Non-Total Unimodularity Neutralized Simplicial Complexes

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(Image: Mmm)

Topologist

Mug: Doughnut is a This Topologist To
character recognition by brain

*imagery: hmm*

Topologist

*Topologist

This is a

Doughnut
AN APPLICATION

(IMAGE: MWW)

access to active site
funnels in proteins —

⑩
AN APPLICATION

...
Given a chain in a simplicial complex, find a homologous chain that is "optimal".

The Optimal Homologous Chain Problem (OCHP)
OHCP
The Optimal Homologous Chain Problem:
Given a chain in a simplicial complex, find a homologous chain that is "optimal" in the same topological class (homology over $\mathbb{Z}$)
OHCp is NP-complete (Garey & Johnson, '79)

* In the same topological class (homology over \( \mathbb{Z} \)) find a homogeneous chain that is "optimal". Given a chain in a simplicial complex, The Optimal Homologous Chain Problem: OHCp
is totally unimodular (TU) (Dey, Finetti, '10)

solvable in poly-time if the boundary matrix

$\ast$

OHP is NP-complete (Dutch, '87, Hirsch, '11)

$\ast$

in the same topological class (homology over $\mathbb{Z}$)

$\leftarrow$

in the same homology class that is "optimal"

"find a homogeneous chain that is "optimal"

"given a chain in a simplicial complex"

"find a chain in a simplicial complex"

Given a chain in a simplicial complex

The Optimized Homologous Chain Problem

OHC
$\wedge$

Topological characterization of $TU$ (DHR'10)

When is the boundary matrix $TU$?

* OChP
Q: What happens when boundary matrix is not TL?

When is the boundary matrix TL? (DHR '00)

\^ Topological characterization of TL (DHR '00)

\begin{align*}
\text{OHP} & \end{align*}
is non-total-unimodularly non-degenerate (NTUN).

Define a condition when simplicial complex K

RESULT
Define a condition when simplicial complex $K$ is non-total-unimodularity neutralized (NTUN)

- boundary matrix is not $Tu$
has an integer optimal solution, and every valid choice of weights, chol and every valid choice of weights, every OHC-P LP, i.e., for every input, every OHC-P LP, i.e., for every input, boundary matrix is not TL, but boundary matrix is not TL, but is non-totally unimodularity non-totally unimodularity non-totally unimodularity non-totally unimodularity

Define a condition when simplicial complex K

RESULT
chains on a simplicial complex

THE SETTING
The setting

Chains on a simplicial complex

1-chains

$A = \begin{bmatrix} 3 & 6 & 4 & 5 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
THE SETTING

chains on a simplicial complex

$\alpha = \frac{3}{6} \cdot 0 - 0$

$\beta = \frac{2}{3} \cdot 0 - 0$

$C = \begin{bmatrix} 6 & 3 \\ -1 & 0 \end{bmatrix}$
cycles

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = b
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = c
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = a
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

- chains
- 1-chains
- A simplicial complex
- The setting
THE SETTING

chains on a simplicial complex

1-chains

cycles

boundary

\[ \begin{align*}
  c &= 2 - 0 \\
  a &= 3 - 0 \\
  b &= 2 - 0 \\
  \end{align*} \]
\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
4 \\
3 \\
2 \\
\end{bmatrix}
= \begin{bmatrix}
2 \\
4 \\
3 \\
2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0_{p,t} \mid 0_{p,t} \mid 0_{p,t} \mid 0_{p,t} \mid 0_{p,t}
\end{bmatrix}
\]

\{ \mathfrak{p}_{t+1} \mid \text{is an m n matrix with entries in } \mathbb{Z} \}

With m p-simplices and n (p+1)simplices in K,

\[
\mathfrak{p}_{t+1} : C_{p+1}(K) \rightarrow C_p(K)
\]

