A COORDINATE-FREE FOUNDATION FOR PROJECTIVE SPACES TREATING PROJECTIVE MAPS FROM A SUBSET OF A VECTOR SPACE INTO ANOTHER VECTOR SPACE

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Abstract

In a related paper [2], the authors have shown that a homeomorphism which preserves convex sets, mapping an open subset of one locally convex topological vector space onto an open subset of another, is a projective map (the quotient of an affine operator by an affine functional). The establishment of this result in its full generality required a treatment of (possibly infinite dimensional) topological projective spaces. The present paper supplies this treatment, developing projective spaces as dual pairs by virtue of the cross-ratio.

The nontopological portion of the paper is valid for projective spaces over (commutative) fields of characteristic different from 2. It is demonstrated in particular, that one-dimensional projective spaces of such type may be described quite simply in a geometric manner using involutions (self-inverse permutations).

1. Introduction

Let V and X be Hausdorff, locally convex, real, topological vector spaces of dimension greater than 1. Motivated by an application [1] in the derivation of optimization algorithms, Ariyawansa, Davidon and McKennon [2] show that a homeomorphism from a connected and open subset of V onto an open subset of X is the restriction of a projective map, if and only if that map preserves convex sets. This characterization is related to the classical fundamental theorem of projective geometry [11], and to the characterization of affine maps on the whole of a real vector space as convexity-preserving maps due to Meyer and Kay [7]. The proof of the characterization in [2] required a coordinate-free foundation for (not-necessarily finite dimensional) projective spaces and projective maps on vector spaces. The lack of such a foundation in the literature motivated us to formulate one, and since we believe it likely to be of use in other contexts as well, we present it here independent of the characterization proved in [2].

One of the most salient and useful aspects of projective geometry is its duality. Projective spaces occur in pairs and, although one of the pair of spaces normally will be prominent in any given application, the other space is always at least dormant, and often plays an active role. The typical situation is that one space P appears as a set of points and the dual space Q as a set of maximal proper subspaces of P. However, it is sometimes useful to reverse the roles of P and Q, just as it is useful...
in the theory of locally convex spaces to reverse the roles of a vector space and its dual space of continuous linear functionals.

It occurred to the authors that a treatment of projective spaces should place both the dual spaces $P$ and $Q$ on an equal status. That accepted, the question became what was the proper way of connecting the two. In vector space duality, each space is associated with a set of linear functionals on its dual space by means of a bilinear functional $\langle \cdot, \cdot \rangle$ defined on the two spaces. It is not so simple with projective spaces because, as will be seen in Theorem 4.29 infra, there are many bilinear functionals connecting $P$ and $Q$, no one of which being distinguished above any of the others.

The proper tool seemed to be the cross-ratio, classically defined on quadruples of distinct points in the scalar field $F$. The scalar field, with a point $\infty$ appended, passes for a minimal nontrivial projective space, and it may be identified with its dual. Thus a quadruple of points in $F$ may be regarded as a pair of points of $P$, plus another pair of points of $Q$. This point of view, when extended to more general projective spaces, turns out to be useful in connecting $P$ to $Q$, and the authors have adopted it in their definition.

The program is to introduce two sets $P$ and $Q$ and to define a cross-ratio function with value denoted by $\left[ p_1, p_2, q_1, q_2 \right]$ for elements $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. Rather than set up a list of axioms for these entities, the authors opted to include as an axiom the existence of a “representation” of them on a vector duality. There are many vector dualities inherent in a projective duality, and so it seemed consistent with the spirit of projective spaces to invoke one for the definition.

The definition is presented and examples are described in §2. Since a vector duality representation is a part of the definition, it is of importance to determine how any two vector duality representations are related: this is effected in Theorem 2.22.

An isomorphism $\varphi$ of projective spaces must preserve the essential projective properties, and those only. These properties are those which are affected by the dual space and the cross-ratio. Thus $\varphi$ must be paired with an isomorphism $\psi$ of the dual space, and the pair $(\varphi, \psi)$ must preserve the cross-ratio. Definitions and first properties of isomorphisms are given in §3.

To any given point $q \in Q$ corresponds a maximal subspace $q^\circ$ of $P$, which is called a dual maximal subspace. The complement $A$ of this dual subspace in $P$ is an affine space over $F$ in a natural way. The vector space which acts on $A$ is realized by invoking the set $\text{Tran}(P; q)$ of all automorphisms $\phi$ of $P$ for which $q^\circ$ is the set of $\phi$-fixed points. If an automorphism of $P$ fixes the elements of $q^\circ$, it is either the identity automorphism $i$, an element of $\text{Tran}(P; q)$, or it fixes exactly one other point (and is said to be a scalar-automorphism). The set $\text{Tran}(P; q) \cup \{i\}$ is an Abelian group, and nonzero scalar multiplication is defined using scalar-automorphisms. Obviously automorphisms $\tau \in \text{Tran}(P; q)$ leave $A$ invariant, and it is under these actions that $A$ is an affine space over $\text{Tran}(P; q) \cup \{i\}$. Details are given in §4.

Similarly, to any given point $p \in P$ corresponds an affine space $B$ in $Q$, the complement of the dual maximal space $p^\circ$. When $p$ is in $A$ and $q$ is in $B$ we say that $(p, q)$ is a standard pair. Specifying a point $p$ of an affine space $A$ permits identification of $A$ with a vector space $V$ having the point $p$ as origin. Similarly, $B$ may be identified with a vector space $W$ having $q$ as its origin. These two vector spaces $V$ and $W$, termed standard vector spaces, are actually in a duality defined...
by the bilinear map
\[ \langle v, w \rangle := [v, p, w, q] - 1 \quad (\forall v \in V, w \in W). \]

These matters are treated in \( \S 4 \) as well.

It is the view of the authors that a projective space is intrinsically more fundamental than a field. Why then do we employ a field to introduce projective spaces? An explanation is given at the end of \( \S 4 \), and an indication of how a field could have been avoided is given.

The initial motivation for this paper was the problem of determining which maps on subsets of (not necessarily finite dimensional) vector spaces were “projective”. In view of the development of \( \S 2 \) through \( \S 4 \), the setting of this problem can be reduced to a function \( \sigma \) which is defined on a subset of a standard vector space \( V \) of \( P \), and which has a standard vector subspace \( X \) of another projective space \( R \) as its range. If such a function \( \sigma \) is the restriction of a projective isomorphism from \( P \) to \( R \), it turns out that it must be the quotient of an affine map of \( V \) into \( X \) with an affine functional of \( V \) into \( F \). A partial converse holds for a class of linear spaces which includes all locally convex spaces and finite dimensional vector spaces. These matters are treated in \( \S 5 \).

In \( \S 6 \), \( F \) is taken to be either the real or the complex field, and topological projective spaces are treated. The definition of a projective topology is given in terms of the scalar-automorphisms mentioned above. It is shown that in a topological projective space each standard vector space is a topological vector space under the relativized topology. Conversely, if a standard vector subspace \( V \) of a projective space is a topological vector space relative to some topology \( T \), there is a unique projective topology on \( P \) of which the relativization to \( V \) is just \( T \).

The culmination of \( \S 6 \) is a theorem which deals with a homeomorphism \( \varphi \) from an open subset of \( V \) onto an open subset of \( X \), \( V \) and \( X \) being standard vector spaces in projective spaces \( P \) and \( R \) respectively. It is shown, that if \( \varphi \) is projective on finite dimensional subspaces, then it is the restriction of a projective isomorphism from \( P \) to \( R \).

The authors freely acknowledge the limitation of the scope of this paper. A complete treatment of a proper foundation of (not necessarily finite dimensional) projective spaces would require a book. We were motivated here by a problem involving functions on vector spaces and our work extends little beyond the essentials to treat that problem. Among the many interesting subjects which suggested themselves to the authors, subjects which were not within the compass of this paper, were that of homomorphisms of projective dualities, correlations (isomorphisms from one projective space onto its dual projective space) with its related subject of conic sections, and applications of the projective setting to operator algebras, in particular to \( C^* \)-algebras.

Another sort of limitation of this paper is in the nature of the projective spaces considered. We consider only Pappian projective spaces without Fano configuration, or equivalently, projective spaces over commutative fields of characteristic different from 2. Since this paper was written to provide a foundation for solving a problem over the real field, the authors had to draw a line at some point regarding the level of generality. It was decided to treat only projective spaces over commutative fields: that is Pappian projective spaces. As indicated above, the cross-ratio is at the heart of our treatment, and the cross ratio requires four collinear points. If a field has characteristic 2 (i.e. \( 1 + 1 = 0 \)), there may be only three. Consequently, we limit our attention only to the case in which the field has characteristic other than 2: that is, projective spaces without Fano configuration.
2. Definitions and Examples of Projective Spaces

**Convention 2.1.** Throughout the paper, the symbol := is used to denote equality by definition.

In the sequel ($\mathbb{F}, +, \cdot$) will be a field with multiplicative identity 1 and additive identity 0. We shall convene as well that the characteristic of $\mathbb{F}$ is not 2, thus $1 + 1 \neq 0$ and so $-1$ is distinct from 1. Let $\mathbb{F}_\infty$ be a set containing $\mathbb{F}$ as a subset and having one additional point $\infty$. We define $\infty \cdot u := u \cdot \infty := \infty$ for all $u \in \mathbb{F}\setminus\{0\}$. We further define $0^{-1} := \infty$ and $\infty^{-1} := 0$.

**Definition 2.2.** Following Schaeffer [10, Chapter 4] we say that a *vector space duality* is a triple $(V,W,\langle ,\rangle)$ consisting of two vector spaces $V$ and $W$ over $\mathbb{F}$, and a map $\langle ,\rangle : V \times W \rightarrow \mathbb{F}$ satisfying

(i) $\langle v_1 + w_2, w_2 \rangle = \langle v_1, w_1 \rangle + \langle v_1, w_2 \rangle$ and $\langle v_1 + w_2, w_1 \rangle = \langle v_1, w_1 \rangle + \langle v_2, w_1 \rangle$, for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$;

(ii) $\langle tv, w \rangle = \langle v, tw \rangle$, for all $v \in V$, $w \in W$ and $t \in \mathbb{F}$;

(iii) for all distinct $w_1, w_2 \in W$, there exists $v \in V$ such that $\langle v, w_1 \rangle \neq \langle v, w_2 \rangle$;

(iv) for all distinct $v_1, v_2 \in V$, there exists $w \in W$ such that $\langle v_1, w \rangle \neq \langle v_2, w \rangle$.

For such a duality we define the *quadra-bracket* $[\cdot,\cdot,\cdot]$ by

$$[v_1, v_2, w_1, w_2] := \frac{\langle v_1, w_1 \rangle \langle v_2, w_2 \rangle}{\langle v_1, w_2 \rangle \langle v_2, w_1 \rangle}$$

(\forall v_1, v_2 \in V \setminus \{0\}) (\forall w_1, w_2 \in W \setminus \{0\})

provided that the fraction makes sense (numerator and denominator not both zero, and $\infty$ being a permitted value.)

**Proposition 2.3.** For a vector space duality $(V,W,\langle ,\rangle)$ with quadra-bracket $[\cdot,\cdot,\cdot]$ we have

(i) $[v_1, v_2, w_1, w_2] = [v_2, v_1, w_2, w_1] = [v_2, v_1, w_1, w_2]^{-1} = [v_1, v_2, w_2, w_1]^{-1}$ for all $v_1, v_2 \in V \setminus \{0\}$ and $w_1, w_2 \in W \setminus \{0\}$;

(ii) $[tv_1, v_2, w_1, w_2] = [v_1, tv_2, w_1, w_2] = [v_1, v_2, tw_1, w_2] = [v_1, v_2, w_1, tw_2]$ = $[v_1, v_2, w_1, w_2]$ for all $v_1, v_2 \in V \setminus \{0\}$, $w_1, w_2 \in W \setminus \{0\}$ and $t \in \mathbb{F} \setminus \{0\}$

provided any of the expressions in (i) are defined.

**Proof.** Direct computation.

**Notation 2.4.** Let $V$ and $W$ denote the families of one-dimensional linear subspaces of $V$ and $W$ respectively, $(V,W,\langle ,\rangle)$ being a vector space duality. For nonzero $v \in V$ and $w \in W$ we define $\mathbf{v}$ and $\mathbf{w}$ to be the linear spans of $\{v\}$ and $\{w\}$ respectively. In view of Part (ii) of Proposition 2.3 we may define

$$[v_1, v_2, w_1, w_2] := [v_1, v_2, w_1, w_2]$$

whenever the latter makes sense for $v_1, v_2 \in V \setminus \{0\}$ and $w_1, w_2 \in W \setminus \{0\}$. 

**Definition 2.5.** Let \((V, W, \langle \cdot, \cdot \rangle)\) and \((X, Y, \langle \cdot, \cdot \rangle)\) be vector space dualities. A *vector duality isomorphism* is a pair \((\chi, \eta)\) such that

(i) \(\chi\) is a linear isomorphism from \(V\) onto \(X\);

(ii) \(\eta\) is a linear isomorphism from \(Y\) onto \(W\);

(iii) \(\langle \chi(v), y \rangle = \langle v, \eta(y) \rangle\) for all \(v \in V\) and \(y \in Y\).

The isomorphism \(\eta\) is said to be the *adjoint* of \(\chi\), and \(\chi\) the adjoint of \(\eta\).

**Definition 2.6.** Let \(P\) and \(Q\) be sets and \([\cdot, \cdot]\) be a \(\mathbb{F}_\infty\)-valued function defined on a subset of \(P \times P \times Q \times Q\) for which there exists a vector duality \((V, W, \langle \cdot, \cdot \rangle)\) and bijective maps \(\nu: P \rightarrow V\) and \(\omega: Q \rightarrow W\) such that \([p_1, p_2, q_1, q_2] = [\nu(p_1), \nu(p_2), \omega(q_1), \omega(q_2)]\) for all \(p_1, p_2 \in P\) and \(q_1, q_2 \in Q\). Then \((P, Q, [\cdot, \cdot])\) will be said to be a *projective duality over* \(\mathbb{F}\) and \((\nu, \omega)\) a vector representation of \((P, Q, [\cdot, \cdot])\) on the vector space duality \((V, W, \langle \cdot, \cdot \rangle)\). We say that \(P\) and \(Q\) are projective spaces, \(Q\) being denominated the dual of \(P\) and vice versa.

A pair \((p, q)\) with \(p \in P\) and \(q \in Q\) will be said to be *standard* if \([p, \overline{p}, q, \overline{q}] \neq 0\) for some \(\overline{p} \in P\) and \(\overline{q} \in Q\). A pair \((p, q)\) that is not standard will be said to be *singular*. We shall refer to the function \([\cdot, \cdot]\) as the *cross-ratio* on the projective duality.

**Notation 2.7.** Let \((V, W, \langle \cdot, \cdot \rangle)\) be a vector space duality. For \(w \in W\) and \(v \in V\) we define

\[
  w^\perp := \{\overline{w} \in V : \langle \overline{w}, w \rangle = 0\} \quad \text{and} \quad v^\perp := \{\overline{v} \in W : \langle v, \overline{w} \rangle = 0\}
\]

respectively. If \(v \in w^\perp\) or \(w \in v^\perp\), we say that \(v\) and \(w\) are *orthogonal.*

**Observation 2.8.** Let \((P, Q, [\cdot, \cdot])\) be a projective duality and \((\nu, \omega)\) a vector representation on \((V, W, \langle \cdot, \cdot \rangle)\). Suppose that \((p_1, q_1)\) is a standard pair with \(p_1 \in P\) and \(q_1 \in Q\), and let \(v_1 \in \nu(p_1)\) and \(w_1 \in \omega(q_1)\) be nonzero. Then \(\langle (v_1, v_1)(v_2, w_2) \rangle / \langle (v_1, w_1)(v_2, w_1) \rangle \neq 0\) for some \(v_2 \in \nu(p_2)\) and \(w_2 \in \omega(q_2)\) where \(p_2 \in P\) and \(q_2 \in Q\). In particular, \(\langle v_1, w_1 \rangle \neq 0\).

Conversely, suppose that \(\langle v_1, w_1 \rangle \neq 0\) for nonzero \(v_1 \in \nu(p_1)\) and \(w_1 \in \omega(q_1)\) with \(p_1 \in P\) and \(q_1 \in Q\). Then \(\langle v_1, v_1, w_1, w_1 \rangle = 1\). Thus for a pair \((p_1, q_1)\) to be standard, one or both of the following are equivalent:

(i) there exist \(v_1 \in \nu(p_1)\) and \(w_1 \in \omega(q_1)\) such that \(\langle v_1, w_1 \rangle \neq 0\);

(ii) \(\langle v_1, w_1 \rangle \neq 0\) for all \(v_1 \in \nu(p_1) \setminus \{0\}\) and \(w_1 \in \omega(q_1) \setminus \{0\}\).

**Motivation 2.9.** In referring to the function \([\cdot, \cdot]\) as cross-ratio on the projective duality in Definition 2.6, our motivation is not so much the form of the quadra-bracket, as it is the result of the following theorem, combined with the classical definition of a cross-ratio on a field.
THEOREM 2.10. Let \((P, Q, [\cdot , \cdot])\) be a projective duality. Let \(p_1, p_2 \in P\) and \(q_1, q_2 \in Q\) be distinct and such that \([p_1, p_2, q_1, q_2]\) is defined. Let \((\nu, \omega)\) be a vector representation of \((P, Q, [\cdot , \cdot])\). Let \(v_1 \in \nu(p_1)\) and \(v_2 \in \nu(p_2)\) be nonzero and distinct, and let \(L\) be the line in \(V\) determined by \(v_1\) and \(v_2\). Let \(v_3\) and \(v_4\) denote the intersection of \(L\) with \(\omega(q_1)^{\perp}\) and \(\omega(q_2)^{\perp}\) respectively. Then, if \(\alpha\) is any affine isomorphism of \(L\) onto \(\mathcal{F}\),

\[
[p_1, p_2, q_1, q_2] = \frac{\alpha(v_1) - \alpha(v_3)}{\alpha(v_1) - \alpha(v_4)} \cdot \frac{\alpha(v_2) - \alpha(v_4)}{\alpha(v_2) - \alpha(v_3)}.
\]

Proof. Since \(\alpha\) is affine (cf. Recollection 4.21 infra),

\[
\alpha(tv_1 + (1-t)v_2) = t\alpha(v_1) + (1-t)v_2 \quad (\forall t \in \mathcal{F}).
\]

Since \(v_1, v_2\) and \(v_3\) are collinear, there exists \(u \in \mathcal{F}\) such that

\[
v_3 = uv_1 + (1-u)v_2.
\]

For nonzero \(w_1 \in \omega(q_1)\) we have

\[
0 = \langle v_3, w_1 \rangle = u\langle v_1, w_1 \rangle + (1-u)\langle v_2, w_1 \rangle \Rightarrow u = \langle v_2, w_1 \rangle / \langle v_2 - v_1, w_1 \rangle \Rightarrow \alpha(v_3) = \frac{\langle v_2, w_1 \rangle \alpha(v_1) + \langle v_1, w_1 \rangle \alpha(v_2)}{\langle v_2 - v_1, w_1 \rangle}.
\]

For nonzero \(w_2 \in \omega(q_2)\) we have similarly

\[
\alpha(v_4) = \frac{\langle v_2, w_2 \rangle \alpha(v_1) + \langle v_1, w_2 \rangle \alpha(v_2)}{\langle v_2 - v_1, w_2 \rangle}.
\]

Hence

\[
\begin{align*}
\alpha(v_1) - \alpha(v_2) &= -(\langle v_1, w_1 \rangle (\alpha(v_1) + \alpha(v_2))/\langle v_2 - v_1, w_1 \rangle, \\
\alpha(v_1) - \alpha(v_4) &= -(\langle v_1, w_2 \rangle (\alpha(v_1) + \alpha(v_2))/\langle v_2 - v_1, w_2 \rangle, \\
\alpha(v_2) - \alpha(v_4) &= -(\langle v_2, w_2 \rangle (\alpha(v_1) + \alpha(v_2))/\langle v_2 - v_1, w_2 \rangle) \text{ and} \\
\alpha(v_2) - \alpha(v_3) &= -(\langle v_2, w_1 \rangle (\alpha(v_1) + \alpha(v_2))/\langle v_2 - v_1, w_1 \rangle)
\end{align*}
\]

whence

\[
\frac{\alpha(v_1) - \alpha(v_3)}{\alpha(v_2) - \alpha(v_3)} : \frac{\alpha(v_1) - \alpha(v_4)}{\alpha(v_2) - \alpha(v_4)} = [v_1, v_2, w_1, w_2] = [p_1, p_2, q_1, q_2].
\]

EXAMPLE 2.11. Let \((V, W, \langle \cdot , \cdot \rangle)\) be a vector space duality. Then \((V, W, \langle \cdot , \cdot \rangle)\), with \(\nu\) and \(\omega\) the identity maps, and \([\cdot, \cdot]\) defined as in Notation 2.4, is a projective duality. We call \((V, W, [\cdot, \cdot])\) the projective duality inherent in the vector space duality \((V, W, \langle \cdot , \cdot \rangle)\).

EXAMPLE 2.12. Let \(V\) be a finite dimensional vector space over \(\mathcal{F}\). The set \(V^*\) of linear functionals on \(V\) is a vector space and \((V, V^*, \langle \cdot, \cdot \rangle)\) is a vector space duality where

\[
\langle v, f \rangle := f(v) \quad (\forall v \in V, f \in V^*).
\]

Every other vector space duality, of which the first coordinate space is \(V\), is isomorphic to \((V, V^*, \langle \cdot, \cdot \rangle)\). Consequently, there incurs no liability in defining the projective duality inherent in \(V\) to be the projective duality inherent in the vector space duality \((V, V^*, \langle \cdot, \cdot \rangle)\). In this case \(V\) will be said to be the projective space inherent in \(V\).
Example 2.13. Let $U$ be a finite dimensional vector space over $\mathbb{F}$. Let $V := U \oplus \mathbb{F}$. For each $v \in V \setminus (U \oplus 0)$ there exists exactly one $u \in U$ such that $u \oplus 1 \in v$, in which case $v$ will be said to be finite. The elements of $U \oplus 0$ are said to be infinite. Thus the finite elements of the projective space inherent in $V$ are in bijective correspondence with the nonzero elements of $U$. We say that $V$ is the standard projective space generated by the vector space $U$.

Example 2.14. Let $\mathbb{F}^{n+1}$ be the vector space of ordered $(n + 1)$-tuples of scalars, $n$ being a fixed positive integer. In each element of the projective space $\mathbb{P}^{n+1}$ inherent in $\mathbb{F}^{n+1}$ there is exactly one $(n + 1)$-tuple of which the last nonzero coordinate is 1. Let $\mathbb{P}(\mathbb{F}; n)$ denote the set of all elements of $\mathbb{F}^{n+1}$ of which the last nonzero coordinate equals 1. We shall call $\mathbb{P}(\mathbb{F}; n)$ projective $n$-space over the field $\mathbb{F}$. This example is a special case of Example 2.13. Following that example, we say that an element of $\mathbb{P}(\mathbb{F}; n)$ of which the $(n + 1)$-st coordinate is 1 is finite and that the other elements are infinite.

Remark 2.15. A fundamental tool for exploiting the duality of projective spaces is the polar of a set. We define this concept and establish that singletons are polars.

