Integer Optimization (Spring 2009) — Homework 3

- The total points (given in parentheses) add up to 130. You will be graded for 120 points.
- **This homework is due in class on Thursday, February 5**.

1. (50) Represent the following sets. You should consider all the following three options.

   (*i*) Model without any extra variables.

   (*ii*) Model with extra continuous variables (here, sometimes you just have to use common sense, not anything learned in MIP modeling).

   (*iii*) Model with extra continuous and 0–1 variables.

   If it is not possible to represent a set as described in one of the options, prove that fact.

   (a) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \leq 1\}$.

   (b) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \geq 1\}$.

   (c) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n |y_i| \geq 1, \ -M \leq y_i \leq M \ \forall i\}$.

   (d) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n c_i|y_i| \leq 1\}$. Here the c_i are scalars, which may be positive or negative.

   (e) $S = \{y \in \mathbb{R}^n \mid \sum_{i=1}^n c_i|y_i| \leq 1, \ -M \leq y_i \leq M \ \forall i\}$. Here the c_i are scalars, which may be positive or negative.

   (f) $S = \{y \in \mathbb{Z}^n \mid Ay \leq b, \ y \neq y^*, \ y^* \in \mathbb{Z}^n \text{ is fixed}\}$.

2. (20) Consider the set S and the formulation P.

$$
\begin{aligned}
S &= \{\boldsymbol{x} \in \{0,1\}^n \,|\, \text{at least } p \text{ of the } \boldsymbol{x}_i\text{'s are } 1\}, \quad \text{and} \\
P &= \{\boldsymbol{x} \in \mathbb{R}^n \,|\, 0 \leq \boldsymbol{x} \leq \boldsymbol{e}, \ \boldsymbol{e}^T\boldsymbol{x} \geq p\},
\end{aligned}
$$

   where $\boldsymbol{e}$ is the n-vector of ones. Assume p, n are positive integers such that $p \leq n$.

   Prove that P is a sharp (or ideal) formulation of S.

3. (60) Let

$$S = \{(x, y_1, y_2, y_3) \in \{0,1\}^4 \mid (x = 1) \Rightarrow \text{ at least two of the } y_i\text{'s are } 1\}.$$

Consider the following inequalities:

$$x \leq y_1 + y_2 \quad (3.1)$$
$$x \leq y_1 + y_3 \quad (3.2)$$
$$x \leq y_2 + y_3 \quad (3.3)$$
$$2x \leq y_1 + y_2 + y_3 \quad (3.4)$$
$$0 \leq x \leq 1 \quad (3.5)$$
$$0 \leq y_i \leq 1 \; (i = 1, 2, 3) \quad (3.6)$$

Now consider the following formulations:

- Formulation 1: bounds ((3.5),(3.6)) and (3.4);
- Formulation 2: bounds ((3.5),(3.6)), (3.4), and (3.1);
- Formulation 3: bounds ((3.5),(3.6)) (3.4), (3.1), and (3.2); and
- Formulation 4: all constraints listed (including bounds).

Prove the following.

(a) Formulations 1,2,3, and 4 are all valid formulations for S.

(b) Formulation 1 is not ideal for S.

(c) Formulation 2 is stronger than Formulation 1.

(d) Formulation 3 is stronger than Formulation 2.

(e) Formulation 4 is stronger than Formulation 3.

(f) Formulation 4 is ideal.