Review of *Linear Algebra and its Applications*, 2nd Edition by Peter D. Lax John Wiley and Sons, 2007

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This is a new edition of the advanced linear algebra text by 2005 Abel Prize winner Peter Lax. The purpose of the book is to provide an introduction to linear algebra that emphasizes the structure of the subject as the theory of linear transformations on vector spaces. A second objective is, in the author's words, "to present a rich selection of analytical results and some of their applications."

What's new?

The second edition differs from the first in several ways. Using the author's words again, "The changes in this second edition are partly to make it more suitable as a text. Terse descriptions, especially in the early chapters, were expanded, more problems were added, and a list of solutions to selected problems has been provided." In addition to this, one new chapter and eight appendices have been added. All told, the book consists of eighteen short chapters (totaling 277 pages), solutions of selected exercises (21 pages), and sixteen appendices (70 pages).

The new Chapter 18, which is entitled *How to calculate the eigenvalues of self-adjoint matrices,* begins with a discussion of the QR factorization. Then the basic (unshifted) QR algorithm for computing the eigenvalues of a real symmetric matrix is introduced. Householder transformations are introduced very briefly, and it is shown how they can be used for two purposes: 1) to compute the QR factorization of a matrix and 2) to transform a real-symmetric matrix to tridiagonal form by an orthogonal similarity transformation. Then it is proved that real tridiagonal form is preserved under iterations of the QR algorithm, and the point is made that this can save a lot of computational effort.

The second half of the Chapter 18 introduces and discusses the (non-periodic) Toda flow, a system of differential equations that is a continuous-time analogue of the QR algorithm. The basic properties of the flow are presented, and Moser's convergence theorem, which parallels the convergence theory of the QR algorithm, is presented.

This is terrific material, and I am glad Lax chose to include it in this new edition, but I do have one complaint. The title of the chapter, *How to compute the eigenvalues ...*, promises too much. It is true that Householder transformations and the reduction to tridiagonal form are discussed —

and these are important computational ideas — but the book stops far short of the state of the art of eigenvalue computation. Shifts of origin to accelerate convergence are crucial in practice, but they are not mentioned here. Nor are we told that the QR algorithm is normally implemented by an implicit-shifting technique that is completely different from what is presented in the book. Finally, it is not mentioned that methods for the *symmetric* problem that are far superior to the QR algorithm have been developed in recent years. I refer to the divide-and-conquer method, the differential qd algorithm, and the MRRR method [1]. A brief overview of these methods is given in [2, § 6.6]. I do not suggest that the author should have included all of this material in the chapter, but I think he should have chosen a more modest title.

In addition to the new chapter, there are eight new appendices. These are brief treatments of a variety of topics that the author chose to leave out of the main text in the interest of not impeding the flow. New topics treated include the fast Fourier transform, the spectral radius theorem, the Lorenz group, the Lyapunov stability criterion, the Jordan canonical form, and more.

What is the book about?

From this point on I will treat the book as if the whole thing were brand new, making no distinction between what is new in the second edition and what was already there.

The book is intended for advanced undergraduates and beginning graduate students, and the author expects that they have had some (but not necessarily a lot of) prior exposure to the concepts of linear algebra. My own observation is that the students had better also be pretty darned good and well prepared; at least they should know how to construct a proof. There are 21 exercises in Chapter 1. The first 19 ask the student to prove something, and the other two ask for proof or disproof. The presentation is abstract, especially in the beginning, but abstract does not imply formal. Lax's proof style is relentlessly informal. Sometimes details are omitted. I found this to be just fine, but it will not work for students who are still trying to figure out what a proof is. Because of the terse statements of many of the exercises, the students will need to invent their own notation in order to work the problems. None of this should be construed as criticism. This is a book for talented, sophisticated students, such as you might expect to find at the Courant Institute. (However, for students whose proof skills are not well developed, the newly added section *Solutions of selected exercises* does offer some consolation.)

The first few chapters of the book lay out the abstract theory of vector spaces over the real or complex numbers and linear transformations on those spaces, including a discussion of quotient spaces and a short chapter on duality. Matrices are introduced in Chapter 4. Then, in Chapter 5, the properties of determinants are developed from the notion of volume. After that it's on to spectral theory and Euclidean structure.

The greatest strength of the book, which sets it apart from most (all?) other linear algebra texts, is its inclusion of a heavy dose of analytical material. Chapter 9 is devoted entirely to the development of the calculus of vector- and matrix-valued functions. This is followed by a chapter on matrix inequalities that makes considerable use of calculus. Several results are proved two different ways. For example, if $0 < M < N$ (in the sense of positive definiteness), then $M^{-1} > N^{-1}$. This is proved first by an algebraic argument using eigenvalues, then by an analytic argument using derivatives.

