Structure of Products

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Structured Eigenvalue Problems
Structured Eigenvalue Problems

Many structures
Structured Eigenvalue Problems

- Many structures
- We deal with structure all the time.
Structured Eigenvalue Problems

- Many structures
- We deal with structure all the time.
- ... some very simple
Structured Eigenvalue Problems

- Many structures
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  - $A$ is real
Structured Eigenvalue Problems

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  - $A$ is real
  - $A$ is Hermitian
Structured Eigenvalue Problems

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- We deal with structure all the time.
- ... some very simple
  - $A$ is real
  - $A$ is Hermitian
  - $A$ is real and Hermitian
Benefits of Respecting Structure
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- faster code
Benefits of Respecting Structure

- faster code
- improved stability, accuracy
Benefits of Respecting Structure

- faster code
- improved stability, accuracy
- preserved structure
Variety of Structures

Some structures are easy to exploit. Hermitian, unitary, and Hamiltonian.

Others are more difficult.
Variety of Structures

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Variety of Structures

Some structures are easy to exploit.

- real
- Hermitian
Variety of Structures

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  - real
  - Hermitian
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Variety of Structures

- Some structures are easy to exploit.
  - real
  - Hermitian
- Others are more difficult.
  - unitary
  - Hamiltonian
The Unitary Case

Shouldn’t this be easy?
The Unitary Case

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Hermitian + Hessenberg = tridiagonal
The Unitary Case

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- Hermitian + Hessenberg = tridiagonal
- Unitary + Hessenberg = ??
The Unitary Case

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- \[ U = G_1 G_2 \cdots G_n \]
The Unitary Case

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  - Hermitian + Hessenberg = tridiagonal
  - Unitary + Hessenberg = ??

- \[ U = G_1 G_2 \cdots G_n \]

- \[ U = \begin{bmatrix} -\gamma_1 & \sigma_1 \\ \sigma_1 & \overline{\gamma}_1 \end{bmatrix} \begin{bmatrix} 1 \\ -\gamma_2 & \sigma_2 \\ \sigma_2 & \overline{\gamma}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -\gamma_3 \end{bmatrix} \]
The Unitary Case

Shouldn’t this be easy?

- Hermitian + Hessenberg = tridiagonal
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\[ U = G_1 G_2 \cdots G_n \]

\[ U = \begin{bmatrix} -\gamma_1 & \sigma_1 & \vline & 1 \\ \sigma_1 & \gamma_1 & \vline & 1 \\ \sigma_1 & \gamma_1 & \vline & 1 \end{bmatrix} \begin{bmatrix} 1 & -\gamma_2 & \sigma_2 & \vline & 1 \\ -\gamma_2 & \sigma_2 & \vline & 1 \end{bmatrix} \begin{bmatrix} 1 & -\gamma_3 \\ \vline & 1 \end{bmatrix} \]

- Gragg, Ammar, Reichel, . . .
The Unitary Case

Shouldn’t this be easy?

Hermitian + Hessenberg = tridiagonal

Unitary + Hessenberg = ??

\[ U = G_1 G_2 \cdots G_n \]

\[ U = \begin{bmatrix}
-\gamma_1 & \sigma_1 \\
\sigma_1 & \gamma_1 \\
1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & -\gamma_2 & \sigma_2 \\
0 & \sigma_2 & -\gamma_2 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -\gamma_3
\end{bmatrix} \]

Gragg, Ammar, Reichel, . . .

This is a product eigenvalue problem.
General Problem
General Problem

Want eigenvalues of \( A = A_k A_{k-1} \cdots A_2 A_1 \)

Don't form \( A \) explicitly. Work with the factors. May have \( A_i \), not \( A_i^* \).

