Digital Computers and Machine Representation of Data

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Computers

Machine computation requires a few ingredients:

1. A means of representing numbers physically
2. A means of manipulating those physical representations to do arithmetic

There are several options

- Mechanical representations - abacus, adding machine
- Analog electric - oscilloscope, analog computer
- Digital electric
- Biological - DNA sequences
Digital Computers

- Represent numbers by electric or magnetic charges
- Easiest, fastest, most failsafe to use binary
- E.g. +12v represents 1; 0v represents 0.
- Thus numbers are represented in binary arithmetic
- Note that this is not inevitable. One could make a computer that represented 0 by 0v, 1 by 1v, 2 by 2v, and so on. Implementation is more difficult.
- Virtually all modern digital computers use binary.
- Each binary digit is called a bit.
Binary Arithmetic

Base 2: 0 is 0

1 1
2 10
3 11
4 100
5 101
6 110
7 111
8 1000
Base $b$ Arithmetic

1. $0, 1, 2, \ldots, b - 1$
2. $10, 11, \ldots, 1(b - 1)$
3. In general $x_kx_{k-1}\ldots x_0$ represents $x_k b^k + x_{k-1} b^{k-1} + \ldots + x_0 b^0$.
4. E.g. In base 7, then
   $666_{7} = 6 \times 7^2 + 6 \times 7^1 + 6 \times 7^0 = 342_{10}$
Binary to Decimal

Convert $10010_2$ to decimal
Five digits, so $k = 4$

$10010_2 = 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 18_{10}$
Convert 423 to binary
What is the largest power of 2 less than 423?
It is $2^8 = 256$. There is one $2^8$ in 423.
Find the remainder: $423 - 2^8 = 167$.
There is one $2^7 = 128$ in 167. Find the remainder:
$167 - 128 = 39$
There is no $2^6 = 64$ in 39, but there is one $2^5 = 32$. The
remainder is $39 - 32 = 7$.
There is no $2^4 = 16$ nor $2^3 = 8$ in 7, but there is one 4, one 2,
and one 1.
Thus $423_{10} = 110100111_2$.
Whew!
Hexadecimal to Decimal

Hexadecimal: base 16
Counting: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10,…
Convert $3FE_{16}$ to decimal
Three digits, so $k = 2$;
$3FE_{16} = 3 \times 16^2 + F \times 16^1 + E \times 16^0 = 768 + 240 + 14 = 1022$
Convert 423 to hex
What is the largest power of 16 less than 423?
It is $16^2 = 256$; There is one $16^2$ in 423.
Find the remainder: 167 again.
There are A 16s in 167, leaving 7 over;
Thus $423_{10} = 1A7_{16}$
Hexadecimal to Binary

We could convert to decimal and then to binary but that would silly!
Each hex digit corresponds to four binary digits: so we can just write each hex digit as binary and line them up.

\[ \begin{align*}
1A7_{16} &= \{0001\}\{1010\}\{0111\} = 110100111_2. \\
AAA_{16} &= \{1010\}\{1010\}\{1010\} = 101010101010_2. \\
D0D_{16} &= \{1101\}\{0000\}\{1101\} = 110100001101_2.
\end{align*} \]
Binary to Hex

Same deal - just group binary digits in fours, from right.

\[110011_2 = \overbrace{0011}^{	ext{binary}} \overbrace{0011}^{	ext{binary}} = 33_{16}\]
\[101110011_2 = \overbrace{0001}^{	ext{binary}} \overbrace{1011}^{	ext{binary}} \overbrace{0011}^{	ext{binary}} = 173_{16}\]
\[10111100000_2 = \overbrace{0101}^{	ext{binary}} \overbrace{1110}^{	ext{binary}} \overbrace{0000}^{	ext{binary}} = 5E0_{16}\]

One could do this to use other bases that are powers of two, e.g. octal.
What is a computer number?

- Some collection of bits.
- Computer memory organized by **bytes** – 8 bits.
- Bytes are organized by **words**, which characterize the architecture of a computer.
- Ancient Apple computers used one-byte words – memory was organized one byte at a time, CPU registers were one byte. . .
- 8-bit operating system
- Early DOS/Windows computers used 16-bits; two bytes
- 70s Unix computers and 90s Windows NT computers used 32-bit
- 80s scientific computers and most modern computers use 64-bit
Integers

- Typically one word
- One bit for sign: 0 → positive, 1 → negative.
- In a 16-bit integer that leaves $2^{15} = 32768$ numbers.
- The larger the number of bits to store the number, the more integers can be represented.
- There is a largest computer integer that a computer can represent.
- Memory addresses are one-word integers
- A 32-bit machine can have at most 4GB of memory
What about non-integers - scientific computation?

- Use decimal notation
- Only finitely many digits
- Must represent very tiny numbers, and very large ones
- Use scientific notation $b \times 10^k$
- $b$ and $k$ are binary, and need a couple of sign bits
- $b$ is called the *mantissa*; $k$ is the *exponent*.
- There are largest and smallest numbers that can be represented.
16-bit Floating Point

- 10-bit mantissa with 1 sign bit
- Integers from -1024 to 1024
- 5 bits for exponent: numbers from 0 to 31, shifted
- $360 \times 10^0$ is in this set.
- $36 \times 10^1$ would be the same number.
- Mantissas are normalized for maximal significance
- Exponents thus run from $-14$ to $+15$.
- Avogadro’s number ($6.022 \times 10^{23}$) cannot be represented in 16-bit floating point.
IEEE 754

- IEEE → Institute of Electrical and Electronics Engineers - standards organization
- 32-bit floating point
  - 23-bit mantissa, 1 sign bit
  - 8-bit exponent, shifted range from -126 to +127
  - Effectively represents about 7 decimal digits
- 64-bit floating point
  - 52-bit mantissa, 1 sign bit
  - 11-bit exponent, shifted range from -1022 to +1023
  - Effectively represents almost 16 decimal digits
Characters

- ASCII - American Standard Code for Information Interchange
- Maps 7-bit integers to characters
Unicode

- ASCII proved limited
- Unicode now supports much larger character sets
- UTF-8, UTF-16, UTF-32
- UTF-8 now dominates the Internet
- UTF-8 is designed so that ASCII is subsumed.