Network Optimization (Fall 2008) – Brief Solutions to Homework 9

1. We first try to find a feasible flow in the given network. For this purpose, we can create an equivalent network with

$$u'_{ij} = u_{ij} - l_{ij} \ \forall \ (i,j) \in A, \text{ and } \ b(i) = \sum_{(j,i)\in A} l_{ji} - \sum_{(i,j)\in A} l_{ij} \ \forall \ i \in N.$$

Then we can solve a max flow problem to establish a feasible flow in this network, or prove that there is no feasible flow (see Application 6.1, pages 169-170 and 193-194 in AMO).

WLOG, let $\boldsymbol{x}$ be a feasible flow in the original network. We have $l_{ij} \leq x_{ij} \leq u_{ij} \ \forall \ (i,j) \in A$. To find the minimum flow, we would like to *push back* as much flow (from t to s), still honoring the bounds. For this purpose, define a residual network with residual capacities defined as follows:

$$r'_{ij} = (x_{ij} - l_{ij}) + (u_{ji} - x_{ji}).$$

The first term is the amount of flow on (i,j) that we can push back still satisfying the lower bound l_{ij}, while the second term is how much *more* flow we can send from j to i along the reverse arc. Recall that the default expression for the residual capacity of arc (i,j) is $r_{ij} = u_{ij} - x_{ij} + x_{ji}$. Hence, we re-write the expression for r'_{ij} as follows:

$$r'_{ij} = (u_{ij} - l_{ij}) - (u_{ij} - x_{ij}) + (u_{ji} - x_{ji}).$$

Defining $u'_{ij} = u_{ij} - l_{ij}$ and $x'_{ij} = u_{ij} - x_{ij}$, the above expression becomes $r'_{ij} = u'_{ij} - x'_{ij} + x'_{ji}$. Now we solve the max flow problem on the network G' defined by u'_{ij} (with zero lower bounds), and the solution for the same will give the corresponding minimum flow. In fact, we can write

$$\begin{aligned} v' &= \sum_{i\in S, j\in \bar{S}} x'_{ij} - \sum_{i\in S, j\in \bar{S}} x'_{ji} \\ &= \left(\sum_{i\in S, j\in \bar{S}} u_{ij} - \sum_{i\in S, j\in \bar{S}} u_{ji} \right) - \left(\sum_{i\in S, j\in \bar{S}} x_{ij} - \sum_{i\in S, j\in \bar{S}} x_{ji} \right) \\ &= \left(\sum_{i\in S, j\in \bar{S}} u_{ij} - \sum_{i\in S, j\in \bar{S}} u_{ji} \right) - v = \text{constant} - v \end{aligned}$$

Thus, $\max\{v'\} \Rightarrow \min\{v\}$. Further, using the max flow-min cut theorem on G', we can write $v' = \min\{u'[S,\bar{S}]\}$. Hence, for any cut $[S,\bar{S}]$, we can then write

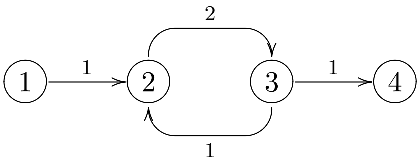
$$\begin{aligned} v' &= \left(\sum_{i\in S, j\in \bar{S}} u_{ij} - \sum_{i\in S, j\in \bar{S}} u_{ji} \right) - v \ \leq \ \left(\sum_{i\in S, j\in \bar{S}} u_{ij} - \sum_{i\in S, j\in \bar{S}} l_{ij} \right) \\ \Rightarrow \ v &\geq \left(\sum_{i\in S, j\in \bar{S}} l_{ij} - \sum_{i\in S, j\in \bar{S}} u_{ji} \right). \end{aligned}$$

Thus, $v = \max \left\{ \sum_{i\in S, j\in \bar{S}} l_{ij} - \sum_{i\in \bar{S}, j\in S} u_{ij} \right\}$, as required.