This is an old problem. Unitary problem is neither the oldest nor the simplest.
General Problem

- Want eigenvalues of $A = A_k A_{k-1} \cdots A_2 A_1$
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Other Examples
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$A$
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- $A$
- $A^*A, AA^*$ (SVD of $A$)
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- $AB^{-1}, B^{-1} A$ (Generalized EVP $A - \lambda B$)
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- $A^*A$, $AA^*$ (SVD of $A$)
- $AB^{-1}$, $B^{-1}A$ (Generalized EVP $A - \lambda B$)
- $AB(CD)^{-1}$ (Van Loan $(1973,75)$)
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- $A^*A$, $AA^*$ (SVD of $A$)
- $AB^{-1}$, $B^{-1}A$ (Generalized EVP $A - \lambda B$)
- $AB(CD)^{-1}$ (Van Loan (1973,75))
- $E_k^{-1}F_kE_{k-1}^{-1}F_{k-1} \cdots E_1^{-1}F_1$
  
  (Periodic linear control systems, Bojanczyk/Golub/Van Dooren (1992), Hench/Laub (1994))
Other Examples

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  (Periodic linear control systems, Bojanczyk/Golub/Van Dooren (1992), Hench/Laub (1994))
- totally-positive, pseudosymmetric, Hamiltonian, ···
Solving by Generic GR algorithm
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- Single matrix case
Solving by Generic GR algorithm

- Single matrix case
- Want eigenvalues of $A$
Solving by Generic GR algorithm

- Single matrix case
- Want eigenvalues of $A$
- Pick “any” function $f$,
  e.g. $f(A) = (A - \mu_1I) \cdots (A - \mu_mI)$
Solving by Generic GR algorithm

- Single matrix case
- Want eigenvalues of $A$
- Pick “any” function $f$, e.g. $f(A) = (A - \mu_1 I) \cdots (A - \mu_m I)$
- Shifts $\mu_1, \ldots, \mu_m$ should approximate eigenvalues.
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- $f(A) = GR$ $G$ nonsingular, $R$ upper triangular
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- $f(A) = GR$ where $G$ nonsingular, $R$ upper triangular
- $\hat{A} = G^{-1} AG$
Solving by Generic GR algorithm

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$\hat{A} = G^{-1} AG$

$\hat{A}$ is “more triangular”
Solving by Generic GR algorithm

- Single matrix case
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- $\hat{A} = G^{-1} AG$
- $\hat{A}$ is “more triangular”
- Repeat as necessary.
GR Algorithm for Products
GR Algorithm for Products

\[ A = A_3 A_2 A_1 \]
GR Algorithm for Products

\[ A = A_3 A_2 A_1 \]

Associated block cyclic matrix

\[
C = \begin{bmatrix}
0 & 0 & A_3 \\
A_1 & 0 & 0 \\
0 & A_2 & 0
\end{bmatrix}
\]

**Theorem (old, easy):** \( \lambda \neq 0 \) is an eigenvalue of \( A \) if and only if its \( k \)th roots \( \lambda^{1/k} \), \( \lambda^{1/k} \omega \), \( \lambda^{1/k} \omega^2 \), \ldots, \( \lambda^{1/k} \omega^{k-1} \) are all eigenvalues of \( C \).
Applying $GR$ to $C$; Choice of Shifts
Applying $GR$ to $C$; Choice of Shifts

- Shifting $A$ by $\mu$ = shifting $C$ by $\mu^{1/k}$
Applying $GR$ to $C$; Choice of Shifts

- Shifting $A$ by $\mu = \text{shifting } C \text{ by } \mu^{1/k}$
- Which $k$th root?
Applying $GR$ to $C$; Choice of Shifts

- Shifting $A$ by $\mu = \text{shifting } C \text{ by } \mu^{1/k}$
- Which $k$th root?
- All of them!

$$\mu^{1/k}, \mu^{1/k}\omega, \ldots, \mu^{1/k}\omega^{k-1}, \quad \omega = e^{2\pi i / k}$$

