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A RECURSIVE TEST FOR P-MATRICES

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Abstract.

The time complexity for testing whether an n-by-n real matrix is a P-matrix is reduced from $O(2^n n^3)$ to $O(2^n)$ by applying recursively a criterion for P-matrices based on Schur complementation. A Matlab program implementing the associated algorithm is provided.

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1 Introduction.

Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is called a *P-matrix* if all of its principal minors are positive. P-matrices arise in a variety of mathematical contexts and applications (see e.g., Berman and Plemmons [1]). The *P-matrix problem*, namely, the problem of testing whether a given matrix A is a P-matrix is of importance in many of these applications. Specifically, it is known that the linear complementarity problem has a unique solution if and only if the associated matrix is a P-matrix. In fact, the P-matrix problem is linked to a finite number of test linear complementarity problems having unique solutions (see Murty [6], Tamir[7] and Kostreva [5]). For certain matrix classes, e.g., the Hermitian matrices and the Z-matrices, the P-matrix problem is computationally straightforward, demanding only low degree polynomial time complexity. Indeed, Hermitian P-matrices are positive definite and the intersection of the Z-matrix class with the P-matrix class comprises the M-matrices; see [1]. However, in its generality, the P-matrix problem seems inevitably of exponential time complexity. As it is shown in Coxson [2], the P-matrix problem is co-NP-complete.

A direct approach to the P-matrix problem is to evaluate all the principal minors of A using standard numerical linear algebra techniques. Since evaluating a principal minor of order k requires $O(k^3)$ operations, we have that the flop

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count for such a task is of order (see e.g., Gould [4])

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = \frac{2^n n^2 (n+3)}{8}. \tag{1.1}$$

A similar approach entails finding the LU factorizations of all possible permutation similarities of A and checking the signs of the pivots. The matrix A is a P-matrix if and only if all these LU factorizations exist and all the pivots encountered are positive. The time complexity of this approach is also $\mathrm{O}(2^n n^3)$.

Our goal in the present note is to improve the above complexity figure. One motivating factor is that the principal minors of a matrix are algebraically dependent and thus one can be hopeful that a significant reduction in the time complexity is possible. Based on an observation about P-matrices (Theorem 3.1), we provide an $\mathrm{O}(2^n)$ algorithm for the P-matrix problem (Algorithm 3.2), which in essence proceeds recursively to transform a P-matrix problem of order n to two P-matrix problems of order $n-1$. It is worth noting at this stage the parallel nature of such an algorithm. In section 4, we include a simple Matlab program that tests for P-matrices, as well as a table of complexity related observations.

2 Notation.

Let n be a positive integer and $A \in \mathbb{R}^{n\times n}$. In the remainder the following notation is used:

- $\langle n\rangle = \{1, 2, \ldots, n\}$. For any $\alpha \subseteq \langle n\rangle$, the cardinality of α is denoted by $|\alpha|$ and $\alpha^c = \alpha \setminus \langle n\rangle$.
- $A[\alpha, \beta]$ is the submatrix of A whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n\rangle$, respectively; the elements of α, β are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1.
- $A[\alpha] = A[\alpha, \alpha]$, $A(\alpha, \beta] = A[\alpha^c, \beta]$; analogously we define $A[\alpha, \beta)$, $A(\alpha, \beta)$ and $A(\alpha)$.
- The *Schur complement* of an invertible principal submatrix $A[\alpha]$ in A is

$$A/A[\alpha] = A(\alpha) - A(\alpha, \alpha](A[\alpha])^{-1} A[\alpha, \alpha).$$

3 The criterion and the algorithm.

We begin with the theoretical context that permits us to devise an $\mathrm{O}(2^n)$ algorithm for the P-matrix problem.

THEOREM 3.1. *Let $A \in \mathbb{R}^{n\times n}$ and* [*] *$\alpha \subseteq \langle n\rangle$ with $|\alpha| = 1$. Then A is a P-matrix if and only if $A[\alpha]$, $A(\alpha)$ and $A/A[\alpha]$ are P-matrices.*

[*]There is a typo at this point of the published version.

PROOF. Without loss of generality, assume that $\alpha = \{1\}$. Otherwise we can consider a permutation similarity of A. If $A = [a_{ij}]$ is a P-matrix, by definition, $A[\alpha]$ and $A(\alpha)$ are also P-matrices. That $A/A[\alpha]$ is a P-matrix is a well known fact (see e.g., [1, Exercise 10.6.1] or [8, Lemma 5.1]).

For the converse, assume that $A[\alpha] = [a_{11}]$, $A(\alpha)$ and $A/A[\alpha]$ are P-matrices. Using $a_{11} > 0$ as the pivot, we can row reduce A to obtain a matrix B with all of its off-diagonal entries in the first column equal to zero. As is well known, $B(\alpha) = A/A[\alpha]$. That is, B is a block triangular matrix whose diagonal blocks are P-matrices. It follows readily that B is a P-matrix. The determinant of any principal submatrix of A that includes entries from the first row of A coincides with the determinant of the corresponding submatrix of B and is thus positive. The determinant of any principal submatrix of A with no entries from the first row coincides with a principal minor of $A(\alpha)$ and is also positive. Hence A is a P-matrix. $\square$

The above result suggests the following algorithm for the P-matrix problem.