2. The idea is to run a modified search (or labeling) algorithm starting at node t that builds a tree rooted at t. At each intermediate step, we mark each node j that is reachable from an already marked node i by a yellow forward arc (i,j), or a green forward arc (i,j), or a green backward arc (j,i). If node s is marked by this search, then the path P from t to s in the tree along with the arc (s,t) gives the cycle (case (1)). If s is not marked, let S be the set of all marked nodes and $\bar{S} = N \setminus S$. Then $[S, \bar{S}]$ forms a t-s cut as described in case (2). Note that there will be no yellow forward arcs, and no green forward or backward arcs in this cut (else the search would have already marked the nodes at the other end of such arc(s)).

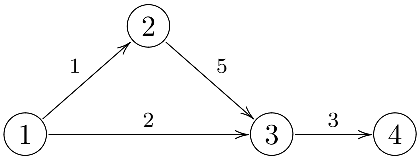
3. If we remove the arc (i,j), or even decrease u_{ij} by 1 unit, the max flow value will decrease. If not, then there will be a maximum flow $\boldsymbol{x'}$ with $x'_{ij} < u_{ij}$. Since none of the other arc capacities were altered, $\boldsymbol{x'}$ will be a maximum flow in the original network as well, which contradicts the assumption that $x_{ij} = u_{ij}$ for *all* max flows. Since the value of the max flow decreases when u_{ij} is decreased, the capacity of all min-cuts decrease as well (by the max flow-min cut theorem). Hence (i,j) must contribute to the capacity of some min cut.

4. (a) FALSE. We can have a circulation between two nodes (x_{ij}'s shown).



   (b) TRUE assuming all $l_{ij} = 0$ in the network. In a maximum flow x, let both $x_{ij} > 0$ and $x_{ji} > 0$ for some pair of nodes i, j, and WLOG, let $x_{ij} \geq x_{ji}$. Then, we can get another maximum flow x' with $x'_{ji} = 0$ and $x'_{ij} = x_{ij} - x_{ji}$. You can find a counterexample when $l_{ij} > 0$ though.

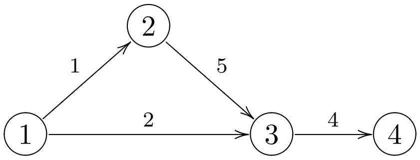
   (c) FALSE. In the following network, both $[\{1\},\{2,3,4\}]$ and $[\{1,2,3\},\{4\}]$ are min-cuts (u_{ij}'s shown).



   (d) FALSE. Consider the trivial case when we have only one directed arc $(2,1)$ in the network with $u_{21} = 1$. The max flow from 1 to 2 is zero. If we make the arc undirected, the max flow becomes 1.

   (e) TRUE. Let $[S_1, \bar{S_1}]$ be a min cut. Then $\sum_{i \in S_1, j \in \bar{S_1}} u_{ij} \leq \sum_{i \in S_2, j \in \bar{S_2}} u_{ij}$ for any cut $[S_2, \bar{S_2}]$. This inequality still holds when all capacities are scaled by a positive number.

   (f) FALSE. In the following network, adding 2 units to each u_{ij} (shown on arcs) changes the min cut from $[\{1\},\{2,3,4\}]$ to $[\{1,2,3\},\{4\}]$.



5. (a) Let $G = (N, A)$ with $N = \{s, t, 1, 2, \ldots, n\}$, and $A = \{(s, i)\} \cup \{(i, t)\}$, $i = 1, \ldots, n$. Setting $S = \{s\} \cup N_1$, $\bar{S} = N \setminus S$ for $N_1 \subseteq \{1, 2, \ldots, n\}$ gives an s-t cut $[S, \bar{S}]$. The number of such s-t cuts is the number of subsets of $\{1, \ldots, n\}$, which is $2^n = 2^{(|N|-2)}$.

   (b) Consider the same network described above. Set $u_{si} = u_{it} = 1$, $i = 1, \ldots, n$. Then the maximum flow value is n with each arc carrying flow equal to its capacity of 1 unit. Now consider the family of cuts $[S, \bar{S}]$, also described above. WLOG, let $N_1 = \{1, \ldots, n_1\}$ for some $n_1 \leq n$. Then the capacity of the cut is

$$u[S, \bar{S}] = \sum_{i=n_1+1}^{n} u_{si} + \sum_{i=1}^{n_1} u_{it} = n - n_1 + n_1 = n,$$

   showing that each cut in this family is a min-cut.