Recall the binomial expansion for

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\]

where \(n\) is a positive integer and

\[\binom{n}{k} \equiv \frac{n!}{k! (n-k)!}\]

with \(n! \equiv n(n-1) \ldots 2 \cdot 1\) and \(0! \equiv 1\).

We shall consider two generalizations of this concept.

**Leibniz’s Rule of Differentiation**

\[(f g)^{(N)} = \sum_{k=0}^{N} \binom{N}{k} f^{(N-k)} g^{(k)}\]

where \((...)^{(k)} \equiv \frac{d^k(...)}{dx^k}\) and \(f^{(0)} \equiv f\),

which will be demonstrated by Mathematical Induction as follows.

a) First, we show that the statement is true for \(N = 1\). From the product rule

\[(f g)^{(1)} = f^{(1)} g^{(0)} + f^{(0)} g^{(1)}\]

while

\[\sum_{k=0}^{1} \binom{1}{k} f^{(1-k)} g^{(k)} = \binom{1}{0} f^{(1)} g^{(0)} + \binom{1}{1} f^{(0)} g^{(1)} = f^{(1)} g^{(0)} + f^{(0)} g^{(1)}\]

since

\[\binom{n}{0} = \binom{n}{n} = 1,\]

hence demonstrating the result.

b) Next, we assume that the statement is true for some integer \(N = p\)

\[(f g)^{(p)} = \sum_{k=0}^{p} \binom{p}{k} f^{(p-k)} g^{(k)}.\]
c) Finally, we test its truth for \( N = p + 1 \). Here

\[
(fg)^{(p+1)} = [(fg)^{(p)}]^{(1)} = \sum_{k=0}^{p} \binom{p}{k} [f^{(p-k)}g^{(k)}]^{(1)}
\]

\[
= \sum_{k=0}^{p} \binom{p}{k} f^{(p+1-k)}g^{(k)} + \sum_{k=0}^{p} \binom{p}{k} f^{(p-k)}g^{(k+1)}
\]

\[
= f^{(p+1)}g^{(0)} + \sum_{k=1}^{p} \binom{p}{k} f^{(p+1-k)}g^{(k)} + \sum_{k=0}^{p-1} \binom{p}{k} f^{(p-k)}g^{(k+1)} + f^{(0)}g^{(p+1)}.
\]

Now, changing the index in the second sum by introducing \( j = k + 1 \), we obtain

\[
\sum_{k=0}^{p-1} \binom{p}{k} f^{(p-k)}g^{(k+1)} = \sum_{j=1}^{p} \binom{p}{j-1} f^{(p+1-j)}g^{(j)} = \sum_{k=1}^{p} \binom{p}{k-1} f^{(p+1-k)}g^{(k)}.
\]

Substituting this back into the previous result yields

\[
(fg)^{(p+1)} = f^{(p+1)}g^{(0)} + \sum_{k=1}^{p} \left[ \binom{p}{k} + \binom{p}{k-1} \right] f^{(p+1-k)}g^{(k)} + f^{(0)}g^{(p+1)}.
\]

It only remains to show that

\[
\binom{p}{k} + \binom{p}{k-1} = \binom{p+1}{k}
\]

which we do as follows:

\[
\binom{p}{k} + \binom{p}{k-1} = \frac{p!}{k!(p-k)!} + \frac{p!}{(k-1)!(p+1-k)!} = \frac{p!}{(k-1)!(p+1-k)!} \left( \frac{1}{k} + \frac{1}{p+1-k} \right)
\]

\[
= \frac{p!}{(k-1)!(p+1-k)!} \left( \frac{p+1}{k} \right) = \binom{p+1}{k},
\]

since then

\[
(fg)^{(p+1)} = f^{(p+1)}g^{(0)} + \sum_{k=1}^{p} \binom{p+1}{k} f^{(p+1-k)}g^{(k)} + f^{(0)}g^{(p+1)} = \sum_{k=0}^{p+1} \binom{p+1}{k} f^{(p+1-k)}g^{(k)},
\]

hence demonstrating its truth for \( N = p + 1 \). We can now conclude from the Principle of Mathematical Induction that Leibniz’s Rule of Differentiation is true for all natural numbers \( N \).
Infinite Binomial Series

\[(1 + v)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} v^n \text{ for } |v| < 1.\]

