Flat Norm Decomposition of Integral Currents

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CURRENTS
Consider integer multiplicities — in GMT, isoperimetric problems, soap bubble conjectures — generalized surfaces with multiplicities & concentration

Currents
CURRENTS

* generalized surfaces with multiplicities & orientation
  — in GMT; isoperimetric problems, soap bubble conjectures
  — we consider \textit{integer} multiplicities
  e.g., 1D-current: collection of oriented curves in $\mathbb{R}^d$
we assume finite mass

mass of d-current = its weighted d-volume

 getCurrent

e.g., 1D-current: collection of oriented curves in \mathbb{R}^d

we consider integer multiplicities

in GMT; isoperimetric problems, soap bubble conjectures

generalized surfaces with multiplicities & orientation

CURRENT5
Hausdorff, fréchet, probématic

Distance between curves
Distance Between Currents

Hausdorff, Frechet, Problematic

mass difference $M(T'_1 - T'_2)$?
Distance Between Currents
\{ s \in (\mathcal{A} + \text{current}) \mid S^m_{\mathcal{F}}(s) \leq \min_{\mathcal{F}} \{ S^m_{\mathcal{F}}(T - 3s) + M_{\mathcal{F}}(S) \} \}^\perp

\text{Flat Norm}

\star
\\text{Flat Norm}
Operations: Add 1 curve; cost = length

Idea: (TD): How to erase T at min cost?

\[ \text{FLAT NORM} \]
Operations: Add 1 curve; cost = length

Idea (TD): How to erase T at min cost?

\[ F(T) = \min \{ 3 + M^p_{(S)} + W^p_{(T - aS)} | S \in \mathcal{S} \} \]
Flat Norm
\[
F(T) = \min \text{ such total cost}
\]

Operations:
- Trace boundary of \( \partial \Omega \) region; cost = area
- Add \( \partial \text{-curve} \); cost = length

Idea (TD): How to erase \( T \) at min cost?

\[
F(T) = \min \left\{ S \mid M(T - s) + W^*(T) \right\}
\]

Flat Norm
Example

* Real distance between $P, Q$: $F(P, Q) = F(P - Q)$
both oriented clockwise

\[ T : \text{unit circle (in IR}^2) \text{, } T^n : \text{inscribed n-gon} \]

\( T : \text{unit circle (in IR}^2) \text{, } T^n : \text{inscribed n-gon} \]

Example
Example
as \( n \to \infty \), \( M(T^n) \to \infty \), but \( H(T^n) \to 0 \).

* Both oriented clockwise

Inscnred n-gon

* In unit circle \((in \mathbb{R}^2)\)

Flat distance between \( P \& Q \):

\[ H(P, Q) = H(P-Q) \]

**Example**
Integral Currents

* oriented rectifiable sets, integer multiplicities,
finite mass, boundary has same properties.
When is the flat norm decomposition finite mass, boundary has same properties, oriented rectifiable sets, integer multiplicities, integral currents?
Integral Currnts

Q: Yes for codimension 1 boundaries?

Q: When is the flat norm decomposition finite mass, boundary has same properties, oriented rectifiable sets, integer multiplicity?
Integral Currents

- Yes in 2D
- Yes for currents in $\mathbb{R}^d$
- Analys framework: Simplicial to continuous

$LTV$ functional - Mangny, Vieille, 2007

Yes for codimension-1 boundaries

- An integral current also integrates?

Q: When is the flat norm decomposition finite mass, boundary has same properties, oriented rectifiable set, integer multiplicities, oriented cycles?
Minimizing fillings. Cheaper by the dozen!

* Related Work

Area filling \( C \) relative to closed curves \( C \) in \( \mathbb{R}^2 \):


\( 2 \cdot \text{Area filling } C \geq \)
Open problem considered by F. Almgren (White 1988):

\[ \text{Area fillings } \text{C} \]

- Area fillings \text{ac} >
- Closed curves \text{c} in \( \mathbb{R}^4 \)

Area minimizing fillings cheaper by the dozen

** RELATED WORK **

- Almgren's conjecture
not simplicial complexes

\[ \text{all faces' and intersections are faces} \]

\[ \text{a collection of simplices that includes} \]

\[ \text{a simplicial complex} \]

* Discretize the problem on a simplicial complex

— Ishim, K. Vixie (2013)

Simplicial Flat Norm
Currents are chains on the simplicial complex.