ALGORITHM 3.2. *P(A)*
1. *Input* $A = [a_{ij}] \in \mathbb{R}^{n \times n}$
2. *If* $a_{11} \leq 0$ *output "A is not a P-matrix" stop*
3. *Evaluate* $A/A[\alpha]$, *where* $\alpha = \{1\}^{\dagger}$
4. *Call* $P(A(\alpha))$
   Call $P(A/A[\alpha])$
5. *Output "A is a P-matrix"*

To study Algorithm 3.2, let $T(n)$ represent the time complexity of the P-matrix problem for $A \in \mathbb{R}^{n \times n}$. By Theorem 3.1 and for any $\alpha \subseteq \langle n \rangle$ with $|\alpha| = 1$, since $A(\alpha)$ and $A/A[\alpha]$ are of order $n-1$, since $A/A[\alpha]$ is a rank one update of $A(\alpha)$ (that requires $2(n-1)^2 + 1$ flops; see Golub and Van Loan [3, p. 19]), and since $A[\alpha]$ is of order 1, we can write

$$T(n) = 2T(n-1) + 2(n-1)^2 + 2 = 2T(n-1) + \mathrm{O}(\,(n-1)^2\,). \tag{3.1}$$

Inductively, (3.1) implies that

$$T(n) = 2^{n-1}T(1) + \mathrm{O}(\,2^{n-2}\,1^2 + 2^{n-3}\,2^2 + \ldots + 2\,(n-2)^2 + (n-1)^2\,).$$

Thus, as $T(1) = 1$,

$$T(n) = 2^{n-1} + 2^{n-1}\mathrm{O}(\sum_{k=1}^{n-1} \frac{k^2}{2^k}\,). \tag{3.2}$$

LEMMA 3.3. *Let m be a positive integer. Then*

$$U_m := \sum_{k=1}^{m} \frac{k^2}{2^k} < 6.$$

†The phrase "where $\alpha = \{1\}$" was omitted but implied in the published version.

PROOF. Notice that

$$\begin{aligned} \frac{U_m}{2} &= U_m - \frac{U_m}{2} = \frac{1}{2} - \frac{m^2}{2^{m+1}} + \sum_{k=2}^{m} \frac{2k-1}{2^k} \\ &< \frac{1}{2} + \sum_{k=2}^{m} \frac{2k-1}{2^k} = \sum_{k=1}^{m} \frac{2k-1}{2^k}. \end{aligned}$$

That is, on letting $V_m = \sum_{k=1}^{m} \frac{2k-1}{2^k}$, we have $U_m < 2V_m$. Similarly it can be shown that

$$\frac{V_m}{2} < \frac{1}{2} + \sum_{k=2}^{m} \frac{2}{2^k} < \frac{3}{2},$$

from which the conclusion of the lemma follows. □

By (3.2) and Lemma 3.3, we now have that

$$T(n) = 2^{n-1} + 2^{n-1}\mathrm{O}(1) = \mathrm{O}(2^n).$$

We have thus shown the following result.

THEOREM 3.4. *The time complexity of Algorithm 3.2 is* $\mathrm{O}(2^n)$.

We conclude this section by noting the parallel nature of Algorithm 3.2, suggesting that an exponential improvement of the time complexity can be achieved with the use of a polynomial number of processors.

4 A Matlab program

The following Matlab recursive M-file implements Algorithm 3.2. If a non P-matrix is encountered in an inner invocation of the program, the information is passed along to the top invocation and the function exits early.

```
function [r]=ptest(a)
% Return r=1 if ‘a’ is a P-matrix (r=0 otherwise)

n=length(a);
if a(1,1)<=0
    r=0;
elseif (n==1)
    r=1;
else
    for i=2:n
      d(i,1)=a(i,1)/a(1,1);
      for j=2:n
        b(i-1,j-1)=a(i,j);
        c(i-1,j-1)=a(i,j)-d(i,1)*a(1,j);
      end
    end
    r1=ptest(b);
    r2=ptest(c);
```

```
    r=r1*r2;
end
```

Let now C_n be the quantity in (1.1) and D_n the flops (as counted by Matlab) performed by Algorithm 3.2 verifying that $A \in \mathbb{R}^{n \times n}$ is a P-matrix. To obtain an understanding of the upper bound of the quantity $D_n/2^n$ and of the improvement achieved with Algorithm 3.2, we run the above program for positive definite matrices of order n ranging from 2 to 25 and observed the following results.

order n	$D_n/2^n$	$C_n/2^n$
2	2	2.5
3	5.3750	6.75
4	9	14
5	12.1562	25
6	14.5938	40.5
7	16.3359	61.25
8	17.5156	88
9	18.2832	121.5
10	18.7676	162.5
11	19.0659	211.75
12	19.2461	270
13	19.3531	338
14	19.4159	416.5
15	19.4522	506.25
16	19.4731	608
17	19.4849	722.5
18	19.4916	850.5
19	19.4954	992.75
20	19.4974	1150
21	19.4986	1323
22	19.4992	1512.5
23	19.4996	1719.25
24	19.4998	1944
25	19.4999	2187.5

Using different Matlab programming techniques, one can improve on the performance of the provided M-file, which was chosen for clarity. Other Matlab implementations, as well as a C program that exhibits a significant running time improvement, are available from the first author.

We comment that Theorem 3.1 and its proof are also valid for complex matrices and thus comparable computational advantages can be realized by applying its result to the complex P-matrix problem.

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