**Boundary Matrix [**\[\mathfrak{p}_{t+1}**]**}
Homologous Chains

$X\rightarrow X$ is homologous to $c$ both go around the same hole
\[ C = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ x = \begin{bmatrix} 2 \\ \vdots \\ 0 \end{bmatrix} \]

\[ [a_2] = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ 1 \]

\[ x = c + [a_2][i] \]

\[ 0 \]

\[ 4 \]

\[ x \quad \text{is homologous to} \quad x \quad \text{both go around the same hole} \]

\[ \text{Homologous Chains} \]
\[ C = \begin{bmatrix} 2 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ x = \begin{bmatrix} 2 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \]

In general \( x = c + [a][b] \)

\( x = c + [a][b] \)

\( x \) is homologous to \( c \)

\( x \) is both homologous to \( c \)

for \( y \in \mathbb{Z}^n \)

Homologous Chains
(OHCP) \iff (m \geq 0) \land \exists z, y \in \mathbb{Z}^n \land x = c + \lfloor a \rfloor y, x 
\leq \min_{\sum_{i=1}^{m} w_i | x_i |} \frac{1}{x} \leq \infty \text{ such that}
\[ \begin{array}{l}
\text{min} \quad \sum_{x \leq 0, y \geq 0} x \cdot y + \sum_{y \leq 0, x \geq 0} x \cdot y \\
\text{subject to} \quad x + y = c + \epsilon \cdot [p_i - \bar{y}] (x - y)
\end{array} \]

\[ \text{(OHCP)} \]

"Piecewise linear"

OHCP as an Integer Program
\[ x' y' \in \mathbb{Z}, y' \in \mathbb{Z}, t' x' \geq 0 \]

\[ t' x' + y' = c + [a + \{ r \}] (y' - y) \]

s.t. \[ x' - x = c \]

\[ \min \{ x' + x \} \leq \min \{ y' \} \]

\[ (m \geq 0) \]

\[ x = c + [a + \{ r \}] y, x \in \mathbb{Z}, y \in \mathbb{Z} \]

\[ \min \{ y \} \geq \min \{ \frac{1}{2} x | x \geq 0 \} \]

\[ \min \{ x' \} \geq \frac{1}{2} \min \{ x \} \]

\[ y' \geq 0 \]

\[ \text{OHC} \]

As an Integer Program

Piecewise Linear

\[ \text{OHC} \]
The constraint matrix of above LP is

\[ \text{Tu Hf } e^{PH} \text{ is Tu}. \]

The constraint matrix of above LP is

\[ x^+ x^+ y^+ y^- \leq 0 \]

s.t. \( x^- x^- = c + [a^{PH}] (y^+ y^-) \)

\[ \min \ \frac{1}{2} w : (x^+ x^-) \]

[OHCP AND TU OF e^{PH}].
OHCP and TU of $[\Theta^p]$.

$[\Theta^p]$ is TU. $\Rightarrow$ OHCP is solvable in poly time when $[\Theta^p]$ is TU. The constraint matrix of above LP is $[\Theta^p]$. $\min \sum w_i (x_i^+ + x_i^-)$ s.t. $x_i^+ - x_i - c + [\Theta^p] (y_i^+ - y_i^-)$ $x_i^+, y_i^+ \geq 0$ $(LP)$
no relative torsion

D HK'10: $[a_{p+1}^p]$ is T U if K has

When is $[a_{p+1}^p]$ T U?
Special cases

K is an orientable manifold

\[ \forall \]

In relative torsion, K has

DHK 10: \([a_1^{+1}] \in \mathbb{T} U \Leftrightarrow K \]

\[ \exists \]

When is \([a_1^{+1}] U \]?
\[
\text{no submatrix } S = 2 \\
\begin{vmatrix}
1 & 1 \\
1 & 1
\end{vmatrix}
\]

\([e_2] \text{ is } T \text{ U if and only if } K \text{ has Mobius strip}(\text{DHK, 11})\)

