

# Network Optimization (Fall 2008) — Final Exam

- This is an *open-book* exam.
  - You are not supposed to discuss this exam with any one (except me, if need be).
  - You must turn in this exam before **noon on Tuesday, December 16**.
1. (20) Consider a network  $G = (N, A)$  with  $u_{ij} = 1 \forall (i, j) \in A$  and  $|N| = n$ . Let  $v^*$  be the value of the maximum flow from node  $s$  to node  $t$ , and let  $d(j)$  be the shortest distance from node  $s$  to node  $j$  in terms of number of arcs. Show that  $d(t) \leq 2n/\sqrt{v^*}$ .
  2. (20) Let  $\mathbf{x}'$  and  $\mathbf{x}''$  be two maximum flows in  $G = (N, A)$  from source node  $s$  to sink node  $t$ . Show that there exists another maximum flow  $\mathbf{x}$  such that  $x_{sj} = x'_{sj}$ ,  $x_{js} = x'_{js}$ , and  $x_{jt} = x''_{jt}$ ,  $x_{tj} = x''_{tj}$ . In other words,  $\mathbf{x}$  is identical to  $\mathbf{x}'$  on the arcs incident to node  $s$ , and is identical to  $\mathbf{x}''$  on the arcs incident to node  $t$ .
  3. (20) Consider the algorithm for finding the minimum spanning tree (MST) that examines the arcs in any order (not necessarily sorted by  $c_{ij}$  values), and builds the tree by adding one arc at a time. If adding the next arc creates a cycle, then it will remove the *costliest* arc from the cycle, i.e., the arc in the cycle with the largest  $c_{ij}$  value. Will this algorithm produce an MST? How will the complexity of this algorithm compare to that of Kruskal's algorithm?
  4. (20) Let  $(\mathbf{x}^*, \boldsymbol{\pi})$  be a set of optimal flows and optimal node potentials for the minimum cost flow problem. An arc  $(i, j)$  in the original network  $G$  is called *upward critical* if increasing  $u_{ij}$  by one unit will strictly decrease, and hence improve, the total cost. Explain how you will determine if an arc  $(i, j)$  is upward critical or not. *Hint: Consider a slight modification of the residual network  $G(x)$ , and look for cycles in this network.*
  5. (20) Prove, or disprove using a counterexample, each of the following claims.
    - (a) Let  $T$  and  $T'$  be MSTs of a given network  $G$ . Let  $f(c)$  and  $f'(c)$  denote the number of arcs in  $T$  and  $T'$ , respectively, with cost equal to  $c$ . Then,  $f(c) = f'(c)$  for all values of  $c$ .
    - (b) For a minimum cost flow problem, all supplies/demands and all arc capacities are even integers. Then there is some optimal solution  $\mathbf{x}$  with each  $x_{ij}$  an even number. (Assume that the problem has at least one finite feasible solution, i.e., it is not unbounded).