Definition 2.16. Let $(P, Q, [\ldots])$ be a projective duality. For $p \in P$ and $q \in Q$ we let $p^o := \{\overline{q} \in Q : (p, \overline{q}) \text{ is singular} \}$ and $q^o := \{\overline{p} \in P : (\overline{p}, q) \text{ is singular} \}$. For $A \subset P$ and $B \subset Q$ we let $A^o := \{\overline{q} \in Q : (\overline{p}, \overline{q}) \text{ is singular} \ (\forall \overline{p} \in A) \}$ and $B^o := \{\overline{p} \in P : (\overline{p}, \overline{q}) \text{ is singular} \ (\forall \overline{q} \in B) \}$. Sets of the form $A^o$ and $B^o$ are called polars.

Theorem 2.17. Let $p$ and $q$ be as in Definition 2.16. Then $\{p\} = (p^o)^o$ and $\{q\} = (q^o)^o$.

Proof. We prove the first equality only—the proof of the second is entirely analogous. That $\{p\} \subset (p^o)^o$ is trivial. Let $\overline{p} \in (p^o)^o$ be generic and assume $\overline{p} \neq p$. Let $(\nu, \omega)$ be a vector representation of $(P, Q, [\ldots])$ and select $v \in \nu(p)$ and $\overline{\nu} \in \nu(\overline{p})$ both nonzero. Since $v$ and $\overline{\nu}$ are linearly independent, there exists $w_1 \in W$ such that

$$0 \neq \langle v, w_1 \rangle \neq \langle \overline{\nu}, w_1 \rangle \neq 0.$$ 

Similarly, since $(\overline{\nu}, w_1)\nu/\langle v, w_1 \rangle$ and $\overline{\nu}$ are linearly independent, there exists $w_2 \in W$ such that

$$0 \neq \langle (\overline{\nu}, w_1)\nu/\langle v, w_1 \rangle, w_2 \rangle \neq \langle \overline{\nu}, w_2 \rangle \neq 0.$$ 

Let $w_3 := \langle v, w_1 \rangle w_2/\langle v, w_2 \rangle$. Thus

$$0 \neq \langle (\overline{\nu}, w_1)\nu/\langle v, w_1 \rangle, w_3 \rangle = \langle \overline{\nu}, w_1 \rangle \neq \langle (\overline{\nu}, w_2)\langle v, w_1 \rangle/\langle v, w_2 \rangle = \langle \overline{\nu}, w_3 \rangle \neq 0$$

and

$$\langle v, w_3 \rangle = \langle v, w_1 \rangle.$$
If \( \langle \overline{v}, w_3 \rangle = \langle \overline{v}, w_1 \rangle \), then from the above follows \( \langle \langle \overline{v}, w_3 \rangle v, \langle v, w_3 \rangle \rangle \neq \langle \overline{v}, w_3 \rangle \) which is absurd. Thus
\[
\langle \overline{v}, w_3 \rangle \neq \langle \overline{v}, w_1 \rangle.
\]
In particular, \( w_3 - w_1 \) is nonzero and so there exists \( \overline{v} \in Q \) with \( w_3 - w_1 \in \omega(\overline{v}) \). Since \( \langle v, w_3 \rangle = \langle v, w_1 \rangle \), it follows that \( \langle v, w_3 - w_1 \rangle = 0 \) and so \( \overline{v} \in p^\circ \). Since \( \langle \overline{v}, w_3 \rangle \neq \langle \overline{v}, w_1 \rangle \) implies \( \langle \overline{v}, w_3 - w_1 \rangle \neq 0 \), it follows that \( \overline{v} \notin \overline{v}' \). The two statements \( \overline{v} \in p^\circ \) and \( \overline{v} \notin \overline{v}' \) however, contradicts \( \overline{v} \in (p^\circ)^\circ \).

**Direction 2.18.** A projective duality is defined by any of its vector representations. It is important to exhibit the relationship between any two such vector representations. We shall do this in Theorem 2.22 *infra*, but first we define some simple notions which will be of use there.

**Definition 2.19.** Let \((P, Q, [\ldots])\) be a projective duality, and \(p_1, p_2\) and \(p_3\) three distinct points in \(P\). The points will be said to be *triangular* if there exists \(q \in Q\) such that exactly one of the pairs \((p_1, q)\), \((p_2, q)\) and \((p_3, q)\) is standard. If the points are not triangular, they are said to be *collinear*. The definitions of triangular and collinear points in \(Q\) are obtained *mutatis mutandis*.

**Theorem 2.20.** Let \((\nu, \omega)\) be a vector representation of a projective duality \((P, Q, [\ldots])\). Let \(p_1, p_2\) and \(p_3\) be three distinct points of \(P\). Then the points are triangular if and only if the subspace generated by \(\nu(p_1) \cup \nu(p_2) \cup \nu(p_3)\) is three dimensional.

**Proof.** Suppose first that the points are triangular. By interchanging symbols if necessary, we may find \(q \in Q\) such that \((p_1, q)\) is standard and both \((p_2, q)\) and \((p_3, q)\) are singular. Then there exist \(v_1 \in \nu(p_1)\), \(v_2 \in \nu(p_2)\) and \(v_3 \in \nu(p_3)\), and \(w \in \omega(q)\) all nonzero such that
\[
\langle v_1, w \rangle = 1 \quad \text{and} \quad \langle v_2, w \rangle = \langle v_3, w \rangle = 0.
\]
Since \(p_2\) and \(p_3\) are distinct, \(v_2\) and \(v_3\) generate a two dimensional subspace of \(V\). It follows from the above that \(\{v_1, v_2, v_3\}\) is linearly independent.

Suppose now that the subspace \(H\) generated by \(\nu(p_1) \cup \nu(p_2) \cup \nu(p_3)\) is three dimensional. Then a basis for \(H\) can be chosen consisting of \(v_1 \in \nu(p_1)\), \(v_2 \in \nu(p_2)\) and \(v_3 \in \nu(p_3)\). Consequently, there exists \(w \in W\) such that (1) is true. Choose \(q \in Q\) such that \(w \in \omega(q)\). Evidently, \((p_1, q)\) is standard and \((p_2, q)\) and \((p_3, q)\) are singular. This means that \(p_1, p_2\) and \(p_3\) are triangular.

**Remark 2.21.** In [12] Tits presents a geometric treatment of finite dimensional projective spaces from the “dual pairs” point of view, based on *buildings*. The interested reader might also see the paper of M"uhlherr [9], which treats buildings of infinite rank (but not infinite dimensional projective spaces per se).

**Theorem 2.22.** Let \((\overline{\nu}, \overline{\omega})\) be a standard pair of a projective duality \((P, Q, [\ldots])\). Let \((\nu, \omega)\) and \((\alpha, \beta)\) be vector representations of \((P, Q, [\ldots])\) onto the vector dualities \((V, W, (\cdot, \cdot))\) and \((X, Y, (\cdot, \cdot))\) respectively. Let \(\overline{\nu} \in \nu(\overline{\nu}), \overline{\nu} \in \alpha(\overline{\nu}), \overline{\omega} \in \omega(\overline{\omega})\) and \(\overline{\omega} \in \beta(\overline{\omega})\) be such that
\[
\langle \overline{\nu}, \overline{\omega} \rangle = 1 = \langle \overline{\nu}, \overline{\omega} \rangle.
\]
Then there exists exactly one vector duality isomorphism \((\chi, \eta)\) of \((V, W, \langle \cdot, \cdot \rangle)\) to \((X, Y, \langle \cdot, \cdot \rangle)\) such that

(i) \(\chi(\overline{v}) = \overline{x}\) and \(\eta(\overline{y}) = \overline{w}\);

(ii) \(\chi(v(p)) = \alpha(p) \ (\forall p \in P)\) and \(\eta(\beta(q)) = \omega(q) \ (\forall q \in Q)\).

Proof. We first establish uniqueness. Suppose that \((\chi_1, \eta_1)\) and \((\chi_2, \eta_2)\) are vector duality isomorphisms satisfying (i) and (ii). Then \(\varphi := \chi_1^{-1} \circ \chi_2\) is a linear isomorphism of \(V\) leaving \(\overline{v}\) fixed and sending each one dimensional subspace onto itself. Such an isomorphism has too many eigenvectors to be anything other than a scalar multiple of the identity. Since it leaves \(\overline{v}\) fixed, it must be the identity. Hence \(\chi_1 = \chi_2\). That \(\eta_1 = \eta_2\) follows analogously.

For each \(\overline{X} \in \alpha(P \setminus q^2)\) denote by \(\rho(\overline{X})\) the element of \(X\) such that

\[
\langle \rho(\overline{X}), \overline{y} \rangle = 1 \quad \text{and} \quad \rho(\overline{X}) \in \overline{X}.
\]

Consider a generic \(v \in V\). Then \(\langle v + (1 - \langle v, \overline{w} \rangle)\overline{v}, \overline{w} \rangle = 1\) and so \(\nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v})\) is in \(P \setminus \overline{v}\) and we may define

\[
\chi(v) := (\langle v, \overline{w} \rangle - 1)\overline{x} + \rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v}).
\]

Evidently

\[
\chi(\overline{tv}) = t\chi(v) \quad (\forall t \in F \setminus \{0\}).
\]

We presently show that \(\chi\) is homogeneous relative to scalar multiplication. Let \(v \in V\) and \(t \in F\) both be nonzero. If \(v = t\overline{v}\) for some \(t \in F\), then

\[
\chi(tv) = \chi(t\overline{v}) = (t\overline{x} - 1)\overline{x} + \rho \circ \alpha \circ \nu^{-1}(t\overline{v} + (1 - t\overline{v})\overline{v}) = t\overline{x} = t\chi(v)
\]

and so we need only consider the case in which \(v\) and \(\overline{v}\) are linearly independent. The points \(\overline{v}, v + (1 - \langle v, \overline{w} \rangle)\overline{v}\) and \(tv + (1 - \langle tv, \overline{w} \rangle)\overline{v}\) are collinear (since \((1 - t)\overline{v} + t(v + (1 - \langle v, \overline{w} \rangle)\overline{v}) = tv + (1 - \langle tv, \overline{w} \rangle)\overline{v}\)). From Theorem 2.20 follows that \(\nu^{-1}(\overline{v}), \nu^{-1}(tv(1 - \langle tv, \overline{w} \rangle)\overline{v})\) and \(\nu^{-1}(tv + (1 - \langle tv, \overline{w} \rangle)\overline{v})\) are collinear in \(P\). Hence \(\alpha \circ \nu^{-1}(\overline{v}), \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v})\) and \(\alpha \circ \nu^{-1}(tv + (1 - \langle tv, \overline{w} \rangle)\overline{v})\) generate a two dimensional subspace of \(X\). The points \(\rho \circ \alpha \circ \nu^{-1}(\overline{v}), \rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v})\) and \(\rho \circ \alpha \circ \nu^{-1}(tv + (1 - \langle tv, \overline{w} \rangle)\overline{v})\) also generate a two-dimensional subspace and, being in the hyperplane \(\overline{v}\) as well, they must be collinear in \(A\). Thus there exists \(u \in F\) such that

\[
\rho \circ \alpha \circ \nu^{-1}(tv + (1 - \langle tv, \overline{w} \rangle)\overline{v}) = u \rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v}) + (1 - u)\overline{v}
\]

Since \(v\) and \(\overline{v}\) are linearly independent, there exists \(w \in W\) such that

\[
\langle \overline{v}, w \rangle = 0 \quad \text{and} \quad \langle v + (1 - \langle v, \overline{w} \rangle)\overline{v}, w \rangle = 1.
\]

It follows that \((\nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v}), \omega^{-1}(w))\) is a standard pair and so \(\alpha(\nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v})), \omega^{-1}(w))\) are not orthogonal, whence follows that \(\rho(\alpha(\nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v}))), \beta(\omega^{-1}(w))\) are not orthogonal. Thus there exists \(y \in \beta(\omega^{-1}(w))\) such that

\[
\langle \rho(\alpha(\nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{v}))), y \rangle = 1.
\]
By definition of $\rho$, we have
\[
\langle \rho(\alpha(\nu^{-1}(v + (1 - \langle v, \overline{w} \rangle))\overline{w})) \rangle_{\overline{y}} = 1.
\]
Since $\langle \overline{v}, \overline{w} \rangle = 0$, it follows that $\langle \nu^{-1}(\overline{w}), \omega^{-1}(w) \rangle$ is singular, whence follows
\[
\langle \overline{v}, y \rangle = 0.
\]
Direct calculation with the above equalities yields
\[
t = [v + (1 - \langle v, \overline{w} \rangle)\overline{w}, (1 - t)\overline{w} + t(v + (1 - \langle v, \overline{w} \rangle)\overline{w}, w]
\]
\[
= [v + (1 - \langle v, \overline{w} \rangle)\overline{w}, tv + (1 - t\langle v, \overline{w} \rangle)\overline{w}, w]
\]
\[
= [\alpha \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{w}), (1 - tv + (1 - (tv, \overline{w}))\overline{w}), \beta \omega^{-1}((\overline{w}), \beta \omega^{-1}(w)]
\]
\[
= [\rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle))\overline{w}, \rho \circ \alpha \circ \nu^{-1}(tv + (1 - (tv, \overline{w}))\overline{w}, \overline{y}, y]
\]
\[
= [\rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle))\overline{w}, \rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle))\overline{w}) + (1 - u)\overline{w}, \overline{y}, y]
\]
\[
u = u
\]
where the third equality follows from Part (ii) of Proposition 2.3. Consequently,
\[
\chi(tv) = (tv, \overline{w}) - 1)\overline{w} + \rho \circ \alpha \circ \nu^{-1}(tv + (1 - \langle v, \overline{w} \rangle)\overline{w})
\]
\[
= ((tv, \overline{w}) - 1)\overline{w} + up \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{w}) + (1 - u)\overline{w}
\]
\[
= ((tv, \overline{w}) - 1)\overline{w} + t\rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{w}) + (1 - t)\overline{w}
\]
\[
= t(v, \overline{w}) - 1)\overline{w} + t\rho \circ \alpha \circ \nu^{-1}(v + (1 - \langle v, \overline{w} \rangle)\overline{w})
\]
\[
= tv(v).
\]
Having established the homogeneity of $\chi$, we now show that $\chi$ preserves midpoints. Let $v_1$ and $v_2$ be distinct nonzero points of $V$. Let $m$ be the midpoint of $v_1$ and $v_2$, and let $n$ be the midpoint of $\chi(v_1)$ and $\chi(v_2)$. We begin with the case: $m \in \Sigma$. First note that from $m \in \Sigma$ follows that $m + (1 - \langle m, \overline{w} \rangle)\overline{w} = \overline{w}$ and therefore that $\rho \circ \alpha \circ \nu^{-1}(m + (1 - (m, \overline{w}))\overline{w}) = \overline{w}$.

Assume that $\overline{w}$ is not the midpoint $z$ of $\rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w})$ and $\rho \circ \alpha \circ \nu^{-1}(v_2 + (1 - \langle v_2, \overline{w} \rangle)\overline{w})$. Since $\rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w}), \overline{y} = 1 = \langle \rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w}), \overline{y} \rangle$, it follows that $\langle z, \overline{y} \rangle = 1 = \langle \overline{w}, \overline{y} \rangle$, which implies that $z$ and $\overline{w}$ are linearly independent. Thus there exists $y \in Y$ such that $\langle \overline{w}, y \rangle = 0$ and $\langle z, y \rangle \neq 0$. Hence
\[
\langle \rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle))\overline{w}, y \rangle \neq \langle \rho \circ \alpha \circ \nu^{-1}(v_2 + (1 - \langle v_2, \overline{w} \rangle))\overline{w}, y \rangle.
\]
Let $\overline{g} \in \omega(\beta^{-1}(y))$ be nonzero. We have $\langle \overline{w}, \overline{g} \rangle = 0$ and so $\langle m, \overline{g} \rangle = 0$. This implies
\[
\langle v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w}, \overline{g} \rangle = -\langle v_2 + (1 - \langle v_2, \overline{w} \rangle)\overline{w}, \overline{g} \rangle.
\]
We have
\[
-1 = \langle v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w}, \overline{g} \rangle / \langle v_2 + (1 - \langle v_2, \overline{w} \rangle)\overline{w}, \overline{g} \rangle
\]
\[
= \langle v_2 + (1 - \langle v_2, \overline{w} \rangle)\overline{w}, \overline{g} \rangle / \langle v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w}, \overline{g} \rangle
\]
\[
= ((v_2 + (1 - \langle v_2, \overline{w} \rangle)\overline{w}, \overline{g} \rangle / \langle v_1 + (1 - \langle v_1, \overline{w} \rangle)\overline{w}, \overline{g} \rangle)
\]
which is absurd. It follows that \( \overline{x} \) is the midpoint of \( \rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle) \overline{w}) \) and \( \rho \circ \alpha \circ \nu^{-1}(v_2 + (1 - \langle v_2, \overline{w} \rangle) \overline{w}) \). Hence

\[
(\chi(v_1) + \chi(v_2))/2 = ((v_1, \overline{w}) - 1)\overline{x} + \rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle) \overline{w}) +
(\langle v_2, \overline{w} \rangle - 1)\overline{x} + \rho \circ \alpha \circ \nu^{-1}(v_2 + (1 - \langle v_2, \overline{w} \rangle) \overline{w})/2
\]

\[= ((v_1 + v_2)/2, \overline{w}) - 1)\overline{x} + \rho \circ \alpha \circ \nu^{-1}(v_2 + (1 - \langle v_2, \overline{w} \rangle) \overline{w})/2 =
\chi((v_1 + v_2)/2).
\]

We now turn to the case: \( m \not\in \overline{x} \) and define

\[\psi(v) := v + (1 - \langle v, \overline{w} \rangle) \overline{w} \quad (\forall v \in V).\]

Evidently \( \psi(v) \) is in \( \overline{x} \) only if \( v \) is—in particular, \( \psi(m) \) is not in \( \overline{x} \) and so there exists \( g \in W \) such that

\[\langle \psi(m), g \rangle = 0 \quad \text{and} \quad \langle \overline{x}, g \rangle = 1.\]

We note that \( \langle \psi(v_1), g \rangle = -\langle \psi(v_2), g \rangle \). The pair \( \nu^{-1}(\psi(m)), \omega^{-1}(g) \) is singular and the pair \( \overline{x}, \omega^{-1}(g) \) is standard, whence follows that there exists \( f \in \alpha \circ \omega^{-1}(g) \) such that

\[\langle \rho \circ \alpha \circ \nu^{-1}(\psi(m)), f \rangle = 0 \quad \text{and} \quad \langle \overline{x}, f \rangle = 1.\]

Since \( v_1, m \) and \( v_2 \) are collinear, so are \( \psi(v_1), \psi(m) \) and \( \psi(v_2) \), which implies that \( \nu^{-1}(\psi(v_1)), \nu^{-1}(\psi(m)) \) and \( \nu^{-1}(\psi(v_2)) \) are collinear in \( P \). Hence \( \rho \circ \alpha \circ \nu^{-1}(\psi(v_1)) \cup \rho \circ \alpha \circ \nu^{-1}(\psi(m)) \cup \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)) \) spans a two dimensional subspace of \( X \)—the intersection of this subspace with the set \( \{ z \in X : \langle z, \overline{y} \rangle = 1 \} \) is thus a line. It follows that \( \rho \circ \alpha \circ \nu^{-1}(\psi(v_1)), \rho \circ \alpha \circ \nu^{-1}(\psi(m)) \) and \( \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)) \) are on this line and so there exists a unique scalar \( t \) such that

\[\rho \circ \alpha \circ \nu^{-1}(\psi(m)) = t \rho \circ \alpha \circ \nu^{-1}(\psi(v_1)) + (1 - t) \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)).\]

Note that \( \psi(m) \) is the midpoint of \( \psi(v_1) \) and \( \psi(v_2) \). That \( \langle \psi(m), g \rangle = 0 \) implies \( \langle \psi(v_1), g \rangle = -\langle \psi(v_2), g \rangle \). Since \( \langle \psi(v_1), \overline{w} \rangle = (1 - \langle v_1, \overline{w} \rangle) \overline{w} = 1, \) it follows that

\[\langle \psi(v_1), g \rangle = \langle \overline{x}, \psi(v_1), \overline{w}, g \rangle = [\overline{x}, \nu^{-1}(\psi(v_1)), \overline{y}, \omega^{-1}(g)] = [\overline{x}, \alpha \circ \nu^{-1}(\psi(v_1)), \overline{y}, f] = \langle \alpha \circ \nu^{-1}(\psi(v_1)), f \rangle.\]

Analogous reasoning shows that

\[\langle \psi(v_2), g \rangle = \langle \overline{x}, \psi(v_2), \overline{w}, g \rangle = [\overline{x}, \nu^{-1}(\psi(v_1)), \overline{y}, \omega^{-1}(g)] = [\overline{x}, \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)), \overline{y}, f] = \langle \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)), f \rangle.\]
Consequently,
\[ \langle \rho \circ \alpha \circ \nu^{-1}(\psi(v_1)), f \rangle = \langle \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)), f \rangle. \]

Observing that \( \langle \psi(m), v_2 \rangle = 1 \) and recalling that \( \langle \psi(m), f \rangle = 0 \), we see that
\[ 0 = [\mathcal{F}, \psi(m), \overline{w}, g] = [\mathcal{F}, \rho \circ \alpha \circ \nu^{-1} \circ \psi(m), \overline{y}, f] = \langle \rho \circ \alpha \circ \nu^{-1}(\psi(m)), f \rangle. \]

Thus \( \langle \rho \circ \alpha \circ \nu^{-1}(\psi(m)), f \rangle = 0 \). Hence
\[
0 = \langle tp \circ \alpha \circ \nu^{-1}(\psi(v_1)) + (1-t) \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)), f \rangle \\
= t[\mathcal{F}, \rho \circ \alpha \circ \nu^{-1}(\varphi(v_1)), \overline{y}, f] + (1-t)[\mathcal{F}, \rho \circ \alpha \circ \nu^{-1}(\varphi(v_2)), \overline{y}, f] \\
= t[\mathcal{F}, \psi(v_1), \overline{w}, g] + (1-t)[\mathcal{F}, \psi(v_2), \overline{w}, g] \\
= t\langle \psi(v_1), g \rangle + (1-t)\langle \psi(v_2), g \rangle \\
= t\langle \psi(v_1), g \rangle - (1-t)\langle \psi(v_1), g \rangle \\
= (2t-1)\langle \psi(v_1), g \rangle \\
\]
whence follows that \( t = 1/2 \). Thus \( \rho \circ \alpha \circ \nu^{-1}(\psi(m)) \) is the midpoint of \( \rho \circ \alpha \circ \nu^{-1}(\psi(v_1)) \) and \( \rho \circ \alpha \circ \nu^{-1}(\psi(v_2)) \). We have
\[
\chi(v_1) + \chi(v_2) = ((v_1, \overline{w}) - 1)\mathcal{F} + \rho \circ \alpha \circ \nu^{-1}(v_1 + (1 - \langle v_1, \overline{w} \rangle)\mathcal{F}) \\
+ ((v_2, \overline{w}) - 1)\mathcal{F} + \rho \circ \alpha \circ \nu^{-1}(v_2 + (1 - \langle v_2, \overline{w} \rangle)\mathcal{F}) \\
= 2((v_1 + v_2)/2, \overline{w}) - 1)\mathcal{F} + 2\rho \circ \nu^{-1}(\psi(m)) \\
= 2\chi(m). \]

We have now determined that \( \chi \) preserves midpoints as well as being homogeneous. For \( v_1, v_2 \in V \) we have
\[
\chi(v_1 + v_2) = 2\chi((v_1 + v_2)/2) = 2((\chi(v_1) + \chi(v_2))/2) = \chi(v_1) + \chi(v_2) \\
\]
which establishes the linearity of \( \chi \). That \( \chi \) is bijective is evident—hence \( \chi \) is a linear isomorphism of \( V \) onto \( X \).

