Integer Optimization (Spring 2009) — Homework 6

- The total points (given in parentheses) add up to 125. You will be graded for 120 points.
- **This homework is due in class on Tuesday, April 7.**

1. (15) **Reduced cost fixing:** Consider the following IP.

$$\begin{array}{lrcl} \max\ z = & 6x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 3x_1 + 3x_2 + 4x_3 & \leq & 5, \\ & x_1,\, x_2,\, x_3 & \in & \{0,1\}. \end{array}$$

Suppose you solve the LP relaxation of this IP, and someone *gives* you the optimal integer solution. Which all variables can you fix at the values that they take in the LP solution? Justify your answers.

2. Consider the **Jeroslow IP** (in its original form) for *odd* n.

$$\begin{array}{lrcl} \max\ z = & -x_{n+1} & & \\ \text{s.t.} & 2x_1 + \cdots + 2x_n + x_{n+1} & = & n \\ & x_j & \in & \{0,1\}, j = 1, \ldots, n+1. \end{array}$$

   (a) (15) Show that even if the optimal solution is known, i.e., the best lower bound $\underline{z}$ is known, branch-and-bound still takes an exponential number of nodes to solve this problem. Will the analysis be different for the cases where the B&B branches first on x_{n+1} and when it branches first on one of the first n x_j's?

   (b) (10) Now consider applying *branching on a constraint* to this IP. Identify a direction vector $\boldsymbol{a} \in \mathbb{Z}^{n+1}$ such that branching on $\boldsymbol{a}^T\boldsymbol{x}$ will solve the IP in only two nodes.

   (c) We now explore whether cuts can help us solve the Jeroslow IP. For each method given below, indicate with justification if it will be helpful when applied to the Jeroslow IP. If yes, indicate how many times should it be applied before we solve the problem. If the method is not helpful, show why it is so.

      i. (10) CG cuts applied by themselves (i.e., without branch-and-bound).

      ii. (10) Lifted cover inequalities (Lovász-Schrijver procedure) applied by themselves (i.e., without branch-and-bound).

3. (20) Solve the following IP using **Chvátal-Gomory (CG) cuts** – you need to add enough CG cuts such that when you solve the resulting LP, you get an integer solution, which should be the optimal solution to the original IP.

$$\begin{array}{lrcl} \min\ z = & 5x_1 + \ 9x_2 + 23x_3 & & \\ \text{s.t.} & 20x_1 + 35x_2 + 95x_3 & \geq & 319, \\ & x_j \geq 0,\ x_j & \in & \mathbb{Z}, j = 1,2,3. \end{array}$$

4. (20) Show that $x_2 + x_4 \leq 20 + 4(y-2)$ is a valid **mixed integer Gomory (MIG) cut** for $X = \{(\boldsymbol{x}, y) \in \mathbb{R}^4_{\geq 0} \times \mathbb{Z}^1_{\geq 0} \,|\, x_1 + x_2 + x_3 + x_4 \leq 10y_1,\ x_1 \leq 13,\ x_2 \leq 15,\ x_3 \leq 6,\ x_4 \leq 9\}$.

5. (25) Consider the set $X = \{\boldsymbol{x} \in \{0,1\}^6 \,|\, 12x_1 + 9x_2 + 7x_3 + 5x_4 + 5x_5 + 3x_6 \leq 14\}$. Consider the inequality $x_3 + x_5 + x_6 \leq 2$ that is valid for $X \cap \{\boldsymbol{x} \,|\, x_1 = x_2 = x_4 = 0\}$. **Lift this inequality** to obtain an inequality $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_4 x_4 + x_3 + x_5 + x_6 \leq 2$ that is valid for X.