldsymbol{p}_1\boldsymbol{x} \,|\text{CKP}_1\} = 31.967, \quad \min\{\boldsymbol{p}_1\boldsymbol{x} \,|\text{CKP}_1\} = 30.102;$
  $\text{width}(\boldsymbol{p}_1, \text{CKP}_1) = 1.865, \quad \text{iwidth}(\boldsymbol{p}_1, \text{CKP}_1) = 1;$

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$$\begin{aligned} 4223 \;\le\; & 274\,x_1 + 407\,x_2 + 680\,x_3 + 953\,x_4 + 1086\,x_5 + \\ & 1089\,x_6 + 1223\,x_7 + 1362\,x_8 + 1360\,x_9 \;\le\; 4224 \end{aligned}$$

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CPLEX 11.0 takes 44 B&B nodes

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- We create variation of CKP with $\mathrm{width}(\boldsymbol{p}_1) > 1$ and $\mathrm{iwidth}(\boldsymbol{p}_1) = \mathbf{2}$; for both branches of $\boldsymbol{p}_1\boldsymbol{x}$, branching on $\boldsymbol{p}_2\boldsymbol{x}$ proves infeasibility

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    CPLEX 11.0 takes 35 B&B nodes for CKP$_2 \wedge \boldsymbol{p}_1\boldsymbol{x} = 32$

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    CPLEX 11.0 takes 35 B&B nodes for $\mathsf{CKP}_2 \wedge \boldsymbol{p}_1\boldsymbol{x} = 32$
  - $\text{width}(\boldsymbol{p}_2, \mathsf{CKP}_2 \wedge \boldsymbol{p}_1\boldsymbol{x} = 33) = 0.158$, $\text{iwidth} = 0$

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    CPLEX 11.0 takes 35 B&B nodes for $\text{CKP}_2 \wedge \boldsymbol{p}_1\boldsymbol{x} = 32$
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    CPLEX 11.0 solves $\text{CKP}_2 \wedge \boldsymbol{p}_1\boldsymbol{x} = 32$ at the root node

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    CPLEX 11.0 solves $\text{CKP}_2 \wedge \boldsymbol{p}_1\boldsymbol{x} = 32$ at the root node
- $\boldsymbol{p}_1$ is *not* preferable to $\boldsymbol{e}_j$ for branching, based on iwidth alone

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  - with $x_j \in \{0, 1\}$, we get *small* a_j's, but CPLEX still struggles
  - e.g., 4-CKP with $n = 30$, $a_{\max} \leq 9000$, CPLEX 9.0 takes ≈ 57 million B&B nodes

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- denoted as $t+1$-CKPs; Recipes to generate $t+1$-CKPs

- computationally hard
  - with $x_j \in \{0, 1\}$, we get *small* a_j's, but CPLEX still struggles
  - e.g., 4-CKP with $n = 30$, $a_{\max} \leq 9000$, CPLEX 9.0 takes ≈ 57 million B&B nodes
  - dynamic programming could be effective (time $= O(n\beta_1)$)?

Computation: 4-CKPs, $n = 30$, $\boldsymbol{u} = \boldsymbol{e}$

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	CKP widths			CKP		CKP$_{-\boldsymbol{p}_1}$		CKP$_{-\boldsymbol{p}_1\boldsymbol{p}_2}$		DKP		RS
#	w_1	w_{21}	w_{312}	BB	TM	BB	TM	BB	TM	BB	TM	BB
1	1.55	1.42	0.92	58,057,939	**u**	2,448,625	126.0	205,814	13.3	11756	0.4	3
2	1.47	1.44	0.90	56,937,604	3484	740,556	41.0	66189	4.6	8708	0.3	1
3	1.57	1.50	0.94	46,187,956	3027	2,005,687	99.4	249,232	14.1	9537	0.3	5
4	1.50	1.53	0.89	55,782,856	**u**	477,707	25.2	252,505	13.7	6496	0.3	4
5	1.49	1.48	0.94	56,313,840	**u**	1,421,719	69.0	334,046	19.0	5527	0.2	3
6	1.50	1.55	0.90	55,597,050	**u**	1,319,626	73.0	257,922	15.0	10520	0.4	15
7	1.50	1.59	0.91	60,453,028	**u**	1,595,424	78.6	151,812	9.1	7336	0.3	6
8	1.57	1.52	0.95	64,409,733	**u**	5,324,924	278.3	310,768	19.2	10360	0.4	6
9	1.50	1.48	0.96	55,491,175	**u**	3,366,436	167.2	312,653	18.0	10061	0.4	5
10	1.49	1.53	0.92	60,307,524	**u**	3,107,323	158.2	443,789	25.6	8227	0.3	68

BB: # B&B nodes, TM: CPU time (sec), **u**: unsolved in **1 hour** time limit, typical instance: $a_{\min} \approx 4000$, $a_{\max} \approx 9000$, β_1, $\beta_2 \approx 65000$; RS: RSRef

Used CPLEX 9.0; instances available at www.wsu.edu/∼kbala

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- when M_i's are big enough, RSRef solves in at most t or 2^t nodes, respectively
- **both** width and iwidth can be **poor** indicators of "good" branching directions

Slides