

Integer Optimization (Spring 2009) — Final Exam

- This is an open-book exam.
- You must not discuss the exam with any one (except me!).
- **This exam is due on Wednesday, May 6, by 5:00 pm.**

1. (20) Consider the two polyhedra in \mathbb{R}^n :

$$P_i = \{\mathbf{x} \in \mathbb{R}^n \mid A_i \mathbf{x} \leq \mathbf{b}^i\}, \quad i = 1, 2.$$

Give an MIP with a single 0–1 variable that could be solved to decide whether $P_1 \cup P_2$ is non-empty. Prove the correctness of your formulation.

You must not assume that upper bounds \mathbf{u}^i exist such that $A_i \mathbf{x} \leq \mathbf{b}^i + \mathbf{u}^i$ hold. Instead, you must introduce a non-negative variable α , and \mathbf{y} , such that $\alpha \mathbf{x}$ (non-linear term) is “represented” by \mathbf{y} . How would you approach the general case where you need to check if $P_1 \cup \dots \cup P_k \neq \emptyset$ for $k \geq 3$?

2. (15) Consider the disjunction

$$\ell_1 \leq x \leq u_1 \vee \dots \vee \ell_n \leq x \leq u_n \tag{F.1}$$

with

$$\ell_1 \leq u_1 \leq \dots \leq \ell_n \leq u_n \tag{F.2}$$

where the constraint $\ell_1 \leq x \leq u_n$ is assumed to be global.

- (a) Write the big-M representation for the disjunction.
 - (b) Show that the sharp representation of the disjunction can be simplified to give a representation with only 2 inequalities and 1 equation.
3. (15) For the set covering problem, we refer to doing the following steps as a *round* of preprocessing.
- Singleton rows – fix the variable that appears alone in a row to 1 (i.e., select the corresponding pole for locating a receiver), remove all rows which got covered by fixing this variable;
 - remove all *dominated* rows; and
 - remove all *dominated* columns.

Construct an instance, i.e., give the meter-pole coverage incidence matrix A , that is as small as possible, and requires at least three rounds to complete the preprocessing.

4. (20) Consider a graph $G = (V, E)$ and the polyhedron

$$P = \{\mathbf{x} \in \mathbb{R}^{|V|} \mid 0 \leq x_i \leq 1 \ \forall i \in V, \ x_i + x_j \leq 1 \ \forall (i, j) \in E\}. \quad (\text{F.3})$$

$\mathbf{x} \in P \cap \mathbb{Z}^{|V|}$ are the incidence vectors of *stable sets* in G . A *clique* in G is a subset of nodes in which every node is connected to every other node. Clearly, for a clique C in G , the inequality

$$\sum_{i \in C} x_i \leq 1 \quad (\text{F.4})$$

is valid. Show that there exists a constant c_1 such that the CG-rank of (F.4) is *at most* $c_1 \log |C|$.

5. (15) Let $\mathbb{L} \in \mathbb{R}^n$ be an n -dimensional lattice with a basis $B = [\mathbf{b}_1, \dots, \mathbf{b}_n]$. Assume that B is LLL-reduced. Given a set of $t \leq n$ linearly independent vectors $\mathbf{x}_1, \dots, \mathbf{x}_t \in \mathbb{L}$, show that

$$\|\mathbf{b}_j\| \leq 2^{(n-1)/2} \max\{\|\mathbf{x}_1\|, \dots, \|\mathbf{x}_t\|\} \quad \text{for } j = 1, \dots, t.$$

6. (15) Let $\mathbf{a} = \mathbf{p}M + \mathbf{r}$, where $\mathbf{p} \in \mathbb{Z}_{>0}^n$, $\mathbf{r} \in \mathbb{Z}^n$, and $M \in \mathbb{Z}_{>0}$ such that $M \gg p_j, |r_j|$. Consider

$$A = \begin{bmatrix} \mathbf{a}^T \\ I \end{bmatrix}, \quad \text{where } I \text{ is the identity matrix.}$$

To prove that the rangespace reformulation of this problem is easy for branch-and-bound, we need to find $n - 1$ linearly independent vectors in $\mathbb{L}(A)$ that are relatively “short” (i.e., do not have terms containing the factor M in them). We can then use the lemma given in problem (5) to derive a contradiction. Prove that there exist $n - 1$ linearly independent vectors in $\mathbb{L}(A)$ with length bounded by $f(p, r)$, where $f(p, r)$ is independent of M .