Simplicial Flat Norm

- Discretize the problem on a simplicial complex that includes all faces, and intersections are simplices that include at least one face.

- Ibrahim, K., Vixie (2013)
chains on a simplicial complex

\[ c = 2 - 0 \]

\[ b = 3 - 0 \]

\[ a = 3 - 0 \]
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & T & 0 & 0 \\
0 & 0 & T & 0 \\
0 & 0 & 0 & T
\end{bmatrix} = \begin{bmatrix} e_2 \end{bmatrix}
\]

\[e_{p+1} \text{ is an mxn matrix with entries in } \mathbb{F}_2 \text{,}
\]

With m p-simplices and n (p+1)-simplices in K,

\[\phi^+_p : C_{p+1}^+(K) \rightarrow C_p(K)\]

Boundary Matrix \[\phi_{p+1}^+\]
Homologous Chains

- x is homologous to c
- x both go around the same hole
Homologous Chains

\[ C = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ X = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \]

\[ [a] = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ x = c - [a][x] \]

\[ x \text{ is homologous to } c \]

\[ x \text{ both go around the same hole} \]
Homologous Chains

For $x \in \mathbb{Z}_n$

$x = c - [a]_1$

In general,

$[\mathbb{Z}_n]^p \rightarrow [\mathbb{Z}_n]$
\[ [I] = S \]

\[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = e^z \]

For \( x \) is homologous to \( t \), \( x = t - [a_2] \)

\[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ t = x \]

\[ \text{Flat Norm decomposition} \]
\[
\begin{align*}
\text{min} & \quad M \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{m} y_j \\
x = t - \lfloor \theta \rfloor \Sigma_s \text{, } x \in Z^m, s \in Z^n
\end{align*}
\]
\[
\begin{align*}
\text{(d)} \quad & (s - s)[s^d e] - t = x - x_+ \\
\text{s.t.} \quad & x_+ - x_- = 0 \quad \begin{array}{c}
s, x \in \mathbb{Z}^n, s \in \mathbb{Z}^n \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{min} \quad & \sum_{i=1}^{n} \omega_i x_i \\
\text{subject to} \quad & \frac{f_i^+ f_i^+}{x_i^+} + \frac{f_i^- f_i^-}{x_i^-} \leq \frac{1}{\mu_i} + \frac{1}{\nu_i} \\
\end{align*}
\]

\[
\begin{align*}
\text{for all } & \quad \mu_i, \nu_i > 0 \\
\text{minimize linearly} \quad & \text{piecewise linear,}
\end{align*}
\]

\text{SFL as an Integer Program}
\( x^+_t, x^-_t \in \mathbb{Z}^n, s^+_t, s^-_t \in \mathbb{Z}^n \)

\[
\begin{align*}
(s^-_t - s^+_t)[1^p e] - t &= x^-_t - x^+_t \\
\text{s.t. } x^+_t - x^-_t &= \sum_{t=1}^m \xi t \leq \frac{1}{t} \sum_{t=1}^m \xi t \\
\end{align*}
\]

\( \min \frac{1}{t} \sum_{t=1}^m \xi t \)

\( x^+_t, x^-_t \in \mathbb{Z}^n, s^+_t, s^-_t \in \mathbb{Z}^n \)

\( \min \frac{1}{t} \sum_{t=1}^m \xi t \)

Piecewise linear

\( \text{SFP as an Integer Program} \)
The constraint matrix of above LP is $T_U$.

