Integer Optimization (Spring 2009) — Homework 2 Solutions

1. (a) Let ℓ_t be the number of units of demand lost in period t. We add to the original formulation in the following ways.
    - i. add the term $\sum_{t=1}^{n-1} b_t \ell_t$ to the objective function;
    - ii. change the balance constraint for each period t to $s_{t-1} + x_t = s_t + d_t - \ell_t$; and
    - iii. set $u_n = 0$.

   (b) We add the following two sets of constraints to the original formulation.
    - i. $\sum_{t=1}^{n} y_t \le T$; and
    - ii. $y_t + y_{t+1} \le 1$, for $t = 1, \ldots, n-1$.

2. Recall the assumptions given in the problem:

$$0 \le x_1 \le M_1, \quad 0 \le x_2 \le M_2, \text{ and} \tag{2.1}$$

$$f_1 \le f_{12}, \quad f_2 \le f_{12}. \tag{2.2}$$

Here is one solution that you could write down using logic (i.e., without directly using any of the standard modeling techniques discussed in class).

Using $y_1, y_2, y_{12} \in \{0,1\}$, write

$$0 \le x_1 \le M_1 (y_1 + y_{12}), \tag{2.3}$$

$$0 \le x_2 \le M_2 (y_2 + y_{12}), \tag{2.4}$$

$$y_1 + y_2 + y_{12} \le 1, \tag{2.5}$$

and then use z_1 in place of $f(x_1, x_2)$ in the minimization objective, where

$$z_1 = f_1 y_1 + f_2 y_2 + f_{12} y_{12}. \tag{2.6}$$

Here, $y_1 = 1$ if $x_1 > 0, x_2 = 0$; $y_2 = 1$ if $x_1 = 0, x_2 > 0$; and $y_{12} = 1$ if $x_1 > 0, x_2 > 0$. When $x_1 > 0, x_2 = 0$, we can have either $y_1 = 1$ or $y_{12} = 1$. Since the objective function is to be minimized, and since $f_1 \le f_{12}$ by assumption (2.2), we get $y_1 = 1$, and the cost f_1 is incurred. The case of $x_1 = 0, x_2 > 0$ is analogous. When we have $x_1 > 0, x_2 > 0$, by the constraint (2.5), we get $y_{12} = 1$, and the cost f_{12} is incurred. Without assumption (2.2), we will incur f_{12} when $x_1 > 0, x_2 = 0$, which is wrong.

Another equivalent "common sense" formulation is obtained by using $y_1, y_2, y_{12}, \in \{0,1\}$ and modeling $y_1 = 1$ when $x_1 > 0$, $y_2 = 1$ when $x_2 > 0$, and $y_{12} = 1$ when both $x_1 > 0$ and $x_2 > 0$. In other words, we need to force $y_{12} = 1 \iff y_1 = 1 \wedge y_2 = 1$. The formulation for this statement is

$$y_1 \ge y_{12}, \; y_2 \ge y_{12}, \quad y_{12} \ge y_1 + y_2 - 1. \tag{2.7}$$

We then write the constraints as

$$\begin{aligned} 0 &\le x_1 \le M_1 y_1, \\ 0 &\le x_2 \le M_2 y_2, \end{aligned} \tag{2.8}$$

and use z_2 in place of $f(x_1, x_2)$ in the minimization objective, where

$$z_2 = f_1y_1 + f_2y_2 + (f_{12} - f_1 - f_2)y_{12}. \tag{2.9}$$

The second formulation is equivalent to the first formulation. Starting with the second formulation, if you replace $y_1 - y_{12}$ by u_1, $y_2 - y_{12}$ by u_2, and y_{12} by u_{12}, where $u_1, u_2, u_{12} \in \{0, 1\}$, you get the first formulation.