We may construct a linear isomorphism \( \eta \) from \( Y \) to \( W \) in a way analogous to the construction of \( \chi \). That (i) and (ii) hold follows from the definitions.

It remains to show that \( \chi \) and \( \eta \) are adjoints of one another. We define the functions \( \sigma \) and \( \lambda \) analogous to \( \rho \) and \( \psi \) respectively. Let \( \nu \in V \) and \( w \in W \) be nonzero. Note from the definition of \( \chi \) that \( \langle \chi(v), \overline{y} \rangle = \langle v, \overline{w} \rangle \) and that by analogy \( \langle \mathcal{F}, \eta(y) \rangle = \langle \mathcal{F}, y \rangle \). Let \( p \in P \) and \( q \in Q \) be generic. Let \( v \in \nu(p) \) and \( y \in \beta(q) \) be nonzero. Then
\[
\langle \chi(v), y \rangle/\langle \mathcal{F}, y \rangle = \langle \mathcal{F}, \overline{y} \rangle \langle \chi(v), y \rangle/\langle \mathcal{F}, y \rangle \langle \chi(v), \overline{y} \rangle \\
= [\mathcal{F}, \chi(v), \overline{y}, y] \\
= [\mathcal{F}, p, \overline{y}, q] \\
= [\mathcal{F}, v, \overline{w}, \eta(y)] \\
= \langle \mathcal{F}, \overline{w} \rangle \langle v, \eta(y) \rangle/\langle \mathcal{F}, \eta(y) \rangle \langle v, \overline{w} \rangle \\
= \langle v, \eta(y) \rangle/\langle \mathcal{F}, y \rangle \langle v, \overline{w} \rangle \\
\]
whence follows that \( \langle \chi(v), y \rangle = \langle v, \eta(y) \rangle \).
3. Projective Isomorphisms

Definition 3.1. Let \((P, Q, [], [])\) and \((R, S, [], [])\) be projective dualities. Let \(\varphi\) be a bijection from \(P\) to \(R\) and let \(\psi\) be a bijection from \(S\) to \(Q\). Suppose that 
\[
[p_1, p_2, \psi(s_1), \psi(s_2)] = [\varphi(p_1), \varphi(p_2), s_1, s_2]
\]
whenever either of the expressions is well-defined. Then \((\varphi, \psi)\) is said to be a projective duality isomorphism and \(\varphi\) and \(\psi\) are said to be projective isomorphisms.

Theorem 3.2. Let \((P, Q, [], [])\) and \((R, S, [], [])\) be projective dualities and let \((\varphi, \psi)\) be a projective duality isomorphism. Let \((\nu, \omega)\) be a vector representation of \((P, Q, [], [])\) on the vector duality \((V, W, [], [])\) and let \((\xi, \zeta)\) be a vector representation of \((R, S, [], [])\) on a vector duality \((X, Y, [], [])\). Let \((\overline{v}, \overline{w})\) be a standard pair in \((P, Q, [], [])\). Let \(\overline{v} \in \nu(\overline{v}), \overline{w} \in \omega(\overline{w}), \overline{x} \in \xi \circ \varphi(\overline{v})\) and \(\overline{y} \in \zeta \circ \psi^{-1}(\overline{w})\) satisfy 
\[
\langle \overline{v}, \overline{w} \rangle = 1 = \langle \overline{x}, \overline{y} \rangle.
\]
Then there exists a unique vector duality isomorphism \((\chi, \eta)\) from \((V, W, [], [])\) to \((X, Y, [], [])\) such that

(i) \(\chi(\overline{v}) = \overline{x}\) and \(\eta(\overline{y}) = \overline{w}\);

(ii) \(\chi(\nu(p)) = \xi(\varphi(p))\) and \(\eta(\zeta(s)) = \omega(\psi(s))\) \((\forall p \in P, s \in S)\).

Proof. Apply Theorem 2.22 to the pair of vector representations \((\nu, \omega)\) and \((\xi \circ \varphi, \zeta \circ \psi^{-1})\) of \((P, Q, [], [])\).

Theorem 3.3. Let \((P, Q, [], [])\) and \((R, S, [], [])\) be projective dualities. Let \((\nu, \omega)\) be a vector representation of \((P, Q, [], [])\) on the vector duality \((V, W, [], [])\) and let \((\xi, \zeta)\) be a vector representation of \((R, S, [], [])\) on a vector duality \((X, Y, [], [])\). Let \((\chi, \eta)\) be a duality isomorphism from \((V, W, [], [])\) to \((X, Y, [], [])\). Define \(\varphi : P \to R\) and \(\psi : S \to Q\) by 
\[
\varphi(p) := \xi^{-1}(\chi(\nu(p)))\quad\text{and}\quad\psi(s) := \omega^{-1}(\eta(\zeta(s)))\quad(\forall p \in P, s \in S).
\]
Then \((\varphi, \psi)\) is a projective duality isomorphism.

Proof. We establish the condition in Definition 3.1 for \((\varphi, \psi)\) to be a projective duality isomorphism. Let \(p_1, p_2 \in P\) and \(s_1, s_2 \in S\) be generic. Let \(v_1 \in \nu(p_1), v_2 \in \nu(p_2), y_1 \in \zeta(s_1)\) and \(y_2 \in \zeta(s_2)\) be nonzero. Then, if the cross-ratio exists,
\[
[\varphi(p_1), \varphi(p_2), s_1, s_2] = [\xi(\varphi(p_1)), \xi(\varphi(p_2)), \zeta(s_1), \zeta(s_2)]
\]
\[
= [\chi(v_1), \chi(v_2), y_1, y_2]
\]
\[
= \langle \chi(v_1), y_1 \rangle \langle \chi(v_2), y_2 \rangle / \langle \chi(v_1), y_2 \rangle \langle \chi(v_2), y_1 \rangle
\]
\[
= \langle v_1, \eta(y_1) \rangle \langle v_2, \eta(y_2) \rangle / \langle v_1, \eta(y_2) \rangle \langle v_2, \eta(y_1) \rangle
\]
\[
= [v_1, v_2, \eta(y_1), \eta(y_2)].
\]

Theorem 3.4. Let \((P, Q, [], [])\) and \((R, S, [], [])\) be projective dualities. Let \((\varphi, \psi)\) and \((\varphi, \tau)\) be projective duality isomorphisms from \((P, Q, [], [])\) to \((R, S, [], [])\). Then \(\psi = \tau\).
Proof. Assume that \( \psi(s_1) \neq \tau(s_1) \) for some \( s_1 \in S \). Let \( q_1 := \psi(s_1) \), \( \sigma := \tau \circ \psi^{-1} \) and denote the identity map on \( P \) by \( \iota \). Then \((\iota, \sigma)\) is a projective duality isomorphism from \((P, Q, \{\ldots\})\) to itself.

Consider the case \( \sigma \circ \sigma(q_1) = q_1 \). We have
\[
[p_1, p_2, q_1, \sigma(q_1)] = [p_1, p_2, \sigma(q_1), \sigma \circ \sigma(q_1)] = [p_1, p_2, \sigma(q_1), q_1] = 1/[p_1, p_2, q_1, \sigma(q_1)]
\]
for all \( p_1 \) and \( p_2 \) for which the expressions make sense. This is manifestly impossible, whence follows that the three elements \( q_1, \sigma(q_1) \) and \( \sigma \circ \sigma(q_1) \) are distinct. Let \( (\nu, \omega) \) be a vector representation of \((P, Q, \{\ldots\})\) and apply Theorem 2.22 to obtain a vector duality isomorphism \((\chi, \eta)\) such that
\[
\chi(\nu(p)) = \alpha(p) \quad (\forall p \in P) \quad \text{and} \quad \eta(\beta(q)) = \omega(q) \quad (\forall q \in Q).
\]
Choose \( w_1 \in \omega(q_1) \), \( w_2 \in \omega(\sigma(q_1)) \) and \( w_3 \in \omega(\sigma \circ \sigma(q_1)) \) all nonzero. Select \( v_1 \in V \) such that \( \langle v_1, w_1 \rangle = 0 \) but \( \langle v_2, w_2 \rangle \) and \( \langle v_1, w_3 \rangle \) are all nonzero. Select \( v_2 \in V \) such that \( \langle v_2, w_1 \rangle = 0 \) but \( \langle v_2, w_2 \rangle \) and \( \langle v_2, w_3 \rangle \) are all nonzero. We have
\[
0 = [v_2, v_1, w_1, w_2] = [\nu^{-1}(w_2), \nu^{-1}(w_1), q_1, \sigma(q_1)] = [\nu^{-1}(w_2), \nu^{-1}(w_1), \sigma(q_1), \sigma \circ \sigma(q_1)] = [v_2, v_1, w_2, w_3] \neq 0.
\]
It follows that the assumption \( \psi(s_1) \neq \tau(s_1) \) was absurd.

**Definition 3.5.** From Theorem 3.4 follows that a projective isomorphism \( \varphi \) can appear in only one projective duality isomorphism \((\varphi, \psi)\). We say that \( \psi \) is the *adjoint to \( \varphi \) and \( \varphi \) is the adjoint to \( \psi \).*

**Definition 3.6.** Let \((P, Q, \{\ldots\})\) be a projective duality. A subset \( A \) of \( P \) is said to be a *subspace of \( P \)* if, whenever three points are collinear and two of them are in \( A \), then the third is also in \( A \). Polars of sets (cf. Definition 2.16 *supra*) are evidently subspaces. A subspace which is a polar is said to be a *dual subspace.* A polar \( q^\circ \) of a point \( q \in Q \) is a *dual maximal subspace.*

**Discussion 3.7.** Let \((P, Q, \{\ldots\})\) be a projective duality. It follows from Theorem 2.17 that the correspondence between \( Q \) and the dual maximal subspaces of \( P \) is bijective. This leads to the possibility of replacing \( Q \) in the definition of projective space by the set of dual maximal subspaces of \( P \). There are however, two objections. The first is esthetic. There is no *a priori* reason to place more emphasis on \( P \) than on \( Q \), or to treat points any differently than dual maximal subspaces.

The second, more practical, has to do with the difficulty of specifying exactly what subspaces in \( P \) are dual maximal subspaces. There are projective dualities which arise in applications, such as those related to infinite dimensional Banach spaces, for which the dual subspaces may be difficult to recognize. There may be maximal proper subspaces (even maximal proper subspaces closed relative to a natural topology) which are not dual maximal subspaces. For finite dimensional spaces, and for certain projective spaces associated with Hilbert Space, the situation is more clear though, as will be seen in Theorem 3.15 and Example 6.16 *infra.*
**Definition 3.8.** Let \((P, Q, [\cdot, \cdot])\) be a projective duality. A line in \(P\) is a minimal subspace of \(P\) containing more than one point.

**Theorem 3.9.** Let \((P, Q, [\cdot, \cdot])\) be a projective duality. Let \(p_1\) and \(p_2\) be distinct elements of \(P\). Then there exists a unique line \(L\) in \(P\) containing \(p_1\) and \(p_2\). Furthermore

\[
L = \{p \in P : p, p_1 \text{ and } p_2 \text{ are collinear}\}.
\]

**Proof.** Let \(L\) be the intersection of all subspaces of \(P\) containing \(p_1\) and \(p_2\). Let \((\nu, \omega)\) be a standard representation of \((P, Q, [\cdot, \cdot])\). The subspace \(H\) of \(V\) generated by \(\nu(p_1) \cup \nu(p_2)\) is evidently two dimensional and a subset of each subspace \(\nu(M)\), \(M\) a subspace of \(P\) containing \(p_1\) and \(p_2\). Thus \(\nu^{-1}(H)\) is the only line in \(P\) which contains \(p_1\) and \(p_2\).

If \(p \in P\), it follows from Theorem 2.20 that \(p, p_1\) and \(p_2\) are collinear if and only if \(\nu(p) \subseteq H\).

**Notation 3.10.** For distinct points \(p_1\) and \(p_2\) in a projective space \(P\), we let \(\text{line}_P(p_1, p_2)\) denote the line in \(P\) containing \(p_1\) and \(p_2\).

**Theorem 3.11.** Let \(L\) be a line in \(P\) and \((P, Q, [\cdot, \cdot])\) a projective duality. Suppose that \(p_1, p_2, p_3\) and \(p_4\) are elements of \(L\) with not both of \(\{p_1, p_3\}\) and \(\{p_2, p_4\}\) singletons. Let \(q_1, q_2, q, \overline{q} \in Q\) be such that \(p_1 \in q_1^q \cap q^\overline{q}\) and \(p_4 \in q_2^q \cap \overline{q}\). Then

\[
[p_1, p_2, q_1, q_2] = [p_1, p_2, q, \overline{q}].
\]

**Proof.** This follows from Theorem 2.10.

**Definition 3.12.** Let \(L\), and \(p_1, p_2, p_3\) and \(p_4\) be as in Theorem 3.11. We define

\[
[p_1, p_2, p_3, p_4] := [p_1, p_2, q, \overline{q}] \quad (\forall q, \overline{q} \in Q : p_3 \in q^q, p_4 \in \overline{q}).
\]

**Theorem 3.13.** Let \((P, Q, [\cdot, \cdot])\) and \((R, S, [\cdot, \cdot])\) be projective dualities. Let \(\varphi\) be a bijection from \(P\) to \(R\). Then \(\varphi\) is a projective isomorphism if and only if the following all hold:

(i) \([p_1, p_2, p_3, p_4] = [\varphi(p_1), \varphi(p_2), \varphi(p_3), \varphi(p_4)]\) for all \(p_1, p_2, p_3, p_4 \in P\) for which \([p_1, p_2, p_3, p_4]\) exists;

(ii) for each \(s \in S\), there exists a unique \(q \in Q\) such that \(\varphi^{-1}(s^q) = q^\overline{q}\);

(iii) for each \(q \in Q\), there exists a unique \(s \in S\) such that \(\varphi^{-1}(s^q) = q^\overline{q}\).

**Proof.** Suppose first that \(\varphi\) is a projective isomorphism and let \(\psi\) be its adjoint. Let \(V, W, X, Y, \nu, \omega, \xi, \zeta, \chi\) and \(\eta\) be as in Theorem 3.2. That (i) holds follows from Definition 3.1 and Definition 3.12. For \(s \in S\), we have \(\varphi^{-1}(s^q) = \nu^{-1}(\chi^{-1}(\xi(s^q)))\). But \(\xi(s^q) = \zeta(s)\overline{q}\), which implies that \(\chi^{-1}(\xi(s^q)) = \chi^{-1}(\zeta(s)\overline{q}) = \chi^{-1}(\zeta(s))\overline{q}\)
\[ \eta(\zeta(s))^{-1} \text{ since } \eta \text{ is the adjoint of } \chi. \] Consequently \( \varphi^{-1}(s^\circ) = \nu^{-1}(\eta(\zeta(s))^{-1}) = \nu^{-1}(\omega(\psi(s)))^{-1} = \psi(s)^\circ. \] This establishes (ii), and (iii) is shown analogously.

Now suppose that (i), (ii) and (iii) hold. For each \( s \in S \) let \( \psi(s) \) be the element of \( Q \) such that \( \varphi^{-1}(s^\circ) = \psi(s)^\circ. \) That \( \psi \) exists and is a bijection from \( S \) onto \( Q \) follows from (ii) and (iii). Let \( p_1, p_2 \in P \) and \( s_1, s_2 \in S \) be such that \( [\varphi(p_1), \varphi(p_2), s_1, s_2] \) exists. Let \( r_3 \) and \( r_4 \) be the intersections of \( s_1^\circ \) and \( s_2^\circ \) with \line_R(\varphi(p_1), \varphi(p_2)), \) and let \( p_3 := \varphi^{-1}(r_3) \) and \( p_4 := \varphi^{-1}(r_4). \) From (i) follows

\[
[\varphi(p_1), \varphi(p_2), s_1, s_2] = [\varphi(p_1), \varphi(p_2), r_3, r_4] = [\varphi(p_1), \varphi(p_2), \varphi(p_3), \varphi(p_4)] = [p_1, p_2, p_3, p_4].
\]

Evidently \( p_3 \) is the intersection of \( \varphi^{-1}(s_1^\circ) = \psi(s_1^\circ) \) with \( \varphi^{-1}(\line_R(\varphi(p_1), \varphi(p_2))) = \line_P(p_1, p_2). \) Similarly, \( p_4 \) is the intersection of \( \varphi^{-1}(s_2^\circ) = \psi(s_2^\circ) \) with \( \varphi^{-1}(\line_R(\varphi(p_1), \varphi(p_2))) = \line_P(p_1, p_2). \) Hence

\[
[p_1, p_2, p_3, p_4] = [p_1, p_2, \psi(s_1), \psi(s_2)].
\]

**Definition 3.14.** A projective duality \( (P, Q, [\ldots, \ldots]) \) is said to be **finite dimensional** if it has a vector representation on a finite dimensional vector duality.

**Theorem 3.15.** Let \( (P, Q, [\ldots, \ldots]) \) and \( (R, S, [\ldots, \ldots]) \) be finite dimensional projective dualities. Let \( \varphi \) be a bijection from \( P \) to \( R. \) Then \( \varphi \) is a projective isomorphism if and only if the following both hold:

(i) \( [p_1, p_2, p_3, p_4] = [\varphi(p_1), \varphi(p_2), \varphi(p_3), \varphi(p_4)] \) for all \( p_1, p_2, p_3, p_4 \in P \) for which \( [p_1, p_2, p_3, p_4] \) exists;

(ii) \( \varphi \) preserves lines.

**Proof.** Since \( \varphi \) is bijective, (ii) is equivalent to

(iii) \( \varphi^{-1} \) preserves lines.

Since \( P \) is finite dimensional (ii) and (iii) of this theorem are equivalent to (ii) and (iii) of Theorem 3.13.

**4. Projective Automorphisms and Vector Subspaces**

**Definition 4.1.** A projective duality isomorphism \((\varphi, \psi)\) for which the domain and range projective dualities are identical is said to be a **projective duality automorphism.** The constituent functions \( \varphi \) and \( \psi \) are said to be **projective automorphisms.**

**Definition 4.2.** Let \( \varphi \) be a projective automorphism with domain \( P. \) Define

\[
\text{Eigen}(\varphi) := \{ p \in P : \varphi(p) = p \}.
\]
Lemma 4.3. Let $(V, W, \langle , \rangle)$ be a vector space duality, and let $\overline{v} \in V$, $\overline{w} \in W$ and $u \in \mathbb{F}$ be such that $1 + u(\overline{v}, \overline{w}) \neq 0$. Define $\rho$ and $\sigma$ by

$$\rho(v) := v + u(v, \overline{v})\overline{v} \quad (\forall v \in V) \quad \text{and} \quad \sigma(w) := w + u(\overline{v}, w)\overline{w} \quad (\forall w \in W).$$

Then $(\rho, \sigma)$ is a vector duality isomorphism. In particular,

$$\langle \rho(v), w \rangle = \langle v, \sigma(w) \rangle \quad (\forall v \in V, w \in W).$$

Proof. Direct calculation.

Theorem 4.4. Let $(P, Q, [\cdot, \cdot, \cdot])$ be a projective duality and let $(\nu, \omega)$ be a vector representation of it on a vector duality $(V, W, \langle , \rangle)$. Let $\varphi$ be as in Lemma 4.3 and $\psi$ a bijection on $P$ such that

$$\nu(\varphi(p)) = \rho(\nu(p)) \quad (\forall p \in P).$$

Then $\varphi$ is a projective automorphism of $P$.

Proof. Define $\psi$ by

$$\psi(q) := \omega^{-1}(\sigma(\omega(q))) \quad (\forall q \in Q)$$

where $\sigma$ is as in Lemma 4.3. The conclusion follows from Lemma 4.3 and Theorem 3.3.

Discussion 4.5. We have represented a projective space as a family of lines through the origin of a vector space. Intrinsically however, a projective space is a set of points rather than lines. In fact, to each standard pair $(o, \infty)$ in a projective duality $(P, Q, [\cdot, \cdot, \cdot])$ corresponds a vector space $P \setminus \{o\}$ in $P$—in this case $P$ may be regarded as a vector space with “points at infinity” (i.e. points in $\infty$) added (which is our motivation for adopting the notation $(o, \infty)$ in this context rather than the $(p, q)$ which has been employed thus far). Our immediate task is to produce the vector operations on $P \setminus \{o\}$.

Theorem 4.6. Let $(o, \infty)$ be a standard pair for a projective duality $(P, Q, [\cdot, \cdot, \cdot])$ and let $u \in \mathbb{F}$ equal neither 0 nor 1. Then there exists exactly one projective automorphism $\varphi$ of $P$ such that both

(i) $\text{Eigen}(\varphi) = \{o\} \cup \infty$;

(ii) $[p, \varphi(p), \infty, q] = u$ for all $p \in P \setminus \text{Eigen}(\varphi)$ and $q \in o$.

Furthermore, if $\psi$ is the dual automorphism of $\varphi$, then

(iii) $\text{Eigen}(\psi) = \{\infty\} \cup o$;

(iv) $[o, p, q, \psi(q)] = u$ for all $q \in Q \setminus \text{Eigen}(\psi)$ and $p \in \infty$.

Proof. Let $(\nu, \omega)$ be a vector representation of $(P, Q, [\cdot, \cdot, \cdot])$ of a vector duality $(V, W, \langle , \rangle)$. Let $\overline{v} \in \nu(o)$ and $\overline{w} \in \omega(\infty)$ be such that $\langle \overline{v}, \overline{w} \rangle = 1$. Define $\chi$ on $V$ by

$$\chi(v) := v + (u^{-1} - 1)\langle v, \overline{w} \rangle \overline{w} \quad (\forall v \in V).$$
and define \( \varphi \) by
\[
\varphi(p) := \nu^{-1}(\chi(\nu(p)) \quad (\forall p \in P).
\]
That (i) holds is evident. Let \( p \in P \setminus \text{Eigen}(\varphi) \) and \( q \in \mathcal{O}^\circ \) be generic. Then, for \( v \in \nu(p) \) and \( w_0 \in \omega(q) \) both nonzero
\[
[p, \varphi(p), \infty, q] = [v, \chi(v), \overline{w}, w_0] = [v, v + (u^{-1} - 1)(v, \overline{w}) \overline{v}, \overline{w}, w_0] = (v, \overline{w})/(\langle v, w_0 \rangle u^{-1}(v, \overline{w})) = u
\]
which establishes (ii). That \( \varphi \) is a projective automorphism follows from Theorem 4.4.

