Integer Optimization (Spring 2009) — Homework 1

- The total points (given in parentheses) add up to 130. You will be graded for 120 points.
- **This homework is due in class on Thursday, January 22**.

1. (20) Let $f_1, \ldots, f_m$ be convex functions from $\mathbb{R}^n$ to R. Define $f(\boldsymbol{x}) = \sum_{i=1}^m f_i(\boldsymbol{x}),\ \forall\, \boldsymbol{x} \in \mathbb{R}^n$. Show that f is a convex function.

2. (30) The infinity norm distance between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is given by

$$\| \boldsymbol{x} - \boldsymbol{y} \|_\infty = \max_{i=1,\ldots,n} |x_i - y_i|.$$

   The *ball* with center $\boldsymbol{c}$ and radius r is then defined as $B(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \| \boldsymbol{x} - \boldsymbol{c} \|_\infty \leq r\}$. Let $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$, where $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. We want to find the ball with the largest radius, which is entirely contained within P. Give a linear programming (LP) formulation for this problem. (When using the usual Euclidean distance instead of the infinity norm, the center of such a *largest* ball is called the *Chebychev center* of P.)

3. (30) You are interested in choosing from a set of seven Investments $\{1, \ldots, 7\}$. Model each of the following conditions using 0-1 variables.

   (a) You cannot invest in all of them.

   (b) You must choose at least one of them.

   (c) Investment 2 cannot be chosen if Investment 3 is chosen.

   (d) Investment 4 can be chosen only if Investment 1 is chosen.

   (e) You must choose either both Investments 1 and 2, or neither.

   (f) You must choose either at least one of Investments 1,2,3, or at least two of Investments 2,3,5,6.

4. (25) A set of n jobs must be carried out on a single machine, which can do only one job at a time. Job j takes p_j hours to complete and has weight w_j, for $j = 1, \ldots, n$. Formulate as a mixed integer program (MIP) the scheduling problem to find the order in which to carry out the n jobs, so as to minimize the weighted sum of their starting times.

5. (25) Give an integer programming formulation for the problem of placing N queens on an $N \times N$ chessboard such that no two queens share any row, column, or diagonal.