- structure preservation principle
Single-shift Case
Single-shift Case

Single shift on $A$: $f(A) = A - \mu I$
Single-shift Case

- Single shift on $A$: $f(A) = A - \mu I$
- corresponds to triple shift on $C$:

$$q(C) = (C - \mu^{1/3} I)(C - \mu^{1/3} \omega I)(C - \mu^{1/3} \omega^2 I)$$
Single-shift Case

- Single shift on $A$: $f(A) = A - \mu I$
- corresponds to triple shift on $C$:

\[ q(C) = (C - \mu^{1/3} I)(C - \mu^{1/3} \omega I)(C - \mu^{1/3} \omega^2 I) \]

\[ q(C) = C^3 - \mu I = f(C^3) \]
A $GR$ iteration on $C$

$C = \begin{bmatrix}
A_{21} & A_{13} \\
A_{21} & A_{32}
\end{bmatrix}$ (notation)
A $GR$ iteration on $C$

$C = \begin{bmatrix} A_{21} & A_{13} \\ A_{32} & \end{bmatrix}$ (notation)

$C^3 = \begin{bmatrix} A_{13}A_{32}A_{21} & A_{21}A_{13}A_{32} \\ A_{32}A_{21}A_{13} & \end{bmatrix}$
A GR iteration on $C$

$$C = \begin{bmatrix} A_{21} & A_{13} \\ A_{32} & \end{bmatrix} \quad \text{(notation)}$$

$$C^3 = \begin{bmatrix} A_{13}A_{32}A_{21} & A_{21}A_{13}A_{32} \\ A_{32}A_{21}A_{13} & \end{bmatrix}$$

$$p(C^3) = \begin{bmatrix} p(A_{13}A_{32}A_{21}) & p(A_{21}A_{13}A_{32}) \\ p(A_{32}A_{21}A_{13}) & \end{bmatrix}$$
$GR$ iteration on $C$ (continued)
GR iteration on $C$ (continued)

$q(C') = p(C^3) = GR, \quad \hat{C} = G^{-1}CG$
$GR$ iteration on $C$ (continued)

$q(C) = p(C^3) = GR$, \[\hat{C} = G^{-1}CG\]

$p(C^3) = \begin{bmatrix} p(A_{13}A_{32}A_{21}) \\ p(A_{21}A_{13}A_{32}) \\ p(A_{32}A_{21}A_{13}) \end{bmatrix}$
**GR iteration on C (continued)**

- $q(C) = p(C^3) = GR$, \quad $\hat{C} = G^{-1}CG$

- $p(C^3) = \begin{bmatrix} p(A_{13}A_{32}A_{21}) \\ p(A_{21}A_{13}A_{32}) \\ p(A_{32}A_{21}A_{13}) \end{bmatrix}$

- $= GR$
**GR iteration on C (continued)**

- \( q(C') = p(C^3) = GR, \quad \hat{C} = G^{-1}CG \)

- \( p(C^3) = \begin{bmatrix} p(A_{13}A_{32}A_{21}) & p(A_{21}A_{13}A_{32}) & p(A_{32}A_{21}A_{13}) \end{bmatrix} \)

- \( = GR = \begin{bmatrix} G_1 & G_2 & R_1 \cr G_3 & R_2 & R_3 \end{bmatrix} \)
The Similarity Transformation

\[ \hat{C} = G^{-1}CG \]

Cyclic structure is preserved.
The Similarity Transformation

\[ \hat{C} = G^{-1}CG \]

\[ = \begin{bmatrix} G_2^{-1}A_{21}G_1 & \quad G_1^{-1}A_{13}G_3 \\ \quad G_3^{-1}A_{32}G_2 \end{bmatrix} \]

\[ = \begin{bmatrix} \hat{A}_{21} & \hat{A}_{13} \\ \quad \hat{A}_{32} \end{bmatrix} \]
The Similarity Transformation