Suppose that \( \sigma \) is another projective automorphism of \( P \) such that (i) and (ii) hold (with \( \varphi \) replaced by \( \sigma \)). Then there exists \( \tau \) such that \( (\sigma, \tau) \) is a projective duality automorphism. By Theorem 3.2 there exists a unique vector duality isomorphism. From Theorem 3.3 follows that each line in \( \overline{w}^{-1} \) is mapped to itself by \( \xi \). It follows that \( \xi \) is a multiple \( f \) of the identity on \( \overline{w}^{-1} \). Let \( z \in \overline{w}^{-1} \) be nonzero and \( v = \overline{v} + z \). We have for each \( w \in \overline{w}^{-1},
\[
u = [v, \xi(v), \overline{w}, w] = \langle v, \overline{w} \rangle \langle \xi(v), w \rangle / (\langle v, w \rangle \langle \xi(v), \overline{w} \rangle)
\]
\[
= \langle \overline{v} + z, \overline{w} \rangle \langle \xi(\overline{v}) + tz, w \rangle / (\langle \overline{v} + z, \overline{w} \rangle \langle \xi(\overline{v}) + tz, \overline{w} \rangle)
\]
\[
= \langle u^{-1} \overline{v} + tz, w \rangle / (\langle z, w \rangle u^{-1} \overline{v} + tz, \overline{w} \rangle)
\]
\[
= t\langle z, w \rangle / (\langle z, w \rangle u^{-1}) = tu
\]
whence follows that \( \xi \) leaves each element of \( \overline{w}^{-1} \) fixed. Thus, for \( v \in V \), the fact that \( v - \langle v, \overline{w} \rangle \overline{v} \) is in \( \overline{w}^{-1} \) implies
\[
\xi(v) = \xi((v, \overline{w}) \overline{v} + (v - \langle v, \overline{w} \rangle \overline{v})) = (v, \overline{w}) u^{-1} \overline{v} + v - \langle v, \overline{w} \rangle \overline{v}
\]
\[
= (u^{-1} - 1)(v, \overline{w}) \overline{v} + v = \chi(v).
\]
Consequently, \( \sigma \) must equal \( \varphi \).

Define \( \eta \) on \( W \) by
\[
\eta(w) := w + (u^{-1} - 1)(\overline{w}, w) \overline{v} \quad (\forall w \in W).
\]

By Lemma 4.3, \( (\chi, \eta) \) is a vector duality isomorphism. From Theorem 3.3 follows that if \( \psi \) is defined by
\[
\psi(q) := \omega^{-1}(\eta(\omega(q))) \quad (\forall q \in Q),
\]
then \( \psi \) is the dual projective isomorphism to \( \varphi \). We have \( \text{Eigen}(\psi) = \{\infty\} \cup \mathcal{O}^\circ \). Let \( q \in Q \setminus \mathcal{O}^\circ \) and choose \( r \in Q \) such that \( \langle \overline{v}, r \rangle = 1 \). For \( p \in \infty^\circ \) we have
\[
[p, q, \psi(q)] = (\overline{v}, r) \langle p, (u^{-1} - 1)(\overline{v}, r) \overline{w} + r \rangle / (((\overline{v}, (u^{-1} - 1)(\overline{v}, r) \overline{w} + r))(p, r))
\]
\[
= (p, r) / u^{-1}(p, r) = u.
\]
DEFINITION 4.7. The function $\varphi$ of Theorem 4.6 will be called the $u$-scalar-automorphism of $P$ associated with the standard pair $(o, \infty)$ and the scalar field element $u \in \mathbb{F}$. We define the 1-scalar-automorphism associated with $(o, \infty)$ to be the identity (the terminology scalar-automorphism is motivated by Corollary 4.26 infra).

In the case $u = -1$ we say that $\varphi$ is the involution of $P$ associated with $(o, \infty)$. In this case, $\varphi = \varphi^{-1}$ and for two points $p_1, p_2 \in P \setminus \infty$ we say that $o$ is an $\infty$-midpoint of $p_1$ and $p_2$ if $\varphi(p_1) = p_2$ (cf. Theorem 4.11 infra).

NOTATION 4.8. Let $(P, Q, [\cdot, \cdot])$ be a projective duality. We write $\text{Aut}(P)$ for the set of all projective automorphisms of $P$. For $q \in Q$ we write $\text{Aut}(P; q)$ for the set of all $\varphi \in \text{Aut}(P)$ such that $\varphi(p) = p$ for each $p \in q^o$. We write $\text{Scal}(P)$ for the set of all scalar-automorphisms on $P$ (including the identity map $i$). For $q \in Q$, we write $\text{Scal}(P; q)$ for $\text{Scal}(P) \cap \text{Aut}(P; q)$. We write $\text{Inv}(P)$ for the set of all involutions of $P$ and $\text{Inv}(P; q)$ for $\text{Inv}(P) \cap \text{Scal}(P; q)$.

THEOREM 4.9. Let $q$ be in $Q$. Then $\text{Aut}(P; q)$ is a subgroup of $\text{Aut}(P)$ (relative to composition) and $\text{Scal}(P; q)$ consists precisely of those elements of $\text{Aut}(P; q)$ which have fixed points not in $q^o$.

Let $\varphi$ be a $u$-scalar-automorphism in $\text{Scal}(P; q)$ and $\psi$ a $t$-scalar-automorphism in $\text{Scal}(P; q)$. If $ut = 1$, then $\varphi \circ \psi$ is not in $\text{Scal}(P; q)$ unless $\varphi \circ \psi$ is the identity. If $ut \neq 1$, then $\varphi \circ \psi$ is a $ut$-scalar-automorphism in $\text{Scal}(P; q)$.

Proof. Suppose that $\lambda \in \text{Aut}(P)$ leaves $\overline{p} \notin q^o$ fixed. Let $(\nu, \omega)$ be a vector representation of $(P, Q, [\cdot, \cdot])$ and let $\overline{\tau} \in \nu(\overline{p})$ be nonzero. By Theorem 3.2 there exists a linear isomorphism $\rho$ of $V$ such that

$\rho(\overline{\tau}) = \overline{\tau}$ and $\rho(\nu(p)) = \nu(\lambda(p)) \quad (\forall p \in P).

Since $\varphi$ leaves the points of $q^o$ fixed, it follows that $\rho$ sends each line in $\nu(q^o)$ to itself. Consequently, $\rho$ is a multiple $d$ of the identity on $\nu(q^o)$. Let $w \in \omega(q)$ be such that $(\overline{\tau}, w) = 1$. Let $p \in P$ be generic, $v \in \nu(p)$ nonzero and define $v := r\overline{\tau} + v_0$ ($r \in \mathbb{F}$ and $v_0 \in \nu(q^o)$). For $q_0 \in q^o$ and $w_0 \in \omega(q_0)$ nonzero we have

$[p, \lambda(p), q_0, q] = [v, \rho(v), w_0, w] = \langle r\overline{\tau} + v_0, w_0 \rangle \langle r\overline{\tau} + dv_0, w \rangle /

(\langle r\overline{\tau} + v_0, w \rangle \langle r\overline{\tau} + dv_0, w_0 \rangle) = \langle v_0, w_0 \rangle r/(rd(v_0, w_0)) = 1/d.$

If $\rho(v) = tv$ for some $t \in \mathbb{F}$, then $r\overline{\tau} + v_0/d = \overline{\tau}$, and so $v_0 = 0$. Hence $\overline{\tau}$ and $\nu(q^o)$ are the eigenspaces of $\rho$ which implies that $\text{Eigen}(\lambda) = \{p\} \cup q^o$. From Lemma 4.3 we see that $\lambda$ is the $(1/d)$-scalar-automorphism associated with $(p, q)$. That the assertion of the first paragraph of the theorem is correct is now evident.

Let $p_\varphi$ and $p_\psi$ be the fixed points not in $q^o$ of $\varphi$ and $\psi$ respectively. If $p_\varphi = p_\psi$, the assertion of the second paragraph of the theorem is trivial, and so we shall suppose that $p_\varphi$ and $p_\psi$ are distinct. Let $(\nu, \omega)$ be a vector representation of $(P, Q, [\cdot, \cdot])$ and let $v_\varphi \in \nu(p_\varphi)$ be nonzero. Choose $w \in \omega(q)$ such that $\langle v_\varphi, w \rangle = 1$ and then choose $v_\psi \in \nu(p_\psi)$ such that $\langle v_\psi, w \rangle = 1$. By Theorem 3.2 there exist linear automorphisms $\rho$ and $\sigma$ of $V$ such that

$\rho(v_\varphi) = v_\varphi, \quad \rho(\nu(p)) = \nu(\varphi(p)) \quad \text{and} \quad \sigma(v_\psi) = v_\psi, \quad \sigma(\nu(p)) = \nu(\sigma(p)) \quad (\forall p \in P).$
For \( v_0 \in \nu(q^o) \) we have \( \rho(v_0) \in \rho(v_0) \) which implies that
\[
\langle \rho(v_0), w \rangle = 0.
\]
Thus, for all \( w_\varphi \in v_\varphi^\perp \)
\[
u = [v_\varphi + v_0, \rho(v_\varphi + v_0), w, w_\varphi]\]
\[
= \langle v_\varphi + \rho(v_0), w_\varphi \rangle / (\langle v_0, w_\varphi \rangle (v_\varphi + \rho(v_0), w))
\]
which implies that \( \rho(v_0) = uv_0 \). Similarly, \( \sigma(v_0) = tv_0 \) for all \( v_0 \in \nu(q^o) \). Let \( \mathcal{E} \) be the 2-dimensional subspace of \( V \) spanned by \( v_\varphi \) and \( v_\psi \). The element \( e := v_\psi - v_\varphi \) is in \( \mathcal{E} \) and, since \( \langle v_\psi - v_\varphi, w \rangle = 1 - 1 = 0 \), \( e \) is in \( \nu(q^o) \) as well. We have
\[
\rho(e) = ue, \quad \rho(v_\varphi) = v_\varphi, \quad \sigma(e) = te \quad \text{and} \quad \sigma(v_\psi) = v_\psi
\]
which implies that \( \rho(\mathcal{E}) = \mathcal{E} = \sigma(\mathcal{E}) \). Let \( \alpha \) and \( \beta \) be the restrictions of \( \rho \) and \( \sigma \) to \( \mathcal{E} \) respectively. The vector \( v_\psi \), being distinct from \( e \), can be written as \( mv_\varphi + ne \) for \( m, n \in \mathbb{F} \) with \( m \neq 0 \).

Suppose that \( ut \neq 1 \), let \( K := nu(1-t)/(1-ut) \) and then let \( d := mv_\varphi + Ke \). We have \( v_\varphi = (v_\psi - ne)/m \) and so \( d = v_\varphi + (K - n)e \). Thus
\[
\alpha \circ \beta(d) = \alpha(v_\varphi + t(K - n)e) = \alpha(mv_\varphi + (n + tK - utn)e) = mv_\varphi + (un + utK - utn)e = mv_\varphi + Ke = d.
\]

Furthermore, \( \alpha \circ \beta(e) = \alpha(te) = ute \). Thus \( \rho \circ \sigma \) has an eigenvector \( d \) not in \( \nu(q^o) \), which implies that \( \varphi \circ \psi \) has a fixed point \( \nu^{-1}(d) \) not in \( q^o \). Hence \( \varphi \circ \psi \) is in \( \text{Scal}(P, q) \). For \( q_1 \in (\nu^{-1}(d))^o \) and \( w_1 \in \omega(q_1) \)
\[
[\nu^{-1}(d + e), \varphi \circ \psi(\nu^{-1}(d + e)), q, q_1] = [d + e, \rho \circ (d + e), w, w_1]
\]
\[
= [d + e, d + ute, w, w_1]
\]
\[
= \langle d, w \rangle \langle ute, w_1 \rangle / (\langle e, w_1 \rangle \langle d, w \rangle)
\]
\[
= ut
\]
which implies that \( \varphi \circ \psi \) is a \( ut \)-scalar automorphism.

Now suppose that \( ut = 1 \). Then all vectors of \( \nu(q^o) \) are fixed by \( \rho \circ \sigma \). If \( \rho \circ \sigma \) is not the identity, it has at most one eigenvector in \( V \setminus \nu(q^o) \), \( \nu(q^o) \) being a maximal subspace of \( V \). Assume \( \rho \circ \sigma \) is not the identity and that \( \rho \circ \sigma(v) = rv \) for \( v \in V \setminus \nu(q^o) \) and \( r \in \mathbb{F} \). Then there exist \( r_0, r_1 \in \mathbb{F} \) and \( v_0, v_1 \in \nu(q^o) \) such that
\[
v_\psi = r_0 v + v_0 = r_1 v_\varphi + v_1.
\]
We have
\[
\alpha \circ \beta(v_\psi) = \alpha \circ \beta(r_0 v + v_0) = r_0 v + v_0
\]
and
\[
\alpha \circ \beta(v_\psi) = \alpha(v_\psi) = \alpha(r_1 v_\varphi + v_1) = r_1 v_\varphi + uv_1
\]
which yields
\[
r_0 (v, w) = r_1 (v_\varphi, w).
\]
Since \( r_0 v + v_0 = r_1 v_\varphi + v_1 \), we have as well
\[
r_0 (v, w) = r_1 (v_\varphi, w)
\]
which implies \( r = 1 \): an absurdity.
NOTATION 4.10. For \( q \in Q \) we write \( \text{Tran}(P; q) \) for the set
\( \{ \varphi \in \text{Aut}(P; q) : \varphi \text{ is not a } u\text{-scalar-automorphism} \} \). Elements of \( \text{Tran}(P; q) \) are called translations of \( P \setminus q^o \). The motivation for this terminology will be apparent in Theorem 4.22 infra.

THEOREM 4.11. Let \( (P, Q, [\ldots]) \) be a projective duality, \( q \in Q \) and \( p_1, p_2 \in P \setminus q^o \). Then \( p_1 \) and \( p_2 \) have a unique \( q \)-midpoint \( o \).

Let \( \varphi \in \text{Tran}(P; q) \) and \( p \in P \setminus q^o \) be generic. Let \( d \) be the \( q \)-midpoint of \( p \) and \( \varphi(p) \). Let \( c \) be the \( q \)-midpoint of \( p \) and \( d \), and let \( e \) be the \( q \)-midpoint of \( d \) and \( \varphi(p) \). Let \( \alpha, \beta \in \text{Inv}(P; q) \) be such that \( \alpha(e) = e \) and \( \beta(e) = e \). Then \( \varphi = \alpha \circ \beta \).

Furthermore, \( \text{Tran}(P; q) = \{ \alpha \circ \beta : \alpha, \beta \in \text{Inv}(P; q) \} \).

Proof. Let \((\nu, \omega)\) be a vector representation of \((P, Q, [\ldots])\) on a vector duality \((V, W, [\ldots])\). Let \( w \in \omega(q) \) be nonzero. Choose \( v_1 \in \nu(p_1) \) and \( v_2 \in \nu(p_2) \) such that \( \langle v_1, w \rangle = 1 = \langle v_2, w \rangle \). Define the linear isomorphism \( \sigma \) of \( V \) by
\[
\sigma(v) := v - \langle v, w \rangle (v_1 + v_2) \quad (\forall v \in V).
\]
Direct calculation shows
\[
\sigma \circ \sigma = \iota, \quad \sigma(v_1/2 + v_2/2) = -(v_1/2 + v_2/2), \quad \sigma(v_1) = -v_2 \quad \text{and} \quad \sigma(v_2) = -v_1.
\]
Let \( o := \nu^{-1}(v_1/2 + v_2/2) \) and define \( \theta \) by
\[
\theta(p) := \nu^{-1}(\sigma(\nu(p))) \quad (\forall p \in P).
\]
Evidently,
\[
\theta = \theta^{-1}, \quad \theta(o) = o, \quad \theta(p_1) = p_2, \quad \theta(p_2) = p_1 \quad \text{and} \quad \theta(p) = p \quad (\forall p \in q^o)
\]
which implies that \( \theta \) is in \( \text{Inv}(P; q) \) and so \( o \) is a midpoint of \( p_1 \) and \( p_2 \).

Suppose \( p_1 \) and \( p_2 \) had another midpoint \( m \). Let \( \eta \) be the element of \( \text{Inv}(P; q) \) such that \( \eta(m) = m, \eta(p_1) = p_2 \) and \( \eta(p_2) = p_1 \). Then \( \eta \circ \theta^{-1} \) is an element of \( \text{Aut}(P; q) \) which leaves \( p_2 \) and \( p_1 \) fixed. By Theorem 4.9, this means that \( \eta \circ \theta^{-1} \) is the identity, and so \( \eta = \theta \).

Let \( \rho \) be a linear automorphism on \( V \) such that
\[
\nu(\varphi(p)) = \rho(\nu(p)) \quad (\forall p \in P).
\]
Since all the elements of \( \nu(q^o) \) are \( \rho \)-eigenvectors, it follows that \( \rho \) is a scalar multiple \( t \) of the identity on \( \nu(q^o) \). By replacing \( \rho \) by \( \rho/t \) if necessary, we may assume that \( \rho \) leaves each element of \( \nu(q^o) \) fixed. Consider \( \overline{\nu} \in V \setminus \nu(q^o) \). Since \( \nu^{-1}(\overline{\nu}) \) is not \( \varphi \)-fixed, \( \rho(\overline{\nu}) \) is not in \( \overline{\nu} \) and so
\[
(\exists r \in F, v_o \in \nu(q^o)) \quad \rho(\overline{\nu}) = r\overline{\nu} + v_o \quad \text{and} \quad v_o \neq 0.
\]
The equality \( \rho(v_o + (r - 1)\overline{\nu}) = v_o + (r - 1)(r\overline{\nu} + v_o) = r(v_o + (r - 1)\overline{\nu}) \) implies that \( \nu^{-1}(v_o + (r - 1)\overline{\nu}) \) is \( \varphi \)-fixed, which requires that \( r = 1 \):
\[
\rho(\overline{\nu}) = \overline{\nu} + v_o.
\]
Choose \( w \in \omega(q) \) such that \( (\mathfrak{v}, w) = 1 \). Define the linear automorphisms \( \zeta \) and \( \xi \) of \( V \) by

\[
\zeta(v) := v - \langle v, w \rangle(v_o/2 + 2\mathfrak{v}) \quad \text{and} \quad \xi(v) := v - \langle v, w \rangle(3v_o/2 + 2\mathfrak{v}) \quad (\forall v \in V).
\]

Direct calculation yields

\[
\zeta(v + v_o/4) = -(v + v_o/4) \quad \text{and} \quad \xi(v + 3v_o/4) = -(v + 3v_o/4).
\]

Define \( \alpha \) and \( \beta \) on \( P \) by

\[
\alpha(p) := \nu^{-1}(\xi(\nu(p))) \quad \text{and} \quad \beta(p) := \nu^{-1}(\zeta(\nu(p))) \quad (\forall p \in P).
\]

Evidently \( \varphi = \alpha \circ \beta \). From Theorem 4.4 follows that \( \alpha \) and \( \beta \) are projective automorphisms. It follows from the above that they are elements of \( \text{Inv}(P; q) \).

That \( \{\alpha \circ \beta : \alpha, \beta \in \text{Inv}(P; q)\} \subset \text{Tran}(P; q) \) is a consequence of Theorem 4.9.

**Definition 4.12.** Let \((P,Q,[\cdot,\cdot])\) be a projective duality and \( q \) an element of \( Q \). For \( \varphi \in \text{Aut}(P; q) \) we define

(i) \( \text{scal}_q(\varphi) := u \) if \( \varphi \) is a \( u \)-scalar-automorphism;

(ii) \( \text{scal}_q(\varphi) := 1 \) if \( \varphi \) is a translation.

We say that \( \text{scal}_q(\varphi) \) is the scalar-automorphism coefficient of \( \varphi \).

**Corollary 4.13** The function \( \text{scal}_q \) of Definition 4.12 is group epimorphism of \( \text{Aut}(P; q) \) onto the multiplicative group of nonzero scalars.

**Proof.** Apply Theorems 4.9 and 4.11.

**Lemma 4.14.** Let \( \alpha, \beta, \gamma \in \text{Inv}(P; q) \) for \( q \in Q \). Then

\[
\gamma \circ \beta \circ \alpha = \alpha \circ \beta \circ \gamma.
\]

**Proof.** From Definition 4.12 follows that \(-1 = \text{scal}_q(\alpha) = \text{scal}_q(\beta) = \text{scal}_q(\gamma)\) and so Corollary 4.13 implies that \( \text{scal}_q(\alpha \circ \beta \circ \gamma) = -1 \), whence follows that \( \alpha \circ \beta \circ \gamma \) is an involution: \( \alpha \circ \beta \circ \gamma = (\alpha \circ \beta \circ \gamma)^{-1} = \gamma \circ \beta \circ \alpha \).

**Theorem 4.15.** Let \((P,Q,[\cdot,\cdot])\) be a projective duality and \( q \) an element of \( Q \). The kernel of \( \text{scal}_q \) is the normal Abelian subgroup \( \text{Tran}(P; q) \cup \{e\} \) of \( \text{Aut}(P; q) \).
Proof. That $\text{Tran}(P; q)$ is the kernel of $\text{scal}_q$ follows from Item (ii) of Definition 4.12. That $\text{Tran}(P; q)$ is Abelian is a consequence of Lemma 4.14 for, if $\sigma$ and $\rho$ are in $\text{Tran}(P; q)$, then the characterization in Theorem 4.11 implies that

$$(\exists \alpha, \beta, \gamma, \delta \in \text{Inv}(P; q)) \quad \sigma = \alpha \circ \beta \quad \text{and} \quad \rho = \gamma \circ \delta$$

which yields

$$\sigma \circ \rho = (\alpha \circ \beta \circ \gamma) \circ \delta = (\gamma \circ \beta \circ \alpha) \circ \delta = \gamma \circ (\beta \circ \alpha \circ \delta) = \gamma \circ (\delta \circ \alpha \circ \beta) = \rho \circ \sigma.$$ 

Remark 4.16. The Abelian group $\text{Tran}(P; q) \cup \{e\}$ is actually a vector space over $F$ relative to a scalar multiplication which we introduce in Definition 4.19 infra. To justify this definition we shall need additional notation and a lemma.

Notation 4.17. For $\varphi \in \text{Scal}(P; q)$ and $u \in F \setminus \{0, 1\}$ we write $\varphi u$ for the element of $\text{Aut}(P; q)$ leaving all the elements of $\text{Eigen}(\varphi)$ fixed and satisfying

$$\text{scal}_q(\varphi u) = u \text{ scal}_q(\varphi).$$

We note in particular, that if $\varphi$ is in $\text{Inv}(P; q)$ and $u = -1$, then $\varphi u$ is the identity map.

Lemma 4.18. Let $\tau \in \text{Tran}(P; q)$ and $u \in F \setminus \{0, 1\}$. Let $\theta, \sigma \in \text{Scal}(P; q)$ be such that

$$\text{scal}_q(\theta) = \text{scal}_q(\sigma) = u.$$

Then

$$\theta \circ \tau \circ \theta^{-1} = \sigma \circ \tau \circ \sigma^{-1}.$$ 

Proof. By Theorem 4.11 we may write $\tau = \alpha \circ \beta$ for $\alpha, \beta \in \text{Inv}(P; q)$. Noting that

$$\text{scal}_q(\theta^{-1} \circ \sigma) = \text{scal}_q(\theta^{-1}) \text{ scal}_q(\sigma) = u^{-1} u = 1$$

we deduce that $\theta^{-1} \circ \sigma$ is a translation and so it is of the form $\gamma \circ \delta$ for $\gamma, \delta \in \text{Inv}(P; q)$. Substituting above,

$$\theta \circ \tau \circ \theta^{-1} \circ (\sigma \circ \tau \circ \sigma^{-1})^{-1} = \theta \circ \tau \circ \theta^{-1} \circ \sigma \circ \tau^{-1} \circ \sigma^{-1} = \theta \circ (\alpha \circ \beta \circ \gamma) \circ \delta \circ (\alpha \circ \beta)^{-1} \circ \sigma^{-1} = \theta \circ (\gamma \circ \beta \circ \alpha) \circ (\delta \circ \beta \circ \alpha) \circ \sigma^{-1} = \theta \circ \gamma \circ \beta \circ \alpha \circ \alpha \circ \beta \circ \delta \circ \sigma^{-1} = \theta \circ \gamma \circ \delta \circ \sigma^{-1} = \theta \circ \theta^{-1} \circ \sigma \circ \sigma^{-1} = e.$$ 

It follows that $\theta \circ \tau \circ \theta^{-1} = \sigma \circ \tau \circ \sigma^{-1}$. 

