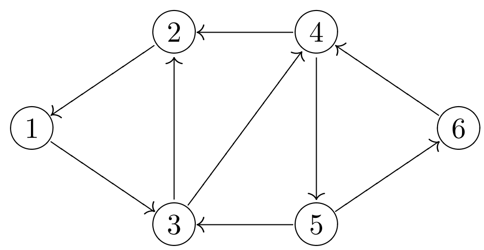
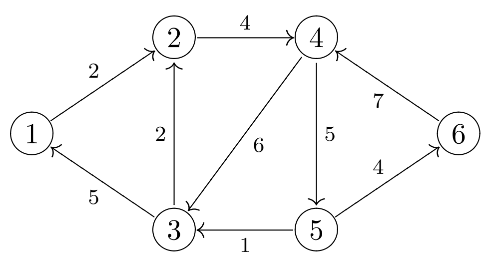
Network Optimization (Fall 2008) — Midterm

- The total points (given in parentheses) add up to 105. You will be graded for 100 points.
- Try to give **brief** answers.
- *"Time is precious."* – anonymous.

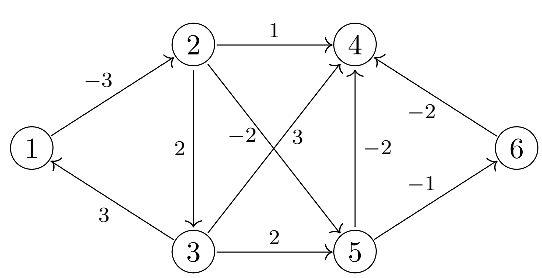
1. (10) Identify all directed cycles in the following graph. Make the graph acyclic by removing the *minimum* number of edges. Give a topological ordering for the resulting acyclic graph. You may use the extra copy of the graph to show the cycles or the topological order.



2. (10) Give a decomposition of the flow (the number on each arc (i, j) is the flow x_{ij}) into flows along paths and cycles. Is this decomposition unique?



3. (12) Find the shortest path from node $s = 3$ to every other node in the graph (the number on arc (i, j) is c_{ij}). Show the optimal shortest path distance labels and the shortest path tree.



4. (15) The cost incurred for a flow x_{ij} along the arc (i,j) in a min-cost flow network is defined as follows.
$$c_{ij}(x_{ij}) = \begin{cases} c^1_{ij}x_{ij} & \text{if } \ l_{ij} \le x_{ij} \le l'_{ij}, \\ c^1_{ij}l'_{ij} + c^2_{ij}(x_{ij} - l'_{ij}) & \text{if } \ l'_{ij} \le x_{ij} \le u_{ij}, \end{cases}$$
where $l_{ij} \le l'_{ij} \le u_{ij}$ and $c^1_{ij} \le c^2_{ij}$. How will you model this problem as a standard min-cost flow problem?

5. (15) Assume that graph $G = (N, A)$ has no negative cost cycles, and $c_{ij} \ge 0 \ \forall \ (i,j) \in A$. Describe an algorithm to find a shortest walk from node s to node t that visits a specified node p. What is the complexity of your algorithm? Will this walk always be a *path*?

6. (12) In each part, choose the best option. Give a **brief** justification.

   (a) Let $\ell_{\text{bfs}}(i), \ell_{\text{dfs}}(i)$, and $\ell_{\text{sp}}(i)$ denote the number of arcs in the path from node s to node i in the breadth-first search tree, the depth-first search tree, and the shortest path tree of a network $G = (N, A)$, respectively. Also define $\ell_{\text{S}}(i) = \min\{\ell_{\text{bfs}}(i), \ell_{\text{dfs}}(i)\}$.

      i. $\ell_{\text{S}}(i) \ge \ell_{\text{sp}}(i) \ \ \forall \ i \in N$.
      ii. $\ell_{\text{S}}(i) \le \ell_{\text{sp}}(i) \ \ \forall \ i \in N$.
      iii. $\ell_{\text{S}}(i) \le \ell_{\text{sp}}(i)$ when $c_{ij} \ge 0 \ \forall \ (i,j) \in A$.
      iv. none of the above.

   (b) Let $d_{\min}$ denote the minimum degree of a node in a graph G. Then G contains a cycle if

      i. $d_{\min} \ge 1$.
      ii. $d_{\min} \ge 2$.
      iii. $d_{\min} = 1$, but at least two nodes have degrees ≥ 2.
      iv. none of the above.

7. (16) For each part, answer TRUE or FALSE. **Justify** your answer.

   (a) $n^{\log \log n} = \Theta(n \log n)$.

   (b) Recall that an arc is *vital* if its removal from the network increases the shortest path distance from node s to node t. A *most vital arc* is a vital arc whose removal produces the maximum increase in the s-t shortest path distance. We can find a most vital arc in a network in $O(mn + n^2C)$ time.

8. (15) In each of t periods, an entrepreneur can buy, sell, or hold for later sale some commodity. In period i, she can buy at most α_i units, can holdover at most β_i units for the next period, and must sell at least γ_i units. Let c_i, h_i, and p_i denote the per unit buying cost, holding cost, and selling price in period i, respectively. Formulate the problem of determining the optimal buy-hold-sell policy (which maximizes overall profit in t periods) as a network flow problem.