We can also use the standard modeling techniques here. A formulation can be obtained by modeling $\text{epi}(f(x_1, x_2))$, which is given by $\text{epi}(f(x_1, x_2)) = P_0 \cup P_1 \cup P_2 \cup P_{12}$, where

$$\begin{array}{rcl} P_0 & = & \{z, x_1, x_2 \,|\, z \geq 0, x_1 = 0, x_2 = 0\}, \\ P_1 & = & \{z, x_1, x_2 \,|\, z \geq f_1, 0 \leq x_1 \leq M_1, x_2 = 0\}, \\ P_2 & = & \{z, x_1, x_2 \,|\, z \geq f_2, x_1 = 0, 0 \leq x_2 \leq M_2\}, \quad \text{and} \\ P_{12} & = & \{z, x_1, x_2 \,|\, z \geq f_{12}, 0 \leq x_1 \leq M_1, 0 \leq x_2 \leq M_2\}. \end{array} \tag{2.10}$$

The recession cones of all the four polyhedra is P_0. Hence we can use the sharp representation of the above disjunction. Choose (z^0, x^0), (z^1, x^1), (z^2, x^2), and (z^{12}, x^{12}) as the variables corresponding to the polyhedra, and let y_0, y_1, y_2, y_{12} be the corresponding binary variables. The sharp representation is given by the two systems of equations listed below.

$$\begin{array}{lll} z^0 \geq 0 & x_1^0 = 0 & x_2^0 = 0 \\ z^1 \geq f_1y_1 & 0 \leq x_1^1 \leq M_1y_1 & x_2^1 = 0 \\ z^2 \geq f_2y_2 & x_1^2 = 0 & 0 \leq x_2^2 \leq M_2y_2 \\ z^{12} \geq f_{12}y_{12} & 0 \leq x_1^{12} \leq M_1y_{12} & 0 \leq x12_2 \leq M_2y_{12} \end{array} \tag{2.11}$$

$$\begin{array}{c} z = z^0 + z^1 + z^2 + z^{12} \\ x_1 = x_1^0 + x_1^1 + x_1^2 + x_1^{12} \\ x_2 = x_2^0 + x_2^1 + x_2^2 + x_2^{12} \\ y_0 + y_1 + y_2 + y_{12} = 1 \end{array} \tag{2.12}$$

This system could be simplified to obtain the first formulation given.

3. Following the notation introduced in class, we use g for the objective function where

$$g = f_0 + s_1x_1 + s_2x_2 + s_3x_3 + (f_{1,r} - f_{1,\ell})y_1, \tag{3.1}$$

with

$$s_1 = \frac{f_{1,\ell} - f_0}{\delta_1}, \quad s_2 = \frac{f_2 - f_{1,r}}{\delta_2}, \quad s_3 = \frac{f_3 - f_2}{\delta_3}, \tag{3.2}$$

and

$$\delta_i = v_i - v_{i-1}, \quad i = 1, 2, 3. \tag{3.3}$$

The forcing constraints are the following.

$$\begin{array}{rcl} x & = & v_0 + x_1 + x_2 + x_3, \\ \delta_i y_i \leq \; x_i & \leq & \delta_i y_{i-1}, \qquad i = 2, 3 \\ y_i & \in & \{0, 1\}, \qquad i = 1, 2, 3. \end{array} \tag{3.4}$$

The constraint that forces $y_1 = 1$ is $x_2 \leq \delta_2 y_1$. Thus, when $x > v_1$, y_1 is forced to 1, and the jump value of $f_{1,r} - f_{1,\ell}$ is added to the objective function. We need the minimization objective to force $y_1 = 0$ when $x < v_1$.

4. (a) Recall that $A \vee (B \wedge C \wedge D) \;\equiv\; (A \wedge B) \vee (A \wedge C) \vee (A \wedge D)$.

$$\begin{aligned}
(L_1 \wedge L_2 \wedge (L_3 \vee L_4)) \vee (L_5 \wedge L_6) \;&\equiv\; (L_1 \wedge (L_2 \vee L_3) \wedge (L_2 \vee L_4)) \bigvee (L_5 \wedge L_6) \\
&\equiv\; ((L_1 \wedge (L_2 \vee L_3) \wedge (L_2 \vee L_4)) \vee L_5) \bigwedge \\
&\;\; ((L_1 \wedge (L_2 \vee L_3) \wedge (L_2 \vee L_4)) \vee L_6) \\
&\equiv\; (L_1 \vee L_5) \wedge (L_2 \vee L_3 \vee L_5) \wedge \\
&\;\; (L_2 \vee L_4 \vee L_5) \wedge (L_1 \vee L_6) \wedge \\
&\;\; (L_2 \vee L_3 \vee L_6) \wedge (L_2 \vee L_4 \vee L_6),
\end{aligned}$$

which is in CNF. The corresponding 0–1 model has six constraints.