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DEFINITION 4.19. For $\tau \in \text{Tran}(P;q)$ and $u \in F \setminus \{0,1\}$, we invoke Lemma 4.18 to define
\[ ut := \theta \circ \tau \circ \theta^{-1} \quad (\forall \theta \in \text{Scal}(P;q) : \text{scal}_q(\theta) = u). \]
We define $0\tau := \iota$ and $1\tau := \tau$.

THEOREM 4.20. Let $(P,Q,[,])$ be a projective duality and $q$ an element of $Q$. Letting scalar multiplication be defined as in Definition 4.19 and letting addition be composition, $\text{Tran}(P;q) \cup \{\iota\}$ is vector space over $F$.

Proof. Let $u$ and $t$ be scalars and $\tau \in \text{Tran}(P;q)$. If $u$ or $t$ is either 0 or 1, then the equality $u(t\tau) = (ut)\tau$ is trivial. Suppose that $u,t \in F \setminus \{0,1\}$ and let $\theta$ be an element of $\text{Scal}(P;q)$ such that $\text{scal}_q(\theta) = u$. Then $\text{scal}_q(\theta t/u) = t$ and $\text{scal}_q(\theta \circ (\theta t/u)) = ut$. We have
\[ u(t\tau) = \theta \circ ((\theta t/u) \circ \tau \circ (\theta t/u)^{-1}) \circ \theta^{-1} = (\theta \circ (\theta t/u)) \circ \tau \circ (\theta \circ (\theta t/u))^{-1} = (ut)\tau. \]

If $\sigma \in \text{Tran}(P;q)$ as well we have
\[ u(\sigma \circ \tau) = \theta \circ \sigma \circ \tau \circ \theta^{-1} = \theta \circ \sigma \circ \theta \circ \theta^{-1} \circ \tau \circ \theta^{-1} = u\sigma \circ ut \]
which establishes the distributive property.

RECOLLECTION 4.21 (cf. Weyl [13, p. 45], Yale [14, Chapter 5]). Let $A$ be a set on which a vector space $V$ acts. For $v \in V$ and $a \in A$ we write $a + v$ for the image of $a$ by the action corresponding to $v$. The set $A$ is said to be an affine space with translation vector space $V$ if
\[ \text{for all } a_1, a_2 \in A \text{ there exists a unique } v \in V \text{ such that } a_2 = a_1 + v. \quad (2) \]

Suppose that $A$ is an affine space with translation vector space $V$. For $a_1, a_2 \in A$ distinct, we denote the element $v$ of (2) by $a_2 - a_1$. For a scalar $u$ and $a_1, a_2 \in A$ we have
\[ a_1 + u(a_2 - a_1) = a_2 + (1 - u)(a_1 - a_2) \]
and we denote that quantity by $(1 - u)a_1 + ua_2$.

A function $\varphi$ from one affine space onto another which preserves actions is an affine map. This is equivalent to
\[ \varphi((1 - u)a_1 + ua_2) = (1 - u)\varphi(a_1) + u\varphi(a_2) \quad (\forall a_1, a_2 \in A) \quad (\forall u \in F). \]

For each $o \in A$, $A$ may be identified with $V$ via the correspondence
\[ A \ni a \mapsto a - o \in V. \]
The addition and scalar multiplication on $A$ thus induced become
\[ a_1 + a_2 = o + ((a_1 - o) + (a_2 - o)) \quad \text{and} \quad ua = o + u(a - o) \quad (\forall a, a_1, a_2 \in A) \quad (\forall u \in F). \]
THEOREM 4.22. Let \((P, Q, [, ,])\) be a projective duality and \(q\) an element of \(Q\). Then \(P \setminus q^o\) is an affine space with translation vector space \(\text{Tran}(P; q) \cup \{\iota\}\), the action of an element \(\tau \in \text{Tran}(P; q)\) on an element \(p \in P \setminus q^o\) being \(\tau(p)\).

Proof. Let \((\nu, \omega)\) be a vector representation of \((P, Q, [, ,])\) on a vector duality \((V, W, (,))\) and suppose that \(p_1 \in P\) is such that \((p_1, q)\) is a standard pair. Let \(v_1 \in \nu(p_1)\) be nonzero, and let \(\varphi\) be in \(\text{Tran}(P; q)\).

Now let \(p_2 \in P \setminus q^o\) be distinct from \(p_1\), and let \(v_2 \in \nu(p_2)\) be nonzero. Let \(m\) be the midpoint of \(v_1\) and \(v_2\). Let \(a\) be the midpoint of \(v_1\) and \(m\), and let \(b\) be the midpoint of \(m\) and \(v_2\). Select \(w_1 \in W\) such that

\[
\langle b, w_1 \rangle = 0 \quad \text{and} \quad \langle v_2, w_1 \rangle = 1.
\]

It follows that \(\langle m, w_1 \rangle = -1\). Define \(\rho\) on \(V\) by

\[
\rho(v) := v + 2\langle v, w_1 \rangle (b - v_2) \quad (\forall v \in V).
\]

Direct calculation shows that \(\rho(v_2) = 2b - v_2 = m\). Since \(1+2\langle v_2 - b, w_1 \rangle = -1 \neq 0\), it follows from Lemma 4.3 that the map \(\varphi\) on \(P\) defined by

\[
\varphi(p) := \nu^{-1}(\rho(\nu(p))) \quad (\forall p \in P)
\]

is a projective automorphism. Evidently \(\varphi\) is in \(\text{Inv}(P; q)\) and

\[
\varphi(\nu^{-1}(v_2)) = \nu^{-1}(\rho(\nu(v_2))) = \nu^{-1}(m).
\]

Similarly, we select \(w_2 \in W\) such that \(\langle a, w_2 \rangle = 0\) and \(\langle m, w_2 \rangle = 1\), define \(\sigma\) on \(V\) by

\[
\sigma(v) := v + 2\langle v, w_2 \rangle (a - m) \quad (\forall v \in V),
\]

and define \(\psi\) on \(P\) by

\[
\psi(c) := \nu^{-1}(\sigma(\nu(p))) \quad (\forall p \in P).
\]

Then \(\psi\) is in \(\text{Inv}(P; q)\) and

\[
\psi(\nu^{-1}(m)) = \nu^{-1}(v_1).
\]

We have

\[
\psi \circ \varphi(p_2) = \psi \circ \varphi(\nu^{-1}(v_2)) = \psi(\nu^{-1}(m)) = \nu^{-1}(v_1) = p_1.
\]

Since \(\psi \circ \varphi\) is in \(\text{Tran}(P; q)\), we have verified (2) of Recollection 4.21.

MOTIVATION 4.23. We have now seen that \(\text{Tran}(p; q) \cup \{\iota\}\) acts on \(P \setminus q^o\). Scalar multiplication on \(\text{Tran}(P; q) \cup \{\iota\}\), as defined in Definition 4.19, is admittedly somewhat cumbersome. When we choose a fixed point in \(P \setminus q^o\), however, and view \(P \setminus q^o\) as a vector space with that fixed point as origin, the scalar multiplication on \(P \setminus q^o\) becomes more simple. This will be seen in Theorem 4.25 infra, but a smooth demonstration requires some new notation (which will be useful in other contexts in the sequel as well).
Notation 4.24. For a standard pair \((p, q)\) and \(u \in \mathbb{F}\) nonzero, \(u_{(p, q)}\) will denote the \(u\)-scalar-automorphism of \(P\) associated with \((p, q)\).

For \(q \in Q\), and \(p_1, p_2 \in q^o\), we write \(\tau_{(p_1, p_2; q)}\) for the element of \(\text{Tran}(P; q)\) which sends \(p_1\) to \(p_2\). Thus, in affine space notation, \(p_2 - p_1 = \tau_{(p_1, p_2; q)}\) and \(p_2 = p_1 + \tau_{(p_1, p_2; q)}(p_1)\).

Theorem 4.25. Let \((P, Q, [\ldots])\) be a projective duality and \((o, \succ\prec)\) a standard pair. Let \(u \in \mathbb{F}\setminus\{0\}\) and \(\overline{p} \in P\setminus q^o\) be generic. Let \(\tau\) be \(\tau_{(o, \prec\succ)}\) the element of \(\text{Tran}(P; \prec\succ)\) such that \(\tau(o) = \overline{p}\). Then

\[
\begin{align*}
u_{(o, \prec\succ)}(\overline{p}) &= (u\tau)(o).
\end{align*}
\]

Proof. For \(u = 1\), the theorem is trivial, so we shall assume that \(u \neq 1\). Let \((\nu, \omega)\) be a vector representation of \((P, Q, [\ldots])\) on a vector duality \((V, W, \langle, \rangle)\). Let \(v_0 \in \nu(o)\) be nonzero and select \(w_{\succ\prec} \in \omega(\succ\prec)\) such that \(\langle v_0, w_{\succ\prec}\rangle = 1\). Since \(\overline{p}\) is in \(P\setminus q^o\), there exists \(\overline{v} \in \nu(\overline{p})\) such that \(\langle \overline{v}, w_{\succ\prec}\rangle = 1\). Define the linear automorphisms \(\alpha\) and \(\lambda_t\) of \(V\) by

\[
\begin{align*}
\alpha(v) &= v + \langle v, w_{\succ\prec}\rangle(\overline{v} - v_0) \quad \text{and} \\
\lambda_t(v) &= v + (t^{-1} - 1)\langle v, w_{\succ\prec}\rangle v_0
\end{align*}
\]

for all \(v \in V\) and \(t \in \mathbb{F}\setminus\{0, 1\}\). We observe that

\[
\begin{align*}
\nu(\tau(p)) &= \alpha(\nu(p)) \quad \text{and} \\
\nu(t(\nu_{(o, \succ\prec)})(p)) &= \lambda_t(\nu(p))
\end{align*}
\]

for all \(p \in P\) and \(t \in \mathbb{F}\setminus\{0\}\). We have

\[
\nu(\nu_{(o, \succ\prec)}(\overline{p})) = \lambda(\overline{p}) = \overline{\nu} + (u^{-1} - 1)v_0 = u\overline{\nu} + (u - 1)v_0.
\]

Moreover \(u\tau = \nu_{(o, \succ\prec)} \circ \tau \circ \nu_{(o, \succ\prec)}^{-1}\) and

\[
\nu_{(o, \succ\prec)}(p) = \lambda_{u} \circ \alpha \circ \lambda_{u^{-1}}(v_0) = u\overline{\nu} + (u - 1)v_0.
\]

by direct computation.

Corollary 4.26. Let \((P, Q, [\ldots])\) be a projective duality, and let \(q \in Q\) be generic. Let \(p_1, p_2 \in P\setminus q^o\) be distinct and \(u \in \mathbb{F}\) be nonzero. Then, using standard notation for the affine space \(P\setminus q^o\), we have

\[
(1 - u)p_1 + up_2 = p_1 + u(p_2 - p_1) = u_{(p_1, q)}(p_2).
\]

Proof. From Recollection 4.21 follows that \((1 - u)p_1 + up_2 = p_1 + u(p_2 - p_1)\). Now apply Theorem 4.25.

Definition 4.27. Let \((P, Q, [\ldots])\) be a projective duality and \((o, \succ\prec)\) a standard pair. By the standard vector space \(P(o, \succ\prec)\) we mean the vector space \(P\setminus q^o\) obtained as in Recollection 4.21 by identifying the point \(o\) of the affine space \(P\setminus q^o\) with an origin of a vector space. We say that \(P\) is the projective completion of each of its standard vector spaces.

From Recollection 4.21, Definition 4.24 and Corollary 4.26 we obtain, for all \(p_1, p_2, p \in P(o, \succ\prec)\) and \(u \in \mathbb{F}\setminus\{0\}\),

\[
p_1 + p_2 = \tau_{(o, p_1, \succ\prec)} \circ \tau_{(o, p_2, \succ\prec)}(o) \quad \text{and} \quad up = u_{(o, \succ\prec)}(p).
\]
Discussion 4.28. Let \( (P, Q, \ldots) \) be a projective duality and let \( (o, \infty) \) be a standard pair. It has just been shown how \( P \setminus o^\circ \) is a vector space. Interchanging the roles of \( P \) and \( Q \) however, it follows that \( Q \setminus o^\circ \) is also a vector space. It is reasonable to consider whether these two vector spaces are connected in duality by some natural bilinear functional \(<,>\). The next theorem shows that this is indeed the case.

**Theorem 4.29.** Let \( (P, Q, \ldots) \) be a projective duality and \( (o, \infty) \) a standard pair. Let \( G \) denote the standard vector space \( P(o, \infty) \) and let \( H \) denote the standard vector space \( Q(\infty, o) \). Define \(<,>\) on \( G \times H \) by
\[
\langle g, h \rangle := [g, o, h, \infty] - 1 \quad (\forall g \in G, h \in H)
\]
Then \((G, H, \langle, \rangle)\) is a vector space duality.

**Proof.** Let \( (v, \omega) \) be a vector representation of \( (P, Q, \ldots) \) on a vector duality \((V, W, \langle, \rangle)\). Choose \( v_0 \in \nu(o) \) and \( w_{\infty} \in \omega(\infty) \) such that \( \langle v_0, w_{\infty} \rangle = 1 \). Let \( g_1, g_2 \in G \) be distinct from \( o \) and from each other. Choose \( v_1 \in \nu(g_1) \) and \( v_2 \in \nu(g_2) \) to satisfy \( \langle v_1, w_{\infty} \rangle = 1 = \langle v_2, w_{\infty} \rangle \). Then \( \langle v_2, w_{\infty} \rangle v_1 / \langle v_1, w_{\infty} \rangle \) is distinct from \( v_2 \) and so there exists \( m \in W \) such that
\[
\langle v_2, w_{\infty} \rangle \langle v_1, m \rangle / \langle v_1, w_{\infty} \rangle \neq \langle v_2, m \rangle.
\]
If \( \langle v_0, m \rangle = 0 \), replace \( m \) by \( m + w_{\infty} \). Then the above equation still holds and \( \langle v_0, m \rangle \neq 0 \). Thus \( \omega^{-1}(m) \) is in \( H \) and
\[
1 + \langle g_2, \omega^{-1}(m) \rangle = [g_2, o, \omega^{-1}(m), \infty] = \langle v_2, m \rangle \langle v_0, w_{\infty} \rangle / (\langle v_2, w_{\infty} \rangle \langle v_0, m \rangle)
\]
\[
\neq \langle v_1, m \rangle / (\langle v_1, w_{\infty} \rangle \langle v_0, m \rangle)
\]
\[
= [g_1, o, \omega^{-1}(m), \infty]
\]
\[
= 1 + \langle g_1, \omega^{-1}(m) \rangle.
\]
For \( h_1, h_2 \in H \) distinct from \( o \) and each other, one finds in a similar manner an element \( g \in G \) such that
\[
\langle g, h_2 \rangle \neq \langle g, h_1 \rangle.
\]
We have established Items (iii) and (iv) of Definition 2.2.

Let \( g_1 \) and \( g_2 \) in \( G \) be generic. Let \( \theta \) and \( \tau \) be the elements of \( \text{Tran}(P; q) \) such that \( \theta(o) = g_1 \) and \( \tau(o) = g_2 \). Let \( v_1 \in \nu(g_1) \) and \( v_2 \in \nu(g_2) \) be such that \( \langle v_1, w_{\infty} \rangle = 1 = \langle v_2, w_{\infty} \rangle \). Let \( \sigma \) and \( \rho \) be defined on \( V \) by
\[
\sigma(v) := v + \langle v, w_{\infty} \rangle (v_2 - v_0) \quad \text{and} \quad \rho(v) := v + \langle v, w_{\infty} \rangle (v_1 - v_0) \quad (\forall v \in V).
\]
Evidently
\[
\nu \circ \theta(p) = \rho \circ \nu(p) \quad \text{and} \quad \nu \circ \tau(p) = \sigma \circ \nu(p) \quad (\forall p \in P).
\]
Let \( h \in H \) be generic and choose \( n \in h \) such that \( \langle v_o, n \rangle = 1 \). Denote by \( '+' \) the addition in \( G \). Then
\[
\langle g_1 + g_2, h \rangle = [g_1 + g_2, o, h, \infty] - 1 = [\theta \circ \tau(o), o, h, \infty] - 1
\]
\[
= [\rho \circ \sigma(v_o), v_0, n, w_{\infty}] - 1.
\]
\[\begin{align*}
\mathcal{V}(v_2, v_5, n, w_{\infty}) - 1 &= \langle v_2 + v_1, n \rangle - 2 \\
\langle v_2, v_5, n, w_{\infty} \rangle - 1 &= ((v_2, n) - 1) + ((v_1, n) - 1) \\
\langle v_2, v_5, n, w_{\infty} \rangle - 1 + ([v_1, v_5, n, w_{\infty}] - 1) &= \langle g_1, h \rangle + \langle g_2, h \rangle.
\end{align*}\]

For \( u \in \mathbb{F} \) we have
\[\mathcal{V} \circ (u \theta)(p) = \mathcal{V} \circ \mathcal{V}(p) \quad (\forall p \in P)\]
where \( \mathcal{V} \) is defined by
\[\mathcal{V}(v) := v + (v, w_{\infty})(u(v_1 - v_5)) \quad (\forall v \in V).\]

We have
\[\begin{align*}
\langle ug_1, h \rangle &= [ug_1, o, h, \infty] - 1 = [(u \theta)(o), o, h, \infty] - 1 \\
&= [\pi(v_5), v_5, n, w_{\infty}] - 1 \\
&= [v_5 + u(v_1 - v_5), v_5, n, w_{\infty}] - 1 \\
&= (1 - u) + u(v_1, n) - 1 \\
&= u([v_1, n] - 1) \\
&= u([v_1, v_5, n, w_{\infty}] - 1) \\
&= u([\mathcal{V}[v_5], v_5, n, w_{\infty}] - 1) \\
&= u([g_1, o, h, \infty] - 1) \\
&= u(g_1, h).
\end{align*}\]

This establishes Item (i) of Definition 2.2. An analogous argument proves Item (ii) of Definition 2.2.

**Definition 4.30.** The vector duality of Theorem 4.29 will be called the standard vector duality induced by the standard pair \((o, \infty)\).

**Remark 4.31.** Given a scalar automorphism \( \varphi \) of a projective space \( P \), how does one determine \( \text{scal}_p(\varphi) \)? The following theorem provides an answer.

**Theorem 4.32.** Let \((P, Q, [\cdot, \cdot])\) be a projective duality. Let \( p \) and \( c \) be distinct points in \( P \) and let \( x, y \in \text{line}_P(p, c) \) be distinct from each other and from \( p \) and \( c \). Let \( q, d \in Q \) be such that \( c = q^\circ \cap \text{line}_P(p, c) \) and \( P = d^\circ \cap \text{line}_P(p, c) \). Then
\[u_{(p, q)}(x) = y \quad \text{where} \quad u := [x, y, q, d].\]

**Proof.** By Part (i) of Theorem 4.6 we have \( u_{(p, q)}(p) = p \) and \( u_{(p, q)}(c) = c \). Since \( u_{(p, q)} \) is an automorphism, it preserves lines: hence \( u_{(p, q)}(\text{line}_P(p, c)) = \text{line}_P(p, c) \). It follows from Theorem 2.10 that \( y \) is the only point \( t \) on \( \text{line}_P(p, c) \) such that \([x, t, q, d] = [x, y, q, d] \). From Part (ii) of Theorem 4.6 we have \([x, u_{(p, q)}(x), q, d] = u = [x, y, q, d] \), whence follows that \( u_{(p, q)}(x) = y \).
Direction 4.33. The major use of vector dualities in this paper is to serve as vector representation spaces of projective dualities. Consider a standard duality \((G, H, \langle, \rangle)\) induced by a standard pair \((o, \varphi)\) in a projective duality \((P, Q, [\cdot, \cdot])\). In a vector representation of \(P\), points in \(P\) correspond to lines in the representation space—thus the representation space must be one dimension greater than that of \(P\). We shall see in §6, the section in which topology is treated, that \(P\) and \(G\) are of the same dimension and \(Q\) and \(H\) are of the same dimension. In this sense then, \(G\) and \(H\) are too small to serve as representation spaces for the projective duality \((P, Q, [\cdot, \cdot])\).

Consequently, one enlarges \(G\) and \(H\) by forming direct sums \(G \oplus F\) and \(H \oplus F\). It shall be seen that there is a natural representation \((\varphi, \psi)\) of \((P, Q, [\cdot, \cdot])\) on \((G \oplus F, H \oplus F, \langle, \rangle)\). The binary operations on the vector spaces \(G\) and \(H\) are carried over by \(\varphi\) and \(\psi\) to operations on \(G \oplus F\) and \(H \oplus F\). To see how this is done, we begin by defining new binary operations on \(G \oplus F\) and \(H \oplus F\).

Notation 4.34. Let \(G\) be a vector space over the field \(F\). We define the binary operations \(\circ\) on \(F \times (G \oplus F)\), \(\cup\) on \((G \oplus F) \times (G \oplus F)\) and \(\ll, \gg\) on \((G \oplus F) \times (G \oplus F)\) by

\[
\begin{align*}
    u_1 \circ (g \oplus u_2) & := (u_1g) \oplus u_2, \\
    (g_1 \oplus u_1) \cup (g_2 \oplus u_2) & := (u_2g_1 + u_1g_2) \oplus (u_1u_2) \quad \text{and} \\
    \ll (g_1 \oplus u_1), (g_2 \oplus u_2) \gg & := \langle u_2g_1, u_1g_2 \rangle + u_2u_1
\end{align*}
\]

for all \(g, g_1, g_2 \in G\) and \(u_1, u_2 \in F\) respectively.

Theorem 4.35. Let \(G\), \(\circ\), \(\cup\) and \(\ll, \gg\) be as in Notation 4.34. Then, for all \(a, b \in G \oplus F\), \(a_1, a_2 \in a\), \(b_1, b_2 \in b\), and \(u \in F\)

\[
\ll a_1, b_1 \gg \ll a_2, b_2 \gg, \quad a_1 \cup b_1 = a_2 \cup b_2 \quad \text{and} \quad u \circ a_1 = u \circ a_2.
\]

Proof. Direct computation.

Definition 4.36. Let \(G\), \(\circ\), \(\cup\) and \(\ll, \gg\) be as in Notation 4.34. We define

\[
\begin{align*}
    u_1 \circ (g \oplus u_2) & := (u_1g) \oplus u_2, \\
    g_1 \oplus u_1 \cup g_2 \oplus u_2 & := (u_2g_1 + u_1g_2) \oplus (u_1u_2) \quad \text{and} \\
    \ll (g_1 \oplus u_1), (g_2 \oplus u_2) \gg & := \langle u_2g_1, u_1g_2 \rangle + u_2u_1
\end{align*}
\]

for all \(g, g_1, g_2 \in G\) and \(u_1, u_2 \in F\) respectively.