$*$

$T_U$ iff $[\mathcal{P}^I]$ is $T_U$.

$$\min \sum_{i} w_i (x^+_i + x^-_i) + \sum_{i} \frac{y_i}{s^+} (s^+_i + s^-_i)$$

s.t. $x^+ - x^- = t - \frac{1}{l} [\mathcal{P}^I] (s^+ - s^-)$

$0 \leq x_i, 0 \leq s^+_i, s^-_i \geq 0$

(LP)
The constraint matrix of above LP is

\[ \begin{align*}
0 \leq x^-_i, s^+_i \leq s^-_i \\
\text{s.t. } & x^-_i \leq t - (s^+_i \leq s^-_i) (p_{i+1}) \\
n \geq \min \left( \frac{f^+}{f^+}, \frac{f^-}{f^-} \right) + (x^+_i \leq t - (s^+_i \leq s^-_i)) \\\n\end{align*} \]

[SFN AND TUL OF [a]]
SFN and TU of \([\alpha_{PH}]\)

The constraint matrix of above LP is

\[
\min \sum_{s,t} w_t (x_s^+ + x_s^-) + \sum_{s} y_s^+ (s^+ + s^-) \\
\text{s.t. } x_s^+ - x_s^- = t - [\alpha_{PH}] (s^+ - s^-) \\
0 \leq x_s^+ \leq x_s^- \leq 0
\]

\((LP)\)

TU iff \([\alpha_{PH}]\) is TU for \(K\) in \(R^d\) (Dey, Hirani, K, 2010)

\([\alpha_{PH}]\) is TU for \(K\) in \(R^d\)

The integral = integral out for codimension-1 simplicial currents

\(<\)
Can we use the simplified result to obtain the continuous result?

\[ \int_{\text{In}}^{\text{Out}} \]
Integral IN = Integral OUT

Could we take simplicial approximations of its simplicial Plott norm decompositions and somehow take the limit of I and obtain the continuous result?

To get a continuous decomposition?
Could we take simplicial approximations of $T$ and somehow take the limit to get a continuous decomposition? Can we use the simplicial result to obtain the continuous result? * Integral IN = Integral OUT?
$K_n$: 2n equilateral triangles

$\text{SFN} \rightarrow \text{FLAT NORM}$
$\ell_n(t) = \frac{\sqrt{3}}{2} \ell(t) \perp \ell(t)$

but as $n \to \infty$, but

$\ell(t -\ell_n) \not\to 0$, but

$K^n$ is an equilateral triangle

$K^n \not\to \text{FLAT NORM}$
... need more sophisticated tools.

\[ \text{If } (r, n) = \begin{cases} \frac{1}{3} & \text{if } (r, t) \\ \frac{2}{3} & \text{if } (t, r) \\ 0 & \text{if } (t, t) \end{cases} \]

\[ \rightarrow \text{ as } u \rightarrow \infty, \text{ but } \]

\[ F(t - n) \rightarrow \infty \] as \( n \rightarrow \infty \), but

\[ K_n : \text{An equilateral triangle} \]

\[ K^n \]

\[ F^n \]

\[ \text{SFIN \leftrightarrow FLAT NORM} \]
\[ f^1(T, p) \geq \epsilon \]
and
\[ M(p) > W(p) + \epsilon \]
\[ W(p) > M(T) + \epsilon \]

For normal current \( I \) in compact KCR, *Fedderer (1969) (4.2.21, 4.2.24) — Polyhedral Approximation
If \( f_t \) is \( (T, p) \) \( \geq \) \( c \)
and \( M(a) / \geq \) \( M(a) + p \) and \( M(p) \geq M(T) + p \)
and \( p \geq \), \( E \) integral polyhedron chain \( \mathbb{P} S \).

For integral current \( T \) in compact \( K \),

- Federer (1969) \((4.2).2, 4.2.24)\) - Polyhedral Approximation
P need not be simplicial

- mass expansion $\rightarrow 0$ as $p \rightarrow 0$, but

$$\ell^1 (T', p) \geq c$$

and

$$M (d p) \geq w (d p) + f$$

and

$$w (p) \geq w (T') + f,$$

for integral current $T$ in compact $CR$.