This is often demonstrated by use of the Maclaurin series

\[f(v) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} v^n\]

since for

\[f(v) = (1 + v)^\alpha, \quad f^{(n)}(0) = \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!}\]

The problem with this approach is that in order for a function to have such a Maclaurin series it is necessary upon considering its Taylor polynomial

\[f(v) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} v^n + R_N(v, \bar{v}) \text{ where } R_N(v, \bar{v}) = \frac{f^{(N+1)}(\bar{v})}{(N+1)!} v^{N+1} \text{ and } |\bar{v}| < 1\]

that

\[\lim_{N \to \infty} R_N(v, \bar{v}) = 0\]

which is difficult to verify for this function due to the behavior of its derivatives as \(\bar{v} \to -1\). Indeed even if the Maclaurin series for \(f(v)\) converges for \(|v| < R\) there is no guarantee that it represents the function in this interval unless the remainder term goes to zero as \(N \to \infty\). As a classical counter example illustrating this concept consider the function

\[f(v) = \begin{cases} e^{-1/v^2} & \text{for } v \neq 0 \\ 0 & \text{for } v = 0 \end{cases},\]

which can be shown to satisfy

\[f^{(n)}(0) = 0 \text{ for all } n \geq 0.\]

Then its formal Maclaurin series

\[\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} v^n \equiv 0 \neq f(v).\]

Hence in order to demonstrate this result properly we must proceed as follows.
Define the function

\[ F(v) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} v^n. \]

We shall first find its interval of convergence by the ratio test with

\[ u_n(v) = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} v^n \text{ for } n \geq 1. \]

Then

\[
\left| \frac{u_{n+1}(v)}{u_n(v)} \right| = \left| \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)(\alpha - n)}{(n + 1)!} v^{n+1} \frac{n!}{\alpha(\alpha - 1) \cdots (\alpha - n + 1)v^n} \right| = \frac{|\alpha - n|}{n + 1} |v|.
\]

Thus

\[
\lim_{n \to \infty} \left| \frac{u_{n+1}(v)}{u_n(v)} \right| = |v| < 1
\]

and hence this power series converges for all \(|v| < 1\). We next wish to demonstrate that \( F(v) \) satisfies the first-order ordinary differential equation

\[ (1 + v)F'(v) = \alpha F(v), \quad F(0) = 1. \]

Since a power series can be differentiated term by term in its interval of convergence

\[
F'(v) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} n v^{n-1} = \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{(n - 1)!} v^{n-1}
\]

\[ = \alpha + \sum_{n=2}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{(n - 1)!} v^{n-1}. \]

Changing the index in the last series by introducing \( j = n - 1 \), we obtain

\[
F'(v) = \alpha + \sum_{j=1}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - j + 1)(\alpha - j)}{j!} v^j
\]

\[ = \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)(\alpha - n)}{n!} v^n. \]

Now multiplying the first series deduced for \( F'(v) \) by \( v \)}
\[ vF'(v) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} n v^n \]

and adding this to the final series for \( F'(v) \) yields the desired differential equation

\[
(1 + v)F'(v) = \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)(\alpha - n + n)}{n!} v^n
\]

\[
= \alpha \left[ 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} v^n \right] = \alpha F(v).
\]

The initial condition \( F(0) = 1 \) follows from direct substitution. Now solving this first-order differential equation and initial condition we find that

\[
F(v) = (1 + v)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} v^n, \quad |v| < 1
\]

completing the demonstration. Finally we particularize this result to \( \alpha = -\frac{1}{2} \) as follows.

\[
(1 + v)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right) \ldots \left(\frac{-2n+1}{2}\right)}{v^n}
\]

\[
= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!} v^n.
\]

Now

\[
1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \ldots \cdot (2n - 1) \cdot (2n)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n) 2^n n!} = \frac{(2n)!}{(2^n n!)^2}
\]

Therefore

\[
(1 + v)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(2^n n!)^2} v^n = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{(2^n n!)^2} v^n, \quad |v| < 1.
\]