Theorem 4.37. Let \((P, Q, [\cdot, \cdot])\) be a projective duality and \((o, \varphi)\) a standard pair. Let \((G, H, \langle, \rangle)\) be the standard duality induced by \((o, \varphi)\). Define \(\varphi\) from \(P\) to \(G \oplus F\) by

\[
\varphi(p) := p \oplus 1 \quad (\forall p \in G) \quad \text{and} \quad \varphi(p) := \{p \oplus 0 : \overline{p} \in \text{line}_p(o, p)\} \quad (\forall p \in \infty^o).
\]

Let \(\circ\) and \(\cup\) be as in Notation 4.36. Then

\[
\varphi(p_1 + p_2) = \varphi(p_1) \cup \varphi(p_2) \quad \text{and} \quad \varphi(up) = u \circ \varphi(p) \quad (\forall p_1, p_2, p \in G) \quad (\forall u \in F).
\]
Define \( \psi \) from \( Q \) to \( H \oplus F \) by

\[
\psi(q) := q \oplus 1 \quad (\forall q \in H) \quad \text{and} \quad \psi(q) := \{ q \oplus 0 : \bar{q} \in \text{line}_Q(\infty, q) \} \quad (\forall q \in \infty).
\]

Then

\[
\psi(q_1 + q_2) = \psi(q_1) \cup \psi(q_2) \quad \text{and} \quad \psi(uq) = u \circ \psi(q) \quad (\forall q_1, q_2, q \in H) \quad (\forall u \in F).
\]

The pair \( (\varphi, \psi^{-1}) \) is a projective duality isomorphism from \( (P, Q, [\ldots]) \) to \( (G \oplus F, H \oplus F, [\ldots]) \) and

\[
\varphi(p, \psi(q)) \Rightarrow [p, o, q, \infty] \quad (\forall p \in G, q \in H).
\]

**Proof.** Verification of the equations in the first paragraph of the theorem is direct. That \( \varphi(p), \psi(q) \Rightarrow [p, o, q, \infty] \) for all \( p \in G, q \in H \) is a direct calculation.

It remains to show that \( \psi^{-1} \) is the adjoint of \( \varphi \): that \( [\varphi(p_1), \varphi(p_2), \psi(q_1), \psi(q_2)] = [p_1, p_2, q_1, q_2] \) for all \( p_1, p_2 \in P \) and \( q_1, q_2 \in Q \) for which the cross-ratio exists. Let \( p_1, p_2 \in G \) and \( q_1, q_2 \in H \) be generic. Let \( (\nu, \omega) \) be a vector representation of \( (P, Q, [\ldots]) \) on a vector duality \( (V, W, \langle \cdot, \cdot \rangle) \). Let \( v_0 \in \nu(o) \) and \( w_{\infty} \in \omega(\infty) \) be such that \( \langle v_0, w_{\infty} \rangle = 1 \). Let \( v_1 \in \nu(p_1), v_2 \in \nu(p_2), w_1 \in \omega(q_1) \) and \( w_2 \in \omega(q_2) \) satisfy

\[
\langle v_1, w_{\infty} \rangle = \langle v_2, w_{\infty} \rangle = \langle v_0, w_1 \rangle = \langle v_0, w_2 \rangle = 1.
\]

We have

\[
[\varphi(p_1), \varphi(p_2), \psi(q_1), \psi(q_2)] = \langle p_1, q_1 \rangle \langle p_2, q_2 \rangle \langle p_2, q_1 \rangle \langle p_1, q_2 \rangle
\]

\[
= [p_1, o, q_1, \infty][p_2, o, q_2, \infty][p_2, o, q_1, \infty][p_1, o, q_2, \infty]
\]

\[
= [v_1, v_0, w_1, w_{\infty}][v_2, v_0, w_2, w_{\infty}][v_2, v_0, w_1, w_{\infty}][v_1, v_0, w_2, w_{\infty}]
\]

\[
= \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle / \langle v_1, w_2 \rangle \langle v_2, w_1 \rangle
\]

\[
= [v_1, v_2, w_1, w_2]
\]

\[
= [p_1, p_2, q_1, q_2].
\]

provided that the cross-ratio exists. The other cases are trivial but tedious.

**Definition 4.38.** Let \( (P, Q, [\ldots]) \) be a projective duality and \( (o, \infty) \) a standard pair. Let \( (G, H, \langle \cdot, \cdot \rangle) \) be the standard duality induced by \( (o, \infty) \). The map \( (\varphi, \psi^{-1}) \) defined in Theorem 4.37 will be called the standard projective duality isomorphism of \( (P, Q, [\ldots]) \) onto \( (G \oplus F, H \oplus F, [\ldots]) \). The maps \( \varphi \) and \( \psi \) will be called the standard projective isomorphisms from \( P \) to \( G \oplus F \) and from \( Q \) to \( H \oplus F \), respectively.

**Philosophy 4.39.** Our treatment of projective spaces depends *ab initio* on that algebraic entity known as a field (or, more precisely, a field of characteristic not equal to 2). For the cross-ratio has the field as its range, and the vector representation of a projective duality requires a field of scalars.
There is however, something incongruous in the adoption of a sophisticated algebraic structure to define what essentially is a geometric concept. Signal evidence of this appears at first appearance of the cross-ratio, where an ad hoc object “infinity” must be adduced to express a quotient of which the denominator has the field value 0.

We have begun with a field because nearly every reader will have some familiarity with one, and because the theory of linear spaces over a field is temptingly available. Although convenient, beginning with a field is nevertheless delusive. What is required for the range of the cross-ratio is a one-dimensional projective space with three distinguished points (which correspond to 0, 1 and $\infty$ in $\mathbb{F}_\infty$). Such an entity seems to the authors to be more geometric than algebraic in nature.

The reader may immediately object, “But how can one define a projective space in terms of a projective space?” The answer is that a minimal projective space can be defined quite simply as a geometrical object (which we shall term a meridian), without any allusion to algebraic structure. We conclude this section by indicating how this can be accomplished.

**Definition 4.40.** By an involution we mean a permutation equal to its own inverse. A nontrivial involution is an involution not equal to the identity. A meridian $(L, \mathcal{F})$ consists of a nonvoid set $L$ and a nonvoid family $\mathcal{F}$ of nontrivial involutions of $L$ satisfying the following three axioms:

(i) for all $(l_1, m_1), (l_2, m_2) \in L \times L$ where $l_1 \neq l_2$, $m_1 \neq m_2$, $m_1 \neq l_2$ and $m_2 \neq l_1$, there exists a unique $\sigma \in \mathcal{F}$ such that $\sigma(l_1) = m_1$ and $\sigma(l_2) = m_2$;

(ii) for all $\alpha, \beta \in \mathcal{F}$ the map $\alpha \circ \beta \circ \alpha$ is in $\mathcal{F}$;

(iii) if $\alpha, \beta, \gamma \in \mathcal{F}$ are such that $\alpha(l) = \beta(l) = \gamma(l)$ for some $l \in L$, then $\alpha \circ \beta \circ \gamma$ is in $\mathcal{F}$.

**Discussion 4.41.** Axiom (i) of Definition 4.40 is a limited “homogeneity” condition. This axiom shows that an element $\sigma$ of $\mathcal{F}$ is determined by its values at two points $l_1$ and $l_2$ (thus the requirement $l_1 \neq l_2$). The map $\sigma$ is injective (which requires $m_1 \neq m_2$). If $m_1 = l_2$ then $\sigma(l_2) = \sigma^{-1}(m_1) = l_1$, or if $m_2 = l_1$, then $\sigma(l_1) = \sigma^{-1}(m_2) = l_2$, and so, in either of these cases, the value of $\sigma(l_2)$ is already determined by the value $\sigma(l_1)$ (whence the requirements $m_1 \neq l_2$ and $m_2 \neq l_1$).

The composition of two involutions $\alpha$ and $\beta$ is an involution only if $\alpha \circ \beta = \beta \circ \alpha$. This is a stringent requirement. It is much more common for the composition $\alpha \circ \beta \circ \gamma$ of three involutions to be an involution. In fact, if $\alpha$ and $\beta$ are involutions, the “similarity” mapping $\alpha \circ \beta \circ \alpha^{-1} = \alpha \circ \beta \circ \alpha$ is always an involution. Axiom (ii) in Definition 4.40 is motivated by these considerations.

Axiom (i) implies that it takes only two points to determine an element of $\mathcal{F}$. If three elements of $\mathcal{F}$ agree at one of these points, then Axiom (iii) of Definition 4.40 requires that their composition be in $\mathcal{F}$. We note that if $\alpha \circ \beta \circ \gamma$ is in $\mathcal{F}$, then $\alpha \circ \beta \circ \gamma = (\alpha \circ \beta \circ \gamma)^{-1} = \gamma \circ \beta \circ \alpha$.

**Theorem 4.42.** Let $(L, \mathcal{F})$ be a meridian and let $\alpha, l$ and $\infty$ be three distinct points in $L$. Let $\mathbb{F} := L \setminus \{\infty\}$. For each $u \in \mathbb{F}$ there exists a unique composition
\(\theta_{(o,w,\infty)}\) of two elements of \(\mathcal{I}\) which keep \(\infty\) fixed and is such that \(\theta_{(o,w,\infty)}(o) = u\).

For each \(t \in \mathbb{F}\setminus\{o\}\) there exists a unique composition \(\pi_{(o,\infty,t)}\) of two elements of \(\mathcal{I}\) such that \(\pi_{(o,\infty,t)}(o)\) keeps \(o\) and \(\infty\) fixed while sending \(l\) to \(t\). Define the operations + and \(\cdot\) by:

\[
\begin{align*}
t + u &= \theta_{(o,w,\infty)}(t) \quad (\forall t, u \in \mathbb{F}); \\
t \cdot u &= \pi_{(o,\infty,t)}(u) \quad (\forall t, u \in \mathbb{F}\setminus\{0\}) \quad \text{and} \quad 0 \cdot u := u \cdot 0 := 0.
\end{align*}
\]

Let \(\mathcal{L}\) be the set of all \((l_1, l_2, l_3, l_4) \in L \times L \times L \times L\) for which no three \(l_i, i = 1, 2, 3, 4\) are identical, and define

\[
[l_1, l_2, l_3, l_4] := \infty \quad (\forall (l_1, l_2, l_3, l_4) \in \mathcal{L} : l_1 = l_4 \text{ or } l_2 = l_3)
\]

and

\[
[l_1, l_2, l_3, l_4] := \frac{(l_1 - l_3)(l_2 - l_4)/((l_1 - l_4)(l_2 - l_3))}{(\forall (l_1, l_2, l_3, l_4) \in \mathcal{L} : l_1 \neq l_4, l_2 \neq l_3)}.
\]

Then

(i) \((\mathbb{F}, +, \cdot)\) is a field (of characteristic not equal to 2) with multiplicative identity \(l\);

(ii) \((L, L, [], \ldots)\) is a projective duality;

(iii) \((\mathbb{F}, +)\) is a standard vector space in \(L\).

**Proof.** The two assertions in the theorem preceding the main assertions (i), (ii) and (iii) follow from the proof of [3, Lemma 3.6]. Assertion (i) is proved in [3, Proof of Theorem 3.7].

To prove (ii) and (iii), let \(V\) and \(W\) be the vector spaces \(\mathbb{F} \oplus \mathbb{F}\) over \(\mathbb{F}\). For each \(\overline{t} \in \mathbb{F}\) let \(\nu(\overline{t}) := \overline{t} \oplus l\), \(\omega(\overline{t}) := -\overline{t} \oplus l\) and let \(\nu(\infty) := l \oplus o\) and \(\omega(\infty) := o \oplus l\). For \((l_1, l_2, l_3, l_4) \in \mathcal{L}\) with \(l_1, l_2, l_3\) and \(l_4\) not equal to \(\infty\) we have

\[
[\nu(l_1), \nu(l_2), \omega(l_3), \omega(l_4)] = [l_1 \oplus l, l_2 \oplus l, -l \oplus l_3, -l \oplus l_4]
\]

\[
= \frac{(l_1 \oplus l, -l \oplus l_3)(l_2 \oplus l, -l \oplus l_4)}{(l_1 \oplus l, -l \oplus l_3)(l_2 \oplus l, -l \oplus l_3)}
\]

\[
= (l_3 - l_1) \cdot (l_4 - l_2)/((l_4 - l_1) \cdot (l_3 - l_2))
\]

\[
= [l_1, l_2, l_3, l_4].
\]

That \([\nu(l_1), \nu(l_2), \omega(l_3), \omega(l_4)] = [l_1, l_2, l_3, l_4]\) for other \((l_1, l_2, l_3, l_4) \in \mathcal{L}\) is a simple computation. Thus \((L, \mathcal{I}, [\ldots])\) is a projective duality (and \((\nu, \omega)\) is a vector representation for it). That \((\mathbb{F}, +)\) is a standard vector space in \(L\) is now trivial.

**Theorem 4.43.** Let \(\mathbb{F}\) be a field of characteristic not equal to 2. Let \(L\) be a projective completion of \(\mathbb{F}\). We shall regard \(\mathbb{F}\) as a subset of \(L\) and write \(\infty\) for the element of \(L\) not in \(\mathbb{F}\). For \(t_1, t_2, t_3 \in \mathbb{F}\) such that \(t_1^2 \neq t_2 t_3\), let \(\sigma_{(t_1, t_2, t_3)}\) be the function on \(L\) defined by

\[
\sigma_{(t_1, t_2, t_3)}(l) := \begin{cases} 
(t_1 l + t_2)/(t_3 l - t_1) & (\forall l \in \mathbb{F}) \\
t_1/t_3 & \text{if } l = \infty \text{ and } t_3 \neq 0 \\
\infty & \text{if } l = \infty \text{ and } t_3 = 0.
\end{cases}
\]

Let \(\mathcal{I}\) be the set of all such functions \(\sigma_{(t_1, t_2, t_3)}\). Then
(i) \((L, \mathcal{P})\) is a meridian;

(ii) if \(\{\infty\} := L \setminus \mathbb{F}, \alpha := 0\) and \(l := 1\), then the operations \(+\) and \(\cdot\) on \(\mathbb{F}\) are just those defined in Theorem 4.42.

**Proof.** See [3, Proof of Theorem 3.9].

5. **Projective Representations and Projective Isomorphisms of Vector Spaces**

**Definition 5.1.** Let \(V\) be a vector space over \(\mathbb{F}\). A linear isomorphism of \(V\) onto a standard vector space \((\mathbb{P}, o, \cdot, \cdot)\) will be called a **projective representation of \(V\)** on the projective space \(\mathbb{P}\).

**Example 5.2.** Let \((V, W, \langle ., \rangle)\) be a vector duality. Consider the vector duality \((V \oplus \mathbb{F}, W \oplus \mathbb{F}, \langle ., \rangle)\) where

\[
\langle v + u, w + t \rangle = \langle v, w \rangle + ut \quad (\forall v, w \in W) \quad (\forall u, t \in \mathbb{F}).
\]

The projective duality inherent in \((V \oplus \mathbb{F}, W \oplus \mathbb{F}, \langle ., \rangle)\) is denoted by \((V \oplus \mathbb{F}, W \oplus \mathbb{F}, \langle ., \rangle)\) (cf. Example 2.11 supra). If \(o\) denotes the origin of \(V\) and \(\infty\) the origin of \(W\), then \((o \oplus 1)\) together with \((\infty \oplus 1)\) comprise a standard pair for \((V \oplus \mathbb{F}, W \oplus \mathbb{F}, \langle ., \rangle)\). Then the standard vector space \((V \oplus \mathbb{F}, o \oplus 1), (\infty \oplus 1)\) consists of the set \(\{v + u : v \in V, u \in \mathbb{F} \setminus \{0\}\}\). The vector binary operations reduce to

\[
v_1 \oplus u_1 + v_2 \oplus u_2 = (u_2 v_1 + u_1 v_2) \oplus (u_1 u_2) \quad (\forall v_1, v_2 \in V) \quad (\forall u_1, u_2 \in \mathbb{F});
\]

\[
t(v \oplus u) = (tv \oplus u) \quad (\forall v \in V) \quad (\forall u, t \in \mathbb{F}).
\]

The **standard projective representation of \(V\) on \(V \oplus \mathbb{F}\)** relative to the duality \((V, W, \langle ., \rangle)\) is the linear isomorphism \(\lambda\) defined by

\[
\lambda(v) := v \oplus 1 \quad (\forall v \in V).
\]

**Definition 5.3.** Let \(\varphi\) and \(\psi\) be projective representations of a vector space \(V\) onto standard vector spaces \((P, p, q)\) and \((R, r, s)\) respectively. We say that \(\varphi\) is **equivalent to \(\psi\)** if there exists a projective isomorphism \(\theta\) of \(P\) onto \(R\) such that \(\theta \circ \varphi = \psi\).

If \(\varphi\) is equivalent to the standard projective representation of \(V\) relative to a duality \((V, W, \langle ., \rangle)\), then we say that \(\varphi\) is a **projective representation of \(V\)** relative to the duality \((V, W, \langle ., \rangle)\).

**Example 5.4.** Let \(V\) be a finite dimensional vector space. Recalling Example 2.13 we have the standard projective space \((V \oplus \mathbb{F}, V^* \oplus \mathbb{F}, \langle ., \rangle)\) generated by \(V\). The **standard projective representation of \(V\) on \(V \oplus \mathbb{F}\)** is defined to be the standard projective representation of \(V\) on \(V \oplus \mathbb{F}\) relative to the duality \((V, V^*, \langle ., \rangle)\).
Example 5.5. In this example we take \( \mathbb{F} \) to be either the field of real numbers or the field of complex numbers. Let \( V \) be vector space over \( \mathbb{F} \) equipped with a topology relative to which it is a topological vector space (i.e. addition and scalar multiplication is continuous). Suppose that the set \( V' \) of continuous linear functionals on \( V \) separates points of \( V \). We describe this situation in abbreviated manner by saying that \( V \) is a dual topological vector space. The standard projective representation of \( V \) on \( V \oplus \mathbb{F} \) is defined to be the standard projective representation of \( V \) on \( V \oplus \mathbb{F} \) relative to the duality \( (V, V', \langle \cdot, \cdot \rangle) \).

Example 5.6. Let \( V \) be a real Hilbert Space. Then the inner product \( \langle \cdot, \cdot \rangle \) on \( H \) provides a vector space duality \( (V, V, \langle \cdot, \cdot \rangle) \). The standard projective representation of the Hilbert Space \( V \) on \( V \oplus \mathbb{F} \) is defined to be the standard projective representation of \( V \) on \( V \oplus \mathbb{F} \) relative to the duality \( (V, V, \langle \cdot, \cdot \rangle) \).

Discussion 5.7. We have seen that a projective space has many subsets which are vector spaces. A projective isomorphism \( \varphi : P \to R \) maps each of these standard vector spaces \( V = (P, o, \infty) \) to a standard vector space of the range and its restriction to the standard vector space is then a linear isomorphism. If the range of \( \varphi \) is viewed as the projective space completion of a standard vector space \( X \) different from \( \varphi(V) \) however, the question arises as to what the character of \( \varphi \) is relative to the operations on \( V \) and \( X \).

This problem typically arises in applications as follows. There are given vector space dualities \( (V, W, \langle \cdot, \cdot \rangle) \) and \( (X, Y, \langle \cdot, \cdot \rangle) \), a subset \( D \) of \( V \), and an injective map \( \sigma \) from \( D \) into \( X \). The question then is what are necessary and sufficient conditions on \( \sigma \) for there to exist a projective representation \( \lambda \) of \( V \) relative to the duality \( (V, W, \langle \cdot, \cdot \rangle) \), a projective representation \( \mu \) of \( X \) relative to the duality \( (X, Y, \langle \cdot, \cdot \rangle) \), and a projective isomorphism \( \varphi \) from \( P \) to \( R \) such that

\[
\mu^{-1} \circ \varphi \circ \lambda(v) = \sigma(v) \quad (\forall v \in D).
\]

A map \( \sigma \) for which the above holds will be said to be a local projective isomorphism of vector spaces in the context of the given vector dualities, or, in abbreviated fashion, a projective map.

The following theorem gives necessary conditions.

**Theorem 5.8.** Let \( (V, W, \langle \cdot, \cdot \rangle) \) and \( (X, Y, \langle \cdot, \cdot \rangle) \) be vector dualities. Let \( \overline{v} \in V \), \( \overline{w} \in W \) and \( M := \{v + w : v \in \overline{v}^\perp\} \). Let \( \sigma : V \setminus M \to X \) be a local projective isomorphism of vector spaces. Then there exists an affine map \( \alpha \) of \( V \) into \( X \) and an affine functional \( a : V \to F \) such that

\[
\sigma(v) = \alpha(v)/a(v) \quad (\forall v \in V \setminus M).
\]

**Proof.** Let \( \lambda \) be a projective representation of \( V \) on a standard projective space \( P \), \( \mu \) a projective representation of \( X \) on a projective space \( R \), and \( \varphi \) a projective isomorphism from \( P \) onto \( R \) such that

\[
\mu^{-1} \circ \varphi \circ \lambda(v) = \sigma(v) \quad (\forall v \in V \setminus (\overline{v} + \overline{w}^\perp)).
\]
We may, and shall, assume that $\lambda$ is the standard projective representation of $V$ on $V \oplus \mathbb{F}$ relative to the duality $(V, W, \langle \cdot, \cdot \rangle)$, and that $\mu$ is the standard projective representation of $X$ on $X \oplus \mathbb{F}$ relative to the duality $(X, Y, \langle \cdot, \cdot \rangle)$. Let $\psi$ be the projective isomorphism such that $(\varphi, \psi)$ is a projective duality isomorphism.

Let $o$ denote the origin in $V$ and choose $\overline{v} \in X$ such that $\varphi(o \oplus 1) = \overline{v} \oplus 1$. Let $\infty$ denote the identity in $W$; then $(o \oplus 1, \infty \oplus 1) = 1$ and by Theorem 3.2 there exists a unique linear isomorphism $\theta$ from $V \oplus \mathbb{F}$ onto $X \oplus \mathbb{F}$ such that

\[
\theta(o \oplus 1) = \overline{v} \oplus 1, \quad \text{and} \\
\theta(v \oplus u) = \varphi(v \oplus u) \quad (\forall v \in V, u \in \mathbb{F}).
\]

Define the maps $\alpha : V \to X$ and $a : V \to \mathbb{F}$ by

\[
\theta(v \oplus 1) = \alpha(v) \oplus a(v) \quad (\forall v \in V).
\]

For each $v \in V \setminus M$ we have

\[
\sigma(v) = \mu^{-1} \circ \varphi \circ \lambda(v) = \mu^{-1} \varphi(v \oplus 1) = \mu^{-1} \theta(v \oplus 1) = \mu^{-1}((\rho(v) \oplus a(v)) \oplus 1) = \alpha(v) / a(v).
\]

We conclude this proof by showing that $\alpha$ and $a$ are affine. Let $v_1, v_2 \in V$ and $u \in \mathbb{F}$ be generic. Then

\[
\alpha(((1 - u)v_1 + uv_2) \oplus ((1 - u)v_1 + uv_2)) = \theta(((1 - u)v_1 + uv_2) \oplus 1) = \theta(((1 - u)v_1 \oplus (1 - u)) + \theta((uv_2) \oplus u) = (1 - u)\theta(v_1 \oplus 1) + u\theta(v_2 \oplus 1) = (1 - u)\alpha(v_1) \oplus (1 - u)a(v_1) + ua(v_2) \oplus ua(v_2) = ((1 - u)\alpha(v_1) + ua(v_2)) \oplus ((1 - u)a(v_1) + ua(v_2)).
\]

**Corollary 5.9.** Let $(V, W, \langle \cdot, \cdot \rangle)$ and $(X, Y, \langle \cdot, \cdot \rangle)$ be vector dualities. Let $\sigma : V \to X$ be a projective local isomorphism of vector spaces. Then there exists a linear isomorphism $\tau$ of $V$ onto $X$ and $\overline{v} \in X$ such that

\[
\sigma(v) = \tau(v) + \overline{v} \quad (\forall v \in V).
\]

**Proof.** Let $\overline{v}$ and $\overline{w}$ be origins of $V$ and $W$ respectively in Theorem 5.8. Then $M$ is void and so $\alpha/a$ being defined on all $V$, it follows that $a$ must be a constant $c$. Since $\alpha/c$ is affine, the function $\tau := (\alpha - \alpha(o))/c$ is linear. That $\tau$ is an isomorphism follows from the fact that $\sigma$ is an isomorphism.