Chapter 11, my favorite chapter, is entitled *Kinematics and Dynamics.* Here we see several more applications involving calculus. Rigid body motions are described in terms of rotation matrices and their skew-symmetric infinitesimal generators. Then fluid motion is described. The polar decomposition of a Jacobian is used to demonstrate that the instantaneous rotation of the fluid is given by the curl of the velocity field. A simpler matrix argument shows that the rate of expansion is given by the divergence of the velocity field. Finally small vibrations of elastic structures are analyzed.

Chapters 12 and 13 discuss convexity and the duality theorem, with indications of how duality theory can be applied to economics and game theory. Chapters 14 and 15 discuss normed linear spaces and linear mappings on them. These chapters could serve as preparation for a course in functional analysis. Everything is done in finite dimensions, but indications of what happens in the infinite-dimensional case are given here and there. Chapter 16 is a brief introduction to Perron-Frobenius theory and Markov chains.

Chapter 17, which is entitled *How to solve systems of linear equations,* begins by motivating and defining the condition number of a matrix. Then the method of steepest descent is described, and it shown that this method converges very slowly if the condition number of the coefficient matrix is large. Then the Chebyshev and minimum residual methods, which have much faster convergence, are developed. The discussion is restricted to symmetric, positive definite matrices.

This is all very interesting material — I hope many people will read it — but the reader may have guessed that I have the same complaint about this chapter as I had about Chapter 18: The title promises more than is actually delivered. Although Chapter 17 contains important practical information about the iterative solution of linear systems, it stops far short of the state of the art. For example, the recurrences that are used in practice are generally pairs of two-term recurrences, whereas the recurrences developed in this book are three-term. More importantly, preconditioners, which are crucial to the success of modern iterative methods, are not mentioned. Once again, I am not complaining about the omission of these topics; I just think that Chapter 17 should have been given a different title.

Contrary to the implicit claims of the titles of Chapters 17 and 18, not to mention some of the hype on the back cover of the book, this is not a book about matrix computations. Gaussian elimination gets five pages at the end of Chapter 4; pivoting is mentioned only in the final paragraph. The classical Gram-Schmidt procedure is described early in the book, but the QR decomposition is not mentioned until Chapter 18. The singular value decomposition gets only passing mention as an afterthought to the polar decomposition at the end of Chapter 10. Schur's theorem — every square matrix is unitarily similar to an upper triangular matrix — appears only in Appendix 10.

So what *is* this book? It is a penetrating theoretical treatment of linear algebra that also showcases an impressive array of applications. By embracing analysis Lax is able to provide a much greater variety than one normally sees in a linear algebra text.

Minor Objections

Some readers will think that my objections to the titles of Chapters 17 and 18 are minor. Aside from that, I was sometimes annoyed by the author's introduction of unorthodox notation. For example, $\frac{123}{132}$ looks like a fraction, but it's actually notation for the permutation that transposes the symbols 2 and 3. The symbol $\{x\}$ looks like the set containing a single element x, but it is actually the coset that contains the vector x. The symbol O is sometimes used to denote an orthogonal matrix or linear transformation. It looks like a zero to me, and in some places it is typeset incorrectly as a zero. By the way, the book contains many, many misprints. Most of them are harmless, but one or two of them caused me considerable puzzlement.

It is claimed in the preface that the conjugate gradient method will be derived in Chapter 17, but the method that is actually derived there is the minimum residual (MINRES) method, which is something else.

Navigation is not easy. When the author refers you to Theorem 17 of Chapter 7, your first task is to find Chapter 7. This is nontrivial because the page headings list the chapter titles but not the chapter numbers. It may be that the easiest way to find chapter 7 is to go to the table of contents and find a page number there. Once you are in chapter 7, you still have some hunting around to do to find Theorem 17.

Final Assessment

I recommend that you and your more talented students read this book. I read most of it carefully, and in the process I learned more than I expected to. If you want to teach a theoretical linear algebra course (with applications interspersed) to a class of bright, motivated graduate students or advanced undergraduates, you should consider using this book as a text. If you are looking for a text on matrix computations, there are other books I could recommend.

References

[1] E. Anderson et al., *LAPACK Users' Guide*. `www.netlib.org/lapack/lug/`.

[2] D. S. Watkins, *Fundamentals of Matrix Computations*, John Wiley and Sons, Second ed., 2002.