Network Optimization (Fall 2008) – Brief Solutions to Homework 1

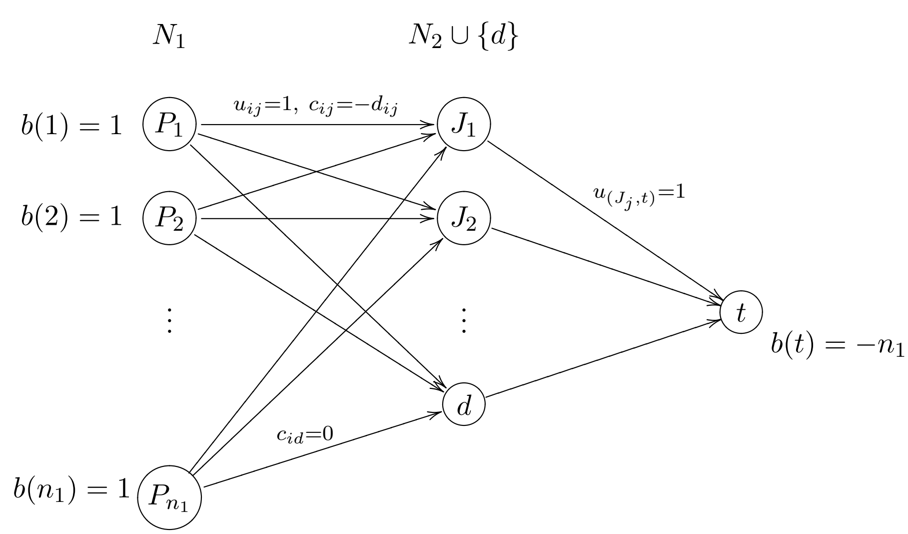
1. For the shortest path problem, add arc (t, s) with $l_{ts} = u_{ts} = 1$, and $c_{ts} = 0$. For the original network, set $l_{ij} = 0$, $u_{ij} \geq 1$. The condition $u_{ts} = 1$ can be relaxed to $u_{ts} \geq 1$ if there are no s–t directed paths in the original network with a negative sum of costs.

   For the assignment problem, add nodes s and t, arcs (s, i) with $l_{si} = 1$ for $i \in N_1$ and arcs (j, t) with $l_{jt} = 1$ for $j \in N_2$, the arc (t, s) with $u_{ts} = \infty,\ c_{ts} = 0$. For the original network, set $l_{ij} = 0$, $u_{ij} \geq 1$. As in the case of the shortest path problem, set $u_{si} = 1$ for $i \in N_1$ and $u_{jt} = 1$ for $j \in N_2$, unless all $c_{ij} \geq 0$ in the original assignment network. (Equivalently, one could set $u_{ts} = |N_1|$ instead of limiting u_{si} and u_{jt} values).

   Similarly, for the transportation problem, add nodes s and t, arcs (s, i) with $l_{si} = u_{si} = s_i$ for $i \in N_1$ (the supply values for the warehouses), arcs (j, t) with $l_{jt} = u_{jt} = d_j$ for $j \in N_2$ (the demand values for the retailers), the arc (t, s) with $u_{ts} = \infty,\ c_{ts} = 0$.

2. Add a dummy supply node d with $b(d) = \sum_{j \in N_2} -b(j) - \sum_{i \in N_1} b(i)$. Add an arc from node d to each demand node $j \in N_2$ with $c_{dj} = p_j$ and $u_{dj} \geq b(d)$.

3. This problem is more general than the assignment problem. First of all, there may be more personnel than there are jobs, or vice versa. Secondly, not all of the d_{ij} values may be positive. Since the goal is to maximize the overall total utility, the optimal assignment may well leave several personnel unassigned (and several jobs unassigned as well). We model this problem as a min-cost flow problem. Assume there are n_1 personnel and n_2 jobs, with $n_1 > n_2$ (the opposite case, where $n_1 < n_2$ is modeled in a similar fashion). Let N_1 and N_2 denote the sets of personnel and jobs, respectively. We construct a graph with node set $N = N_1 \cup (N_2 \cup \{d\}) \cup \{t\}$. The node d is an extra (dummy) node added to the jobs set N_2. The arc set consists of arcs (i, j) for every $i \in N_1,\ j \in N_2 \cup \{d\}$, and arcs $(j, t)\ \forall\ j \in N_2 \cup \{d\}$. We set $b(i) = 1 \,\forall\, i \in N_1,\ b(t) = -n_1$, and $b(j) = 0$ for all other nodes j. The upper bounds are set as follows. $u_{ij} \geq 1\ \forall\ i \in N_1, j \in N_2 \cup \{d\}$ and $u_{jt} = 1\ \forall\ j \in N_2,\ u_{dt} \geq n_1$. All lower bounds are set to zero. We set $c_{ij} = -d_{ij}\ \forall\ i \in N_1, j \in N_2$, and all other costs are set as zero (including the costs of the arcs (i, d) going from each personnel node to the dummy node). If the unit supply from personnel node i is routed through the dummy node, then that personnel i is not assigned to any job. (in the figure, P_i and J_j represent personnel i and job j, respectively).



4. Seat-sharing problem

   (a) The answer is NO. To (possibly) convert the seat-sharing network flow problem to a circulation problem, one would add a (source) node s and draw a directed arc to each family f_i with $l_{si} = b(i)$. One would also add the arc (t, s) with $u_{ts} = \infty$ and $c_{ts} = -1$ (say). All the nodes are set as transshipment nodes. In such a model, members from all the families are mixed together at nodes t and s. Setting $l_{si} = b(i)$ alone will not make sure that the particular $b(i)$ persons belonging to family f_i are exactly the ones that get routed through the arc (s, i). Due to this possible mixing, two (or more) persons from the same family could end up in the same car.

   (b) To the min-cost flow model given in class, add costs $c_{jt} = -u(j)$ for the arcs (j, t).

5. Create a graph with $n+1$ nodes numbered $1, 2, \ldots, n+1$. Draw arcs (i, j) for all $i < j$, and set the cost as $-c_{ij}$. Find the shortest path from node 1 to node $n+1$. The arcs (i, j) which are part of the shortest path represent the lines in the best decomposition and contains the words numbered i to $j-1$.

6. If we assume that parts of woods available (w_{ij}) that are not harvested in one year are not carried over to the subsequent years, then we can model this problem as a max-flow problem. We have a node for each forest unit (f_i, for $i = 1, \ldots, p$), one node for each year $(1, \ldots, k)$, and the start and terminus nodes (s, t). The bounds are illustrated in the figure below. All lower bounds not mentioned are set to zero.