* Federer (1969) (4.2.21, 4.2.24)

Polyhedral Approximation
Simplicial deformation

\[ F_T p \leq \Delta_c \left[ M(T) + \varphi_M(\lambda T) \right] \]

\[ M(p) \leq \varphi_M(T) \triangleleft + \Delta_c M(\lambda T) \]

\[ M(p) \leq \lambda M(T) + \Delta_c M(\lambda T) \]

\[ \text{s.t.} \]

\[ \text{can be defined to chains} \]

- Ibrahim, K. Vixie (2013)
Simplicial Deformation
Simplicial deformation

\[ F(t, p) = \Delta C^1[M(t)] + 3M(e(t)) \]

\[ \text{s.t. } C^0(e(t)) = C^2[M(e(t))] \]

\[ M(p) \supset \text{c.m}(e(t)), \text{ and } M(p) \supset C^2[M(e(t))] \]

\[ \Delta \rightarrow 0, \beta \]

\[ \text{simplicial complex } K \text{ s.t. } \]

\[ \text{can be deformed to chains } \partial \text{ p in a} \]

\[ \text{Ibraghim, K'Vixie (2013) } \]

Simplicial deformation
Main Result: Overview
Main Result: Overview
Main Result: Overview

\[ P \approx x^s + 3s \]

\[ P \approx x^s \]

\[ x^s \approx x^{s+s} \]

\[ s \approx s \]

But difference is probably small.
MAIN RESULT: OVERVIEW

* Find simplicial complex $K_\varepsilon$ with mass expansion $L$ indep. of $\varepsilon$ by triangulating $S$.

$P_\varepsilon \times S \approx \text{polyh. approx}$

$P_\varepsilon \times S \approx \text{polyh. approx}$

$P_\varepsilon + \varepsilon S \approx \text{polyh. approx}$

$P_\varepsilon + \varepsilon S \approx \text{polyh. approx}$

But difference is probably small.

Ps + s + S. Probably small.

Ps + s + S.
Simplified flat norm vs continuous flat norm

\[ \lim_{t \to 0} \| f(t) \|_{R^k} = 0, \quad (R^k, \text{e}^1) \]

Main Result: Overview
\[ P^g \text{ has integral SFN decomposition } \leq \text{ so does } T \text{ under flat norm} \]

Simplified flat norm \& continuous flat norm

\[ \lim_{t \to 0} \| P^g(T) \| = \lim_{t \to 0} \| F^g(P^g) \| \]

\[ \forall \sigma, \| \sigma \|_{P^g} = \| \sigma \|_{SFN} \]

\text{Main Result: Overview}
inequality pushes to corners of small mass. Is $K$ that is regular in most places, with $K \in C^\infty$? Conjecture: There exists substitution so does $T$ under flat norm.

$P_\delta$ has integral SFR decomposition $\Rightarrow$ simplicial flat norm $\Rightarrow$ continuous flat norm. Show $\lim_{T \to \infty} R(T) \in K_\delta (P)$, i.e.,

Main Result: Overview
**MAIN RESULT: OVERVIEW**

* Show \( \frac{1}{F(T)} = \lim_{s \to 0} \frac{1}{F_k(P_s)} \), i.e., \( P_s \) has integral SFN decomposition as continuous flat norm.

* Simplicial flat norm \( \Rightarrow \) so does \( T \) under flat norm.

* Conjecture: There exists subdivision of \( K_s \) that is regular in most places, with irregularity pushed to "corners of small mass." True in 2D (Shewchuk 2002).
Open Questions

? Extending result to higher dimensions?
- our framework works in any dimension, assuming subdivision conjecture holds
Other regularization methods?

Assuming subdivision convergence holds — our framework works in any dimension?

Extending result to higher dimensions?

Open Questions
Open Questions

- Extending result to higher dimensions?
- Our framework works in any dimension, assuming subdivision conjecture holds.
- Other regularization methods?
- Codimension 2?
Other questions where discrete = continuous?

Codimension 2?

Other regularization methods?

Assuming subdivsion convergence holds – our framework works in any dimension?

Extending result to higher dimensions?