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Cyclic structure is preserved.
GR Steps Effected on Products

\[ p(A_{13}A_{32}A_{21}) = G_1 R_1, \]

\[ \hat{A}_{13}\hat{A}_{32}\hat{A}_{21} = G_1^{-1}(A_{13}A_{32}A_{21})G_1 \]
GR Steps Effected on Products

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\[ p(A_{21}A_{13}A_{32}) = G_2 R_2, \]

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GR Steps Effected on Products

\[ p(A_{13}A_{32}A_{21}) = G_1 R_1, \]
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\[ p(A_{21}A_{13}A_{32}) = G_2 R_2, \]
\[ \hat{A}_{21} \hat{A}_{13} \hat{A}_{32} = G_2^{-1}(A_{21}A_{13}A_{32})G_2 \]

\[ p(A_{32}A_{21}A_{13}) = G_3 R_3, \]
\[ \hat{A}_{32} \hat{A}_{21} \hat{A}_{13} = G_3^{-1}(A_{32}A_{21}A_{13})G_3 \]
Implicit Implementation

\[ C = \begin{bmatrix}
A_{21} & A_{13} \\
A_{31} & A_{32}
\end{bmatrix} \]
Implicit Implementation

\[ C = \begin{bmatrix}
  A_{21} & A_{13} \\
  A_{31} & A_{32}
\end{bmatrix} \]

Reduce to condensed form
Implicit Implementation

\[ C = \begin{bmatrix} A_{21} & A_{13} \\ A_{31} & A_{32} \end{bmatrix} \]

- Reduce to condensed form
- \( A_{13} \) Hessenberg, \( A_{21}, A_{32}, \ldots \) triangular
Looks like nice generalization of standard reduction to Hessenberg form, but . . .
Looks like nice generalization of standard reduction to Hessenberg form, but . . .

Looks like nice generalization of standard reduction to Hessenberg form, but . . .


shuffle the rows and columns (perfect shuffle)
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Periodic GR by Bulge Chasing
Periodic GR by Bulge Chasing

Do $GR$ iterations on the product.
Periodic GR by Bulge Chasing

- Do $GR$ iterations on the product.
- work with the factors only
Periodic GR by Bulge Chasing

- Do $GR$ iterations on the product.
- work with the factors only
- Bojanczyk/Golub/Van Dooren
- Hench/Laub
Looks like a nice generalization of implicit $GR$ algorithm, but . . .
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Looks like a nice generalization of implicit $GR$ algorithm, but . . .


shuffle the rows and columns
Looks like a nice generalization of implicit $GR$ algorithm, but . . .


shuffle the rows and columns

Do triple-shift $GR$ on shuffled form
Looks like a nice generalization of implicit $GR$ algorithm, but . . .


shuffle the rows and columns

Do triple-shift $GR$ on shuffled form

Shifts: $\mu^{1/3}, \mu^{1/3} \omega, \mu^{1/3} \omega^2$
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Example: SVD

\[
\begin{bmatrix}
0 & A^T \\
A & 0
\end{bmatrix} =
\begin{bmatrix}
a_1 & b_1 & a_2 & b_2 & a_3 \\
a_1 & b_1 \\
a_2 & b_2 \\
a_3
\end{bmatrix}
\]
Double-shift QR with shifts $1 = 2$, $1 = 2 = \text{Golub/Kahan QR for SVD (1965)}$
Double-shift $QR$ with shifts $+\mu^{1/2}$, $-\mu^{1/2}$
Double-shift $QR$ with shifts $+\mu^{1/2}, -\mu^{1/2}$

= Golub/Kahan $QR$ for SVD (1965)
Other Examples

I would have liked to discuss the following additional topics in the context of product eigenvalue problems.

- skew-symmetric eigenvalue problem
- generalized eigenvalue problem
- differential qd algorithm for symmetric eigenvalue problem
- HR algorithm for pseudosymmetric matrices
- and more.
Other Examples

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Thank you for your attention.