**Corollary 5.10.** Let $(V, W, \langle \cdot, \cdot \rangle)$ and $(X, Y, \langle \cdot, \cdot \rangle)$ be vector dualities. Let $\overline{v} \in V$, $\overline{w} \in W \setminus o$ and $M := \{\overline{v} + v : v \in \overline{w}^{-1}\}$. Let $\sigma : V \setminus M \to X$ be a local projective isomorphism of vector spaces which cannot be extended to all of $V$. Then there exists a linear homomorphism $\tau$ of $V$ into $X$, a constant $\overline{v} \in X$, a linear functional $\ell : V \to \mathbb{F}$, and $c \in \mathbb{F}$ such that

\[
\sigma(v) = (\tau(v) + \overline{v})/(\ell(v) + c) \quad (\forall v \in V \setminus M).
\]
Proof. That \( \sigma \) cannot be extended to all of \( V \) implies that the function \( a \) of Theorem 5.8 is not a constant.

Remark 5.11. We have a partial converse to Corollary 5.10, which seems to include most applications.

Theorem 5.12. Let \( V \) and \( X \) be vector spaces over a field \( F \) of equal dimension. Suppose that either

(i) \( V \) and \( X \) are finite dimensional; or

(ii) \( V \) and \( X \) are dual topological vector spaces over either the real or the complex field.

Let \( \tau \) be a linear homomorphism of \( V \) into \( X \), \( \overline{x} \) an element of \( X \), \( \ell \) a linear functional on \( V \) and \( c \) a scalar such that \( \{ v \in V : \tau(v) = \overline{x}, \ell(v) = c \} \) is void and that \( \tau/\ell \) is injective on \( V \). Then, in Case (i) the function \( \sigma \) defined by

\[
\sigma(v) := (\tau(v) - \overline{x})/(\ell(v) - c) \quad (\forall v \in V : \ell(v) \neq c)
\]

is a local projective isomorphism of vector spaces in the context of the vector dualities \((V, V', \langle \cdot, \cdot \rangle)\) and \((X, X', \langle \cdot, \cdot \rangle)\) where \( V' \) and \( X' \) denote the sets of linear functionals on \( V \) and \( X \) respectively. In Case (ii) also \( \sigma \) is a local projective isomorphism of vector spaces provided that \( \sigma \) is homeomorphic and \( V' \) and \( X' \) denote sets of continuous linear functionals on \( V \) and \( X \) respectively.

Proof. If \( c = 0 \), then \( \ell \) is not identically 0 and we can find \( \overline{x} \in V \) such that \( \ell(\overline{x}) = 1 \). The function \( \sigma \) is a projective map if it is its translate:

\[
\sigma'(v) := \sigma(v - \overline{x}) \quad (\forall v \in V : \ell(v) \neq c).
\]

Letting \( x' := \tau(\overline{x}) + \overline{x} \) and \( c' := 1 \), we have

\[
\sigma'(v) = (\tau(v) - x')/(\ell(v) - c') \quad (\forall v \in V : \ell(v) \neq c)
\]

whence follows that we may (and shall) take \( c \) to be nonzero.

Let \( M := \{ v \in V : \ell(v) = c \} \). Then \( M \) is either void or a maximal subspace of \( V \) (closed in Case (ii)). Let \( \lambda \) be the standard projective representation of \( V \) on \( V \oplus F \) and \( \mu \) the standard projective representation of \( X \) on \( X \oplus F \) relative to the vector dualities \((V, V', \langle \cdot, \cdot \rangle)\) and \((X, X', \langle \cdot, \cdot \rangle)\) respectively. Define \( \varphi \) on \( V \oplus F \) and \( \psi \) on \( X' \oplus F \) by

\[
\begin{align*}
\varphi(v \oplus u) &:= (\tau(v) - ux) \oplus (\ell(v) - uc) \quad (\forall v \oplus u \in V \oplus F); \\
\psi(f \oplus t) &:= (f \circ \tau + t\ell) \oplus (-f(x) - tc) \quad (\forall f \oplus t \in X' \oplus F).
\end{align*}
\]

For \( v_1 \oplus u_1, v_2 \oplus u_2 \in V \oplus F \) and \( f_1 \oplus t_1, f_2 \oplus t_2 \in X' \oplus F \) we have

\[
\begin{align*}
&[v_1 \oplus u_1, v_2 \oplus u_2, \psi(f_1 \oplus t_1), \psi(f_2 \oplus t_2)] \\
= & [v_1 \oplus u_1, v_2 \oplus u_2, (f_1 \circ \tau + t_1 \ell) \oplus (-f_1(x) - t_1 c), (f_2 \circ \tau + t_2 \ell) \oplus (-f_2(x) - t_2 c)] \\
= & f_1 \circ \tau(v_1) + t_1 \ell(v_1) + u_1 (-f_1(x) - t_1 c) \\
= & f_2 \circ \tau(v_1) + t_2 \ell(v_1) + u_1 (-f_2(x) - t_2 c) \\
& f_2 \circ \tau(v_2) + t_2 \ell(v_2) + u_2 (-f_2(x) - t_2 c) \\
& f_1 \circ \tau(v_2) + t_1 \ell(v_2) + u_2 (-f_1(x) - t_1 c) \\
= & [\tau(v_1) - u_1 x) \oplus (\ell(v_1) - u_1 c), (\tau(v_2) - u_2 x) \oplus (\ell(v_2) - u_2 c), f_1 \oplus t_1, f_2 \oplus t_2] \\
= & [\varphi(v_1 \oplus u_1), \varphi(v_2 \oplus u_2), f_1 \oplus t_1, f_2 \oplus t_2].
\end{align*}
\]
which proves that \((\varphi, \psi)\) is a projective duality isomorphism. On \(V \setminus M\) we have
\[
\mu^{-1} \circ \varphi \circ \lambda(v) = \mu^{-1}(\tau(v) - x) \oplus (\ell(v) - c)
\]
\[
= (\tau(v) - x)/(\ell(v) - c)
\]
\[
= \sigma(v).
\]

6. Topological Projective Spaces

**Convention 6.1.** In this section we shall take \(F\) to be either the field of real numbers or the field of complex numbers.

**Remark 6.2.** Infinite dimensional vector spaces usually appear in applications bearing a topology of some sort, and it is this topology which allows the analyst to circumvent to some degree the difficulties incurred by the lack of a finite basis. For projective spaces the situation is similar.

Topologies associated with vector spaces are said to be vector space topologies provided the operations of addition and scalar multiplication are continuous. Projective spaces of course do not come equipped with these binary operations (at least not defined on the entire space). Supplying this want is the family of scalar automorphisms introduced in §4.

**Definition 6.3.** Let \((P, Q, [\ldots])\) be a projective duality, and let \(T\) be a topology on \(P\). Then \((P, T)\) will be said to be a topological projective space provided that

(i) \(\Psi : F \times (P \setminus q_1) \times (P \setminus q_2) \ni (u, p_1, p_2) \mapsto u(p_1, q_1)(p_2) \in (P \setminus q_1)\) is continuous for all \(q_1 \in Q\);

(ii) \(P \ni p_1 \mapsto [p_1, p_2, q_1, q_2]\) is continuous on its domain of definition, for all \(p_2 \in P\) and \(q_1, q_2 \in Q\).

We say that \(T\) is a projective space topology for \(P\).

**Theorem 6.4.** Let \((P, T)\) be a topological projective space. Then

(i) each scalar-automorphism is a homeomorphism;

(ii) the map \(P \times P \times P \ni (p_1, p_2, p) \mapsto \tau_{(p_1, p_2; q)}(p) \in P\) is continuous for all \(q \in Q\).

**Proof.** Evidently each scalar-automorphism is continuous. Since the inverse of a scalar-automorphism is a scalar-automorphism, each element of \(\text{Scal}(P)\) is in fact a homeomorphism.

For \(p_1, p_2 \in P \setminus q\) we write \(p_1/2 + p_2/2\) for the midpoint of \(p_1\) and \(p_2\), \(3p_1/4 + p_2/4\) for the midpoint of \(p_1\) and \(p_1/2 + p_2/2\), and \(p_1/4 + 3p_2/4\) for the midpoint of \(p_1/2 + p_2/2\) and \(p_2\). From Theorem 4.11 we have \(\tau_{(p_1, p_2; q)} = (-1)(3p_1/4 + p_2/4; q) \circ (-1)(p_1/4 + 3p_2/4; q)\). That Part (ii) holds is now evident.
PREVIEW 6.5. It was shown in §4 that a projective space is invested with many standard vector spaces. It happens that in topological projective spaces, these standard vector spaces are topological vector spaces.

We have also seen that each vector space (viewed as half of a vector duality) has a projective completion, a projective space in which the vector space lies as a standard vector space. If that vector space is a topological vector space, the projective completion is a topological projective space in which the vector space is dense. We proceed to the demonstration.

**Theorem 6.6.** Let \((P, T)\) be a topological projective space and let \(\langle P, T \rangle\) be a standard pair. Then the standard vector space \(V = (P, T, R)\), with the topology relativized from \(P\), is a topological vector space.

Furthermore, \(V\) is an open dense subset of \(P\).

**Proof.** For \(v_1, v_2 \in V\) we have \(v_1 + v_2 = \tau_{(P, v_1, T)} \circ \tau_{(P, v_2, T)}(p)\). From Part (ii) of Theorem 6.4 follows that addition is continuous. For \(u \in F\) we have \(up = u_{(P, T)}(p)\) by Definition 4.27. It follows that scalar multiplication is continuous.

To show that \(V\) is open, we prove that \(P \setminus V = \emptyset\) is closed. Let \(p\) be a net in \(P\) with limit \(p\). Assume \(p\) is not in \(P\). Let \(\varphi\) be an element of \(\text{Inv}(P, T)\) which leaves some other point of \(V\) fixed than \(p\). Then \(\varphi\) is continuous and \(p = \varphi(p)\) converges to \(\varphi(p) \neq p\) which is absurd. Hence \(P \setminus V\) is closed.

Let \(p_1 \in P\) be generic. Let \(q \in Q\) be such that \(p_1 \neq q\). Let \(p_2 \in \text{line}_P(\langle P, p_1 \rangle\) be different from \(p_1\) and \(\text{line}_P(\langle P, p_1 \rangle \cap q\). For each \(u \in F \setminus \{0, 1\}\), \(u_{(p_2, q)}(p_1)\) is not \(p_1\) and so in \(V\). As \(u\) converges to \(1\), \(u_{(p_2, q)}(p_1)\) converges to \(1_{(p_2, q)}(p_1) = p_1\). Hence \(V\) is dense in \(P\).

**Theorem 6.7.** Let \((P, T)\) be a topological projective space and let \((p, q)\) be a standard pair. Let \((V, W, \langle \cdot, \cdot \rangle)\) be the standard vector duality induced by \((p, q)\) (cf. Theorem 4.29). For each \(w \in W\), define \(w^{(p, q)}\) on \(V\) by

\[
w^{(p, q)}(v) := \langle v, w \rangle \quad (\forall v \in V).
\]

Then \(\{w^{(p, q)} : w \in W\}\) is a subspace of the set \(V'\) of continuous linear functionals on \(V\).

**Proof.** Follows from Definition 6.3 and the fact that \(\langle v, w \rangle = [v, p, w, q] - 1\).

**Theorem 6.8.** Let \((V, T)\) be a dual topological vector space with identity \(i_V\). Let \(W\) be a vector subspace of \(V'\) which separates points and let \(i_W\) be the identity of \(W\). Consider the projective duality \((V \oplus F, W \oplus F)\). Write \(\lambda\) for the standard representation of \(V\) onto \(V \oplus F\) (cf. Example 5.2). Let \(q_1 \in W \oplus F\) be generic and define \(\Psi\) by

\[
\Psi : (F \setminus \{0\}) \times (V \oplus F \setminus q_1) \times (V \oplus F \setminus q_2) \ni (u, p_1, p_2) \mapsto u_{(p_1, q_1)}(p_2) \in V \oplus F
\]

and define \(\Phi : F \times V \times V \to F \times V \times V\) by

\[
\Phi(u, v_1, v_2) := (u, \lambda(v_1), \lambda(v_2)) \quad (\forall u \in F)(\forall v_1, v_2 \in V).
\]

Then \(\lambda^{-1} \circ \Psi \circ \Phi\) is continuous on \(Q\).
Proof. Consider first the case: \( q_1 = iW \oplus 1 \). Choose \( v_1, v_2 \in V \) distinct and \( u \in F \setminus \{0\} \). Then, writing ‘+’ for addition on \( \lambda(V) \), we have by Corollary 4.26

\[ \Psi \circ \Phi(u,v_1,v_2) = u(\lambda(v_1),q_1)(\lambda(v_2)) = \lambda(v_1) + u(\lambda(v_2) - \lambda(v_1)) \]

which implies

\[ \lambda^{-1} \circ \Psi \circ \Phi(u,v_1,v_2) = v_1 + u(v_2 - v_1). \]

This implies continuity since \( V \) is a topological vector space.

We now suppose that \( q_1 \neq iW \oplus 1 \). Recall from the first paragraph of the proof of Theorem 4.6, that if \( (\nu,\omega) \) is a vector representation of \( (P,Q,[,\ldots,]) \), then \( u_{(\nu,\omega)}(z) = \nu^{-1}(\omega(\nu(z))) \) where \( \psi(z) = z + (u^{-1} - 1)(z,d)c \) with \( c \in \nu(p) \) and \( d \in \omega(q) \) such that \( (c,d) = 1 \). Here we take \( \bigoplus F \) for \( P \), \( \bigoplus F \) for \( Q \), the identity maps for \( \nu \) and \( \omega \), \( q_1 \) for \( q \), and we let \( d \) be any nonzero element of \( q_1 \). For \( p_1 \in (\bigoplus F \setminus q_1^2) \) we shall write \( p_1' \) for the element of \( p_1 \) such that \( (p_1',d) = 1 \). Choose \( w \in W \) and \( t \in F \) such that \( d = w \oplus t \). For \( v \in V \) we have \( \lambda(v,d) = (v \oplus 1, w \oplus t) = (v,w) + t \), whence follows that \( \lambda(v)' = (v \oplus 1)/(\langle v, w \rangle + t) \). Putting all this together we have

\[
\begin{align*}
\lambda^{-1} \circ \Psi \circ \Phi(u,v_1,v_2) &= \\
&= \lambda^{-1} \circ u_{(\lambda(v_1),q_1)}(\lambda(v_2)) \\
&= \lambda^{-1}(v_1 + (u^{-1} - 1)(v_1 \oplus 1, w \oplus t)(v_2 \oplus 1)/(\langle v_2, w \rangle + t)) \\
&= \lambda^{-1}(v_1 + (u^{-1} - 1)((v_1, w) + t)(v_2 \oplus 1)/(\langle v_2, w \rangle + t)) \\
&= \lambda^{-1}((v_1 + v_2(u^{-1} - 1)((v_1, w) + t)v_2/(\langle v_2, w \rangle + t) + 1) + 1) \\
&= (v_1 + v_2(u^{-1} - 1)((v_1, w) + t)v_2/(\langle v_2, w \rangle + t))/ \\
&= (v_1 + v_2(u^{-1} - 1)((v_1, w) + t)v_2/(\langle v_2, w \rangle + t)).
\end{align*}
\]

Since \( V \) is a topological vector space, it follows that \( \lambda^{-1} \circ \Psi \circ \Phi \) is continuous.

**Definition 6.9.** Let \( V \) be a topological vector space. Let \( V \oplus F \) bear the direct sum topology. The family \( \bigoplus F \cup \{0\} \) is a quotient space of \( V \oplus F \). The restriction of the quotient topology to \( \bigoplus F \) will be called the **quotient topology** on \( \bigoplus F \).

**Theorem 6.10.** The projective space \( V \oplus F \) relative to its quotient topology is a topological projective space.

*Proof.* Let \( \Psi \) and \( \Phi \) be as in Theorem 6.8. Let \( O \) be an open subset of \( V \oplus F \setminus q_1^2 \). Then \( \{ z \in V \oplus F : z \in O \} \) is open in \( V \oplus F \) and so \( \lambda^{-1}(\{ z \in V \oplus F : z \in O \}) \) is open in \( V \). Since \( \lambda^{-1} \circ \Psi \circ \Phi \) is continuous (by Theorem 6.8),

\[ (\lambda^{-1} \circ \Psi \circ \Phi)^{-1}(\lambda^{-1}(\{ z \in V \oplus F : z \in O \})) = (\Psi \circ \Phi)^{-1}(O) \]

is open in \( F \times (\bigoplus F \setminus q_1^2) \times (\bigoplus F \setminus q_1^2) \).

Let \( p \in O \) be generic, and choose \( (u,v_1,v_2) \in F \times V \times V \) such that \( \Psi \circ \Phi(u,v_1,v_2) = p \). Since \( (\Psi \circ \Phi)^{-1}(O) \) is open, there exist an open neighborhood \( N_a \)
of \( u \) in \( F \) and open neighborhoods \( N_{u_1} \) of \( v_1 \) in \( V \) and \( N_{u_2} \) of \( v_2 \) in \( V \) such that \( N_u \cap N_{v_1} \cap N_{v_2} \subset (\Psi \circ \Phi)^{-1}(O) \). Note that \( \Phi(N_u \times N_{v_1} \times N_{v_2}) = N_u \times \lambda(N_{v_1}) \times \lambda(N_{v_2}) \) and so \( \{ (u, \text{1}, \text{1}, \text{1}) : u \in F, \text{1} \in N_{u_1}, \text{1} \in N_{v_1}, \text{1} \in N_{v_2} \} \subset \Psi^{-1}(O) \). Since \( N_{v_1} \) is open in \( V \), \( \{ (\text{1}, \text{1} : \text{1} \in N_{v_1} \} \) is open in \( V + F \) hence \( \{ (\text{1} : \text{1} \in N_{v_1} \} \) is open in \( V + F \). Similarly \( \{ (\text{1}, \text{1} : \text{1} \in N_{v_1} \} \) is open in \( V + F \). Thus \( \{ (u, \text{1}, \text{1}, \text{1}) : u \in F, \text{1} \in N_{v_1}, \text{1} \in N_{v_2} \} \) is an open subset of \( \Psi^{-1}(O) \) which contains \( p \). It follows that \( \Psi \) is continuous, which establishes (i) of Definition 6.3.

Let \( v_1 \oplus u_1, v_2 \oplus u_2 \in V + F \) and \( w_1 \oplus t_1, w_2 \oplus t_2 \in W + F \) be such that \( [v_1 \oplus u_1, v_2 \oplus u_2, w_1 \oplus t_1, w_2 \oplus t_2] \) is defined. We have

\[
[v_1 \oplus u_1, v_2 \oplus u_2, w_1 \oplus t_1, w_2 \oplus t_2] = \frac{(v_1, w_1) + (v_2, w_2) + u_1 t_1}{(v_1, w_2) + u_1 t_1} + \frac{(v_2, w_2) + u_2 t_2}{(v_1, w_2) + u_2 t_2} \cdot \frac{(v_1, w_1) + u_2 t_1}{(v_1, w_1) + u_1 t_1}
\]

Let \( \varepsilon > 0 \). Choose neighborhoods \( N_{v_1} \) of \( v_1 \) and \( N_{u_1} \) of \( u_1 \) such that

\[
\left| \frac{(v_1, w_1) + (u_1 t_1)}{(v_1, w_2) + u_1 t_1} \right| < \varepsilon \quad (\forall v \in N_{v_1}, u \in N_{u_1}).
\]

The set \( N_{v_1} + N_{u_1} \) is an open neighborhood of \( v_1 \oplus u_1 \) in \( V + F \) and

\[
[v \oplus u, v_2 \oplus u_2, w_1 \oplus t_1, w_2 \oplus t_2] - [v_1 \oplus u_1, v_2 \oplus u_2, w_1 \oplus t_1, w_2 \oplus t_2] < \varepsilon \quad (\forall v \oplus u \in N_{v_1} + N_{u_1}).
\]

This shows that Item (ii) of Definition 6.3 holds.

**Remark 6.11.** Other authors have considered topologies on projective spaces, beginning with Kolmogorov [5], but until the work of Bourbaki [4] only spaces of two- and three-dimensions seem to have been treated.

Bourbaki [4] treated the general finite dimensional case over the real field by (in our terminology) assigning to \( P \) the inductive topology induced by a projective representation of \( P \). It is a consequence of Theorem 6.10 that Bourbaki’s topology is consistent with ours.

The approach of Bourbaki [4] requires showing that the inductive topology is independent of the particular representation employed to induce it.

Lenz [6] appears to have been the first to approach the general finite dimensional case from a geometric point of view. In our terminology, a *central projection* (relative to a standard pair \((p, q)\)) is the map taking each \( p \in P \setminus \{p\} \) to the intersection of \( q^* \) with the line through \( p \) and \( q \). Lenz [6] defines a projective topology to be a nontrivial topology on \( P \) relative to which each central projection is continuous. He shows that if \( P \) is Hausdorff and compact, then his topology is the same as that obtained using the algebraic approach employed by Bourbaki [4], and that the field is locally compact. Conversely, if the field is locally compact, the algebraic approach yields a projective topology in the sense of Lenz [6] and \( P \) is compact. Thus, in the case of the real or complex field (which is locally compact), our topology is consistent with that of Lenz [6].

Misfeld [8] defined projective topologies for finite dimensional projective spaces over skew fields. Interestingly, his method simultaneously provides \( P \) and the dual \( Q \) with topologies. The topology of Misfeld [8] is a projective topology in the sense of Lenz [6] ([8, Satz 3.3]). In the case of the real or complex fields, the topology...
of Misfeld is compact [8, Satz 5]. It follows that the topology of Misfeld [8] is also consistent with ours.

**Theorem 6.12.** Let \((P, Q, [], []\)) be a projective duality. Let \(T \) and \(U \) be two topologies on \(P \) such that \((P, T) \) and \((P, U) \) are topological projective spaces. Suppose that \(V \) is a standard vector space in \(P \) and that the restrictions of \(T \) and \(U \) to \(V \) are identical. Then \(T = U \).

