Math 466/566 (Fall 2008) The Seat-sharing problem (AMO 1.8, page 21)

Several families are planning a shared car trip on scenic drives in the Cascades in Washington. To minimize the possibility of any quarrels, the organizers of the trip want to assign individuals to cars so that **no two members of a family are in the same car**. Formulate this problem as a network flow problem.

Assume that there are p families, and that the family i has $b(i)$ members (for $1 \leq i \leq p$). Also, assume that there are q cars, and that car j can hold up to $u(j)$ persons (for $1 \leq j \leq q$). To start with, assume that there is enough room to accommodate all families, i.e., $\sum_{1 \leq j \leq q} u(j) \geq \sum_{1 \leq i \leq p} b(i)$.

A setup similar to that of the transportation problem can be imagined here, with two subsets of nodes – the **family** nodes being the supply nodes ($N_1 = \{f_1, \ldots, f_p\}$), and the **car** nodes being the demand nodes ($N_2 = \{c_1, \cdots, c_q\}$). The supply on the family node f_i is $b(i)$. To model the restriction that no two persons from one family should be assigned to the same car, we draw one arc from each family node f_i to every car node c_j with upper bound $u_{ij} = 1$. Thus, there will be a total of pq such arcs.

One could put a demand of $-u(j)$ on the car node c_j if $\sum_{j \in N_2} u(j) = \sum_{i \in N_1} b(i)$, and the model will be complete. The more realistic situation occurs when there are more seats available in the cars than there are people to be accommodated, and hence we may not be able to satisfy certain demands completely (in other words, certain cars will have unfilled seats). To model this situation, the car nodes are set as transshipment nodes. We add another node t and draw arcs from each car node c_j to t with an upper bound of $u(j)$. Then we put $b(t) = -\sum_{i \in N_1} b(i)$ to complete the network flow model. This construction makes sure that we can assign up to (or possibly less than) $u(j)$ persons in car c_j, and at the same time, all the family members are assigned to some car.