\([\bar{e}_2] \text{ is } T \text{ U if and only if } K \text{ is an orientable manifold}\)

Special cases

no relative torsion

[\bar{e}_2^{2+}] \text{ is } T \text{ U if and only if } K \text{ has}

\text{When IS [} e_2^{1+} \text{] T U?}
Characterizing non-TH neuromorphic \[ K \] prove several intermediate results toward \[ \textit{Properties of OChP Lp} \]
Properties of OCP Lp

Study properties of basic solutions, rather than BLFs — easier to characterize characterizing non-TL networks. Proof several intermediate results toward
Theorem 9.7: If \( z \) is a basic solution, then there exists a unique basic feasible solution \( z^* \) equivalent to \( z \).

Study properties of basic solutions, rather than BFS - easier to characterize.

Characterizing non-TL neighborhoods and several intermediate results toward properties of OCP LP.
Fractional basic solution? When can there not exist a unique basic feasible solution? 

Theorem: \( \exists z \) is a basic solution, there exists a unique basic feasible solution, \( z \). That is, "equivalent" to \( z \). 

If \( z \) is a basic solution, then \( \exists \) easier - easier to characterize than properties of basic solutions, rather than properties of LP. LP characterized non-Tr. 

Properities of LP
OHCP?  : OHCP instance with input chain c_c_e

Elementary chain = unit vector e

* OHCP and ELEMENTARY CHAINS
Theorem OHCp: OHCp instance with input chain \( C = e \) has a fractional basic solution if \( C \neq 0 \) and OHCp has a fractional basic solution if there is an i.s.t. fractional basic solution. If there is an i.s.t. fractional basic solution \( i \), then OHCp has a basic chain with input chain \( C = e \).

* Elementary chain = unit vector
OHCP and Elementary Chains

* elementary chain $\equiv$ unit vector $e_i$

OHCP$_i$: OHCP instance with input chain $c = e_i$

Theorem OHCP with input chain $c$ has a fractional basic solution iff there is an $i$ s.t. $c_i \neq 0$, and OHCP$_i$ has a fractional basic solution

$\Rightarrow$ specify characterization of non-TU neutralized for elementary chains
Theorem Every OHCP LP on K has an integer optimal solution if every elementary chain in each relative torsion in K has a "neutralizing chain" in K.
Theorem: Every OHCF LP on $K$ has a non-trivial basic solution.

Convex combination of the integral basic solutions ensures that every fractional basic solution is a

Every OHCF LP on $K$ has a "neutraizing chain" in $K$. Theorem: Every OHCF LP on $K$ has a non-trivial basic solution.
K is non-TU neutralized (NTU)

ensures that every fractional basic solution is a convex combination of two integral basic solutions.

in K has a "neutralizing chain" in K.

Every OHCP LP on K has an integer optimal solution if every elementary chain in each relative tension

Theorem.
TU OR NOT ?
a disc whose boundary consists of an odd number of red (heavy/many red)
edges of a chain
Property of $K - K'$ holds for every choice of weights on simplices and input chain

$\text{Non-Top Neuroalized K}$
If you do not appear in objective function

relevant objective function vector

right hand side vector and every

optimal solution exists for

an integral optimal solution, but

not integral, but

\[ \text{TCP LP} \]

property of \( K - K \) holds for every choice

\[ \text{NON-TRIVIALIZED K} \]
OPEN QUESTIONS

- In low dimensions?
- Connection to TDI
- Efficient algorithm to check NTUN?

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Open Questions

- Efficient algorithm to check NTU?
- Connection to TDI
- In how dimensions?
- Other instances where LP polytope is not integral, but every LP has an integral optimal solution?
All objective functions (y, y' included) works in practice – triangle-edge case in R³ with no holes or voids.

Integrated optimal solution?

Integrated optimal, but every LP has an LP ployable is other instances where LP ployable is in low dimensions?

Connection to TDI

Efficient algorithm to check NTU? * OPEN QUESTIONS