**Proof.** Let \(O \subseteq T \) and \(p \in O \) be generic. We shall show that there exists \(S \subseteq U \) such that \(p \in S \subseteq O \). This will imply that \(T \subseteq U \). Interchanging the roles of \(T \) and \(U \), we shall have \(U \subseteq T \) as well, which will imply that \(U = T \).

First suppose that \(p \in V \). From Theorem 6.6 we know that \(V \subseteq T \cap U \). Thus \(V \cap O \subseteq T \). Since \(T \) and \(U \) agree on \(V \), it follows that \(V \cap O \subseteq U \). Letting \(S \subseteq V \cap O \), we have \(p \in S \subseteq O \).

Now suppose that \(p \notin V \). Let \(o \) be the origin in \(V \) and \(v \) an element of \(\text{line} \(_P(o, p)\setminus \{o, p\}\). Choose \(q \in Q \) such that \(v \in q^-\). The involution \(\varphi := -1_{(o, q)}\) fixes only \(o\) and \(v\) on \(\text{line} \(_P(o, z)\). Since \(\varphi \) is a homeomorphism, \(\varphi(O) \) is a \(T\)-open neighborhood of \(\varphi(p)\). Hence \(V \cap \varphi(O) \) is a \(T\)-open neighborhood of \(\varphi(p)\), and so a \(U\)-open neighborhood as well, which will imply that \(\varphi(V \cap \varphi(O)) = p\). Letting \(S := \varphi(V \cap \varphi(O)) \), we have \(p \in S \subseteq O \).

**Definition 6.13.** Let \((P, Q, [], []\)) and \((R, S, [], [],)\) be projective dualities. Let \(T_1 \) and \(T_2 \) be projective topologies for \(P \) and \(R \) respectively. Then a projective isomorphism from \(P \) to \(R \) which is a homeomorphism will be called a \emph{topological projective isomorphism}.

**Theorem 6.14.** Let \((P, T) \) be a topological projective space and let \((p, q) \) be a standard pair. Let \(V \) be the standard vector space \((P, p, q) \) and let \(A \) be the relativized topology \(\{O \cap V : O \subseteq T \} \). Let \(V \oplus F \) have the quotient topology \(M \) inherited from \(V \oplus F \) endowed with the direct sum topology. Let \(\varphi \) be the standard projective isomorphism of \(P \) onto \(V \oplus F \). Then \(\varphi \) is a topological projective isomorphism.

**Proof.** Let \(U \subseteq T \) be the topology \(\{\varphi^{-1}(O) : O \subseteq M \} \). It will suffice to show that \(U = T \). Let \(B \subseteq U \) be the relativization of \(U \) to \(V \). From Theorem 6.10 we know that \((P, U) \) is a topological projective space. From Theorem 6.12 we know that \(U \) equals \(T \) if \(A \) equals \(B \). We shall show that \(A = B \).

Let \(v \subseteq N \) be a net in \(V \) which is \(A\)-convergent to an element \(v \) of \(V \). Then \(v + 1 \) converges to \(v + 1 \) in \(V \oplus F \), whence follows that \(v + 1 \) \(M\)-converges to \(v + 1 \) in \(V \oplus F \). Hence \(v \) \(B\)-converges to \(v \). This shows that \(B \subseteq A \).

Let \(O \subseteq A \) and \(\mathfrak{r} \subseteq O \) be generic. Since \((V, A) \) is a topological vector space, the map \(V \times F \ni (v, u) \mapsto uv \in V \) is continuous and so there exist \(N \) \(_1 \) \(1 \) in \(F \setminus \{0\} \) and \(N \mathfrak{r} \) of \(\mathfrak{r} \) in \(V \) such that \(N \mathfrak{r} \mathfrak{n} \subseteq O \) and \(\mathfrak{r} \subseteq A \). Then \(N \mathfrak{r} \oplus (N \mathfrak{n})^{-1} \) is open in \(V \oplus F \) and so \(L := \{x \oplus u : u \in N \mathfrak{n}^{-1}, v \in N \mathfrak{r} \} \) is in \(M \). Thus \(\varphi^{-1}(L) \) is in \(U \) and we have

\[
\mathfrak{r} \subseteq \varphi^{-1}(L) = \varphi^{-1}(\{uv \oplus 1 : u \in N \mathfrak{n}, v \in N \mathfrak{r} \}) \subseteq N \mathfrak{n} \mathfrak{r} \subseteq O.
\]

This implies that \(A \subseteq B \), whence follows that \(A = B \).
**Discussion 6.15.** Let \((P, T)\) be a topological projective space and \(V\) a standard vector space in \(P\). We know that \(V\) is a topological vector space, its topology being \(\{V \cap O : O \in T\}\). If \(U\) is any other projective topology on \(P\) such that the two inherited topologies on \(V\) are the same, then it follows from Theorem 6.12 that \((P, U)\) is topologically isomorphic to \((P, T)\).

Now let \(V\) be a dual topological vector space, *a priori* independent of any projective space. A projective representation of \(V\) on a topological projective space will be said to be a *topological projective representation* if it is a homeomorphism onto a standard vector space. Two equivalent topological projective representations will be said to be *topologically equivalent* if the associated projective isomorphism of the topological projective spaces is a homeomorphism. It follows from Theorem 6.12 that if two topological projective representations of \(V\) are equivalent, then they are topologically equivalent. Let \(\sigma\) be the standard projective representation of \(V\) on \(V \oplus F\) (as in Example 5.2). As in Theorem 6.10, \(V \oplus F\) is a topological projective space under its natural topology. Theorem 6.14 states that \(\sigma\) is a topological projective representation. A topological projective space associated with any topological projective representation of \(V\) will be called a *topological projective completion of \(V\)*. A topological projective completion of \(V\) associated with a topological representation equivalent to \(\sigma\) will be called a *standard topological projective completion of \(V\)*.

Let \((P, Q, [\cdot, \cdot])\) be a projective duality and \(T\) a projective topology on \(P\). If \((P, Q, [\cdot, \cdot])\) is a standard topological projective duality completion of one of its standard vector spaces, it is so for all of them. In such a case we shall refer to \((P, Q, [\cdot, \cdot])\) as a *standard topological projective duality*.

**Example 6.16.** Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert Space and let \((P, T)\) be a standard topological completion of \(H\). We shall say that \((P, T)\) is a *Hilbert Projective Space*. The vector duality associated with the Hilbert space is \((H, H', \langle \cdot, \cdot \rangle)\). Hence the projective duality associated with the projective space \(P\) is \((P, P', [\cdot, \cdot])\).

In this example, the dual maximal subspaces are more amenable than in general. To demonstrate we shall assume that \((P, P', [\cdot, \cdot])\) is \((H \oplus F, H \oplus F', [\cdot, \cdot])\). The general dual maximal subspace of \(P\) is the polar of a point \(i + u\) of \(H \oplus F\). We have \((i + u)^0 = \{z \oplus t : 0 = \langle z \oplus t, i + u \rangle\}\). If \(h\) is the identity \(o\) of \(H\), we have \((i + u)^0 = P \setminus H\). If \(h \neq o\), then

\[ (i + u)^0 = \{z : \langle z, h \rangle = -u\} \cup \{z : \langle z, h \rangle = 0\}. \]

Thus, viewing \(H\) as a subset of \(P\), a generic dual maximal subspace is of the form

\[ \{z \in H : \langle z, h \rangle = u\} \cup \{\text{line}_{P}(o, z) \setminus H : \langle z, h \rangle = 0\}. \]

In other words, the dual maximal subspaces of \(P\) are the closures in \(P\) of the closed hyperplanes in \(H\).

**Theorem 6.17.** Let \((V, T_V)\) and \((X, T_X)\) be dual topological vector spaces and let \(\alpha : V \to X\) be a topological affine isomorphism. Let \((P, T_P)\) and \((R, T_R)\) be standard topological projective completions of \((V, T_V)\) and \((X, T_X)\) respectively. Let \(\lambda\) be a standard representation of \(V\) on \(P\) and \(\mu\) a standard representation of \(X\) on \(R\). Then there exists a unique topological projective isomorphism \(\varphi\) of \(P\) onto \(R\) such that

\[ \varphi \circ \lambda(v) = \mu \circ \alpha(v) \quad (\forall v \in V). \]
Proof. We may (and shall) take $P$ to be $V \oplus \mathbb{F}$, $R$ to be $X \oplus \mathbb{F}$, and $\lambda$ and $\mu$ to be the respective standard projective representations. By changing the origin of $X$ to the image by $\alpha$ of the origin of $V$, we may (and shall) assume that $\alpha$ is linear. Define
\[
\varphi(v \oplus u) := \alpha(v) \oplus u \quad (\forall v \in V, u \in \mathbb{F})
\]
(the linearity of $\alpha$ ensures that $\varphi$ is well-defined). That $\varphi$ is a homeomorphism is evident.

Define $\psi : X' \oplus \mathbb{F} \to Y' \oplus \mathbb{F}$ by
\[
\psi(f \oplus u) := f \circ \alpha \oplus u \quad (\forall f \in X', u \in \mathbb{F})
\]
(linearity providing well-definedness again). For $v_1, v_2 \in V$, $f_1, f_2 \in X'$, and $u_1, u_2, t_1, t_2 \in \mathbb{F}$ we have (when the quadra-bracket exists)
\[
[v_1 \oplus u_1, v_2 \oplus u_2, \psi(f_1 \oplus t_1), \psi(f_2 \oplus t_2)] = (f_1 \circ \alpha(v_1) + u_1 t_1)(f_2 \circ \alpha(v_2) + u_2 t_1) - (f_2 \circ \alpha(v_1) + u_1 t_2)(f_1 \circ \alpha(v_2) + u_2 t_1) = [\varphi(v_1 \oplus u_1), \varphi(v_2 \oplus u_2), f_1 \oplus t_1, f_2 \oplus t_2]
\]
which shows that $(\varphi, \psi)$ is a projective duality isomorphism.

That $\psi$ is unique follows from the density of $\nu(V)$ in $P$.

**Discussion 6.18.** Let $V$ and $X$ be dual topological vector spaces and $\sigma$ a homeomorphism from a subset $D$ of $V$ onto a subset $E$ of $X$. Let $(P, Q, [., .])$ and $(R, S, [., .])$ be standard projective duality completions of $V$ and $X$ respectively: for the purpose of this discussion we shall regard $V$ and $X$ as being subsets of $P$ and $R$ respectively. Recall from Discussion 5.7 that $\sigma$ is said to be a local projective isomorphism in the context of the dualities $(V, V', (., .))$ and $(X, X', (., .))$ provided it is the restriction to $D$ of a projective isomorphism $\varphi$ of $P$ onto $R$. If $\varphi$ is a topological projective isomorphism, then $\sigma$ will be said to be a **standard topological local projective isomorphism**.

Since the property of being a standard topological local projective isomorphism is of some importance in applications, the problem of finding sufficient conditions for topological local projectivity arises. A plausible such condition, the determination of which avoids troublesome topological considerations, is that $\sigma$ be locally projective on finite dimensional subspaces. We show in Theorem 6.20 below that this condition is in fact sufficient when $D$ and $E$ are open.

**Theorem 6.19.** Let $(P, Q, [., .])$ be a projective duality and let $E$ be a finite dimensional vector subspace of a standard vector space in $P$. Let $G$ be the set of lines in $P$ which intersect $E$ in more than one point, and let $H$ be the subspace of $P$ consisting of the union of all the lines in $G$. Then there is a unique map $\theta$ of $H$ onto $E \oplus \mathbb{F}^*$ which preserves lines and of which the restriction to $E$ is the standard representation of $E$ in the projective space $E \oplus \mathbb{F}^*$.

Proof. Let $o$ be the origin of $E$. Then each point in $H \setminus E$ lies on one line in $P$ passing through $o$ and intersecting $E$ in more than one point. Conversely, each line in $E$ passing through $o$ is the intersection of a line in $H$ with $E$. Let $\sigma : H \setminus E \ni h \mapsto \sigma(h)$ denote this correspondence. To each point $e \oplus 0$ in $E \oplus 0$
corresponds a unique line $e$ of $E$ passing through the origin, and each such line corresponds to such a point: let $\rho : E \oplus 0 \ni (e \oplus 0) \mapsto \rho(e \oplus 0)$ denote that correspondence. Define

$$\theta(h) := h \oplus 1 \quad (\forall h \in E) \quad \text{and} \quad \theta(h) := \rho^{-1} \circ \sigma(h) \quad (\forall h \in H \setminus E).$$

Lines in $H \setminus E$ are the unions of planes of lines in $E$ passing through the origin. Evidently $\theta$ preserves lines. The uniqueness of $\theta$ follows from the fact that lines are preserved.

**Theorem 6.20.** Let $(P, Q, [. , .])$ and $(R, S, [. , .])$ be projective dualities and let $T_P$ and $T_R$ be projective topologies on $P$ and $R$ respectively relative to which $(P, Q, [. , .])$ and $(R, S, [. , .])$ are standard topological projective dualities. Let $V$ and $X$ be standard vector spaces in $P$ and $R$ respectively. Let $D$ be an open subset of $V$, $E$ an open subset of $X$, and $\sigma : D \to E$ a bijection such that

(i) for each finite dimensional subspace $E$ of $V$, the restriction of $\sigma$ to $E \cap D$ is a locally projective isomorphism of the vector space $E$ into the linear span of $\sigma(E \cap D)$;

(ii) $\sigma$ is homeomorphism;

Then $\sigma$ is a standard topological local projective isomorphism.

**Proof.** Let $\mathcal{E}$ denote the family of all finite dimensional subspaces $E$ of $V$ such that $E \cap D$ is nonvoid. For each $E \in \mathcal{E}$, Theorem 6.19 ensures that we may identify the smallest projective subspace $E^P$ in $P$ containing $E$ with a projective space containing $E$ as a dense subspace.

Let $E \in \mathcal{E}$ be generic, write $E^{\sigma}$ for the linear span in $R$ of $\sigma(E \cap D)$, and write $E^{\sigma R}$ for the smallest projective subspace of $R$ of which $E^{\sigma}$ is a subset. Thus $E^{\sigma R}$ may be identified with a projective completion of $E^{\sigma}$. By Hypothesis (i) there exists a unique projective isomorphism $\sigma^E$ of $E^P$ onto $E^{\sigma R}$ such that $\sigma^E$ and $\sigma$ agree on $E \cap D$.

Now let $E, G \in \mathcal{E}$. Then the linear span $H$ of $E \cup G$ in $V$ is an element of $\mathcal{E}$ and the uniqueness of $\sigma^H$ implies that it agrees with $\sigma^E$ on $E^P$ and with $\sigma^G$ on $G^P$. It follows that $\sigma^E$ and $\sigma^G$ agree on $E^P \cap G^P$. Thus the union $\sigma$ of all the maps $\sigma^E$, $E \in \mathcal{E}$, is a function. Since each point in $P$ is on a line through $D$, and each point in $R$ is on a line through $E = \sigma(D)$, it follows that $\sigma$ is a bijection from $P$ onto $R$. Evidently $\rho$ and $\rho^{-1}$ preserve lines, intersections of lines, projective subspaces and maximal projective subspaces.

Let $o$ and $\infty$ denote the origins of the vector spaces $V$ and $A$ respectively.

To show that $\rho$ is continuous, we take a generic point $p_1$ of $P$ and show that the inverse of any open neighborhood $N_r$ of $r := \rho(p_1)$ contains an open neighborhood of $p_1$. If $p_1$ is in $D$, then $N_r \cap E \in T_R$ and so $\sigma^{-1}(N_r \cap E) \in T_P$—thus $p_1 \in \rho^{-1}(N_r \cap E) = \sigma^{-1}(N_r \cap E) \subset \rho^{-1}(N_r)$. Now suppose that $p_1$ is in $P \setminus D$, which means that $r$ is in $\bar{R} \setminus E$. Let $e$ be in $E$. Since $E$ is open, there are two other distinct points $r_1, r_2 \in E \cap \operatorname{line}_R(e, r)$ which are also distinct from $e$. Let $s_1, s_2 \in S$ be such that $s_1 \in \operatorname{line}_R(e, r) = \{r_1\}$ and $s_2 \in \operatorname{line}_R(e, r) = \{r_2\}$. We may (and shall) choose $s_2$ such that $\rho(o) \notin s_2$. From Theorem 4.32 we know that

$$[r, e, s_1](r_1, s_2)(r) = e$$

where $[r, e, s_2, s_1](r_1, s_2)$ is the scalar-automorphism associated with scalar $[r, e, s_2, s_1]$. Since $(R, T_R)$ is a topological projective space, the dual subspace $s_2^*$ is $T_R$-closed.
in $R$. Hence $s^2_2 \cap E$ is $T_P$-closed as a subset of $E$. Since $\sigma$ is a homeomorphism, $\sigma^{-1}(s^2_2 \cap E)$ is $T_P$-closed as a subset of $D$. Furthermore
\[(\rho^{-1}(s^2_2) \cap V) \cap D = \rho^{-1}(s^2_2) \cap D = \rho^{-1}(s^2_2 \cap E) = \sigma^{-1}(s^2_2 \cap E).
\]
Since $s^2_2$ is a maximal subspace of $R$ not containing $\rho(o)$, it follows that $\rho^{-1}(s^2_2)$ is a maximal subspace of $P$ not containing $o$. Hence $\rho^{-1}(s^2_2) \cap V$ is a maximal affine subspace of $V$, which implies that $(\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2)$ is a maximal vector subspace of $V$. Since $O$ is open in $V$, it follows that $O - \rho^{-1}(r_2)$ is an open neighborhood of the identity in $V$. We have
\[
(\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2)) \cap (D - \rho^{-1}(r_2)) = ((\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2)) = (\rho^{-1}(s^2_2) \cap E) - \rho^{-1}(r_2).
\]
which is relatively $T_P$-closed in $D - \rho^{-1}(r_2)$ since $\sigma^{-1}(s^2_2 \cap E)$ is $T_P$-closed as a subset of $D$. It follows from [10, Chapter I, §4.2] that $(\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2)$ is a closed linear subspace of $V$, whence follows that there exists $f \in V'$ such that
\[f^{-1}(0) = (\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2).
\]
If $f(\rho^{-1}(r_2)) = 0$, then $(\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2)) = ((\rho^{-1}(s^2_2) \cap V) - \rho^{-1}(r_2)) + \rho^{-1}(r_2) = \rho^{-1}(s^2_2) \cap V$ and so $\rho^{-1}(s^2_2) \cap V$ would be a subspace of $V$—but $\rho^{-1}(s^2_2)$ does not contain $o$ by choice of $s_2$. Hence $f(\rho^{-1}(r_2))$ is nonzero and we may (and shall) choose $f$ such that $f(\rho^{-1}(r_2)) = 1$. It follows that
\[f^{-1}(1) = \rho^{-1}(s^2_2) \cap V.
\]
Since $(P, Q, [\ldots])$ is a standard topological projective completion of $(P, T_P)$, it follows from Theorem 4.29 that there exists $q_1 \in Q \setminus o^o$ such that
\[f(v) = \langle v, q_1 \rangle = [v, o, q_1, \varnothing] - 1 \quad (\forall v \in V).
\]
We have
\[q_1^o \cap V = \{v \in V : [v, o, q_1, \varnothing] = 1\}
\]
whence follows that
\[q_1^o \cap V = \rho^{-1}(s^2_2) \cap V.
\]
The sets $q_1^o$ and $\rho^{-1}(s^2_2)$ being maximal subspaces of $P$ of which the intersections with $V$ are identical, it follows that
\[q_1^o = \rho^{-1}(s^2_2).
\]
The function $\eta := \rho^{-1} \circ [r, e, s_2, s_1]_{(r_1, s_2)} \circ \rho$ has $\rho^{-1}(r_1)$ as its set of fixed points. Let $E'$ denote the set of $E \in \mathcal{E}$ such that line$_2(r, e) \subset E^{\sigma R}$ and $[r, e, s_2, s_1]_{(r_1, s_2)}(E^{\sigma R}) = E^{\sigma R}$. Then, for $E \in \mathcal{E}'$, the restriction of $[r, e, s_2, s_1]_{(r_1, s_2)}$ to $E^{\sigma R}$ is a $[r, e, s_2, s_1]$-scalar-automorphism of $E^{\sigma R}$. For $E \in \mathcal{E}$ define $\eta_E := \sigma_{E'}^{-1} \circ [r, e, s_2, s_1]_{(r_1, s_2)} \circ \sigma_E = \sigma_{E'}^{-1} \circ [r, e, r_2, r_1]_{(r_1, s_2)} \circ \sigma_E$. Since $\sigma_E$ is a projective isomorphism from $E^\sigma$ to $E^{\sigma R}$ and the restriction of $[r, e, s_2, s_1]_{(r_1, s_2)}$ to $E^{\sigma R}$ is a $[r, e, s_2, s_1]$-scalar-automorphism of $E^{\sigma R}$, it follows that $\eta_E$ is a $[r, e, s_2, s_1]$-scalar-automorphism of $E^{\sigma}$. We have $\text{Eigen}(\eta_E) = \{p_1\} \cup (q_1^o \cap E')$, whence follows that $\eta_E$ is the restriction of $[r, e, s_2, s_1]_{(p_1, q_1)}$ to $E^\sigma$. Consequently, $\eta = \ldots$
Let \( C := \rho^{-1}(r, e, r_2, r_1)(r_1, s_2)(N_r) \). Since \( [r, e, r_2, r_1](r) = e \in E \), it follows that \( r \) is in \( \rho(C) \) and so \( p_1 \) is in \( C \). We have

\[
C = \rho^{-1}(r, e, r_2, r_1)(E \cap [r, e, r_2, r_1](r_1, s_2)(N_r))
\]

Since \( [r, e, r_2, r_1](r_1, s_2) \) is a homeomorphism, it follows that \( [r, e, r_2, r_1](r_1, s_2)(N_r) \) is open. Since \( E \) is open and \( \sigma \) is continuous, the set \( \sigma^{-1}(E \cap [r, e, r_2, r_1](r_1, s_2)(N_r)) \) is open. Since \( \eta = [r, e, s_2, s_2](p_1, q_1) \) is a \( T_\rho \)-homeomorphism, it follows that \( C = \eta \circ \sigma^{-1}(E \cap [r, e, r_2, r_1](r_1, s_2)(N_r)) \) is in \( T_\rho \).

We have shown that \( \rho \) is continuous. The proof that \( \rho^{-1} \) is continuous is entirely analogous. Thus \( \rho \) is a homeomorphism.

It remains to show that \( \rho \) is a projective isomorphism. Let \( s \in S \) be generic. Then \( s^2 \) is a dual subspace of \( R \) and so some scalar automorphism \( \delta \) maps it into another dual subspace which intersects \( E \). Arguing as above with \( s^2 \), we see that \( \rho^{-1}(\delta(s^2)) \) is a dual subspace of \( P \). Arguing as above with \( [r, e, s_1, r_2](r_1, s_2) \), we see that \( \rho^{-1} \circ \delta \circ \rho \) is a scalar-automorphism of \( P \). Thus \( (\rho^{-1} \circ \delta^{-1} \circ \rho) \circ \rho^{-1}(\delta(s^2)) = \rho^{-1}(s^2) \) is a dual subspace of \( P \). We have shown that \( \rho^{-1} \) preserves dual subspaces. Since \( \rho \) is bijective, \( \rho \) also preserves dual subspaces. It follows from Theorem 3.13 that \( \rho \) is a projective isomorphism.
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