$$\begin{aligned}
&x_1 + x_5 \geq 1, \quad x_2 + x_3 + x_5 \geq 1, \quad x_2 + x_4 + x_5 \geq 1, \\
&x_1 + x_6 \geq 1, \quad x_2 + x_3 + x_6 \geq 1, \quad x_2 + x_4 + x_6 \geq 1.
\end{aligned}$$

(b) We break the equivalence into two implications.

$$\begin{aligned}
L_1 \vee \cdots \vee L_m \Rightarrow J_1 \wedge \cdots \wedge J_n \;&\equiv\; \neg(L_1 \vee \cdots \vee L_m) \bigvee (J_1 \wedge \cdots \wedge J_n) \\
&\equiv\; (\neg L_1 \wedge \cdots \wedge \neg L_m) \bigvee (J_1 \wedge \cdots \wedge J_n) \\
&\equiv\; (\neg L_1 \vee (J_1 \wedge \cdots \wedge J_n)) \bigwedge \cdots \bigwedge \\
&\;\; (\neg L_m \vee (J_1 \wedge \cdots \wedge J_n)) \\
&\equiv\; \textstyle\bigwedge_i \bigwedge_j (\neg L_i \vee J_j), \qquad \text{which is in CNF.} \\
J_1 \wedge \cdots \wedge J_n \Rightarrow L_1 \vee \cdots \vee L_m \;&\equiv\; \neg(J_1 \wedge \cdots \wedge J_n) \bigvee (L_1 \vee \cdots \vee L_m) \\
&\equiv\; (\neg J_1 \vee \cdots \vee \neg J_n) \bigvee (L_1 \vee \cdots \vee L_m),
\end{aligned}$$

which is in CNF. Writing the constraints corresponding to the two statements in CNF, the 0–1 model for the original statement is:

$$1 - x_i + y_j \geq 1 \;\;\Leftrightarrow\;\; x_i \leq y_j \quad \forall i, j$$

$$\sum_{j=1}^{n} (1 - y_j) + \sum_{i=1}^{m} x_i \geq 1 \;\;\Leftrightarrow\;\; -\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j \leq n - 1$$

5. **Assumption 2** $\Rightarrow$ **Assumption 1**:
Assumption 2 says the recession cones are same. In other words, the inequalities defining the recession cones are equivalent. Hence, by Farkas' lemma, there exist non-negative multipliers with which we can derive one system from another.

$$A_i \boldsymbol{x} \leq 0 \Rightarrow A_j \boldsymbol{x} \leq 0 \;\; \forall\, i \neq j.$$

$$\Rightarrow \;\exists\, \boldsymbol{v}^{ij} \geq 0 \mid A_j = \boldsymbol{v}^{ij} A_i.$$

$$\text{Thus,} \;\; A_i \boldsymbol{x} \leq \boldsymbol{b}^i \;\;\Rightarrow\;\; A_j \boldsymbol{x} \leq \boldsymbol{v}^{ij} \boldsymbol{b}^i,$$

$$\text{so we can choose } \boldsymbol{u}^j \text{ s.t.} \quad \boldsymbol{b}^j + \boldsymbol{u}^j \geq \boldsymbol{v}^{ij} \boldsymbol{b}^i \;\forall\, i \neq j.$$

Assumption 1 $\Rightarrow$ **Assumption 2**:
We use the contrapositive argument here. Assume the recession cones are not all the same. Let $\text{rec}(P_1) \neq \text{rec}(P_2)$. Then $\exists\, \boldsymbol{x}^* \in \text{rec}(P_2)/\,\text{rec}(P_1)$. Hence, there is one row $\boldsymbol{a}^T$ in A_1 such that $\boldsymbol{a}^T \boldsymbol{x}^* > 0$. Then, as $\lambda \to \infty$, for all $\boldsymbol{x} \in P_2$ we get

$$A_2(\boldsymbol{x} + \lambda \boldsymbol{x}^*) \leq \boldsymbol{b}^2, \quad \boldsymbol{a}^T(\boldsymbol{x} + \lambda \boldsymbol{x}^*) \to \infty$$

Hence, there does not exist $\boldsymbol{u}^1$ such that $\boldsymbol{x} \in P_2 \Rightarrow A_1 \boldsymbol{x} \leq \boldsymbol{b}^1 + \boldsymbol{u}^1$. So, **Assumption 1** is violated if **Assumption 2** is violated.

Thus **Assumption 1** $\Longleftrightarrow$ **Assumption 2**.