MAPPING AND PRESERVER PROPERTIES OF THE PRINCIPAL PIVOT TRANSFORM

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Abstract. The principal pivot transform (PPT) is a transformation of a matrix A tantamount to exchanging a fixed set of entries among all of the domain-range vector pairs of A. First in this paper, mapping properties of the PPT applied to certain matrix positivity classes are identified. These classes include the (almost) P-matrices and (almost) N-matrices, arising in the linear complementarity problem. Second, a fundamental property of PPTs is proved, namely, that PPTs preserve the rank of the Hermitian part. Third, conditions for the preservation of left eigenspaces by a PPT are determined.

Key words. Principal pivot transform; Schur complement; P-matrix; N-matrix; Hermitian part; left eigenspace.

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1. Introduction. Suppose that $A \in \mathcal{M}_n(\mathbb{C})$ (the n-by-n complex matrices) is partitioned in blocks as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.1}$$

and further assume that A_{11} is an invertible submatrix of A. Consider the matrix

$$B = \begin{bmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12} \\ A_{21}(A_{11})^{-1} & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{bmatrix}, \tag{1.2}$$

which is known as the principal pivot transform (PPT) of A relative to A_{11} and is related to A as follows: For every $x = (x_1^T, x_2^T)^T$ and $y = (y_1^T, y_2^T)^T$ in $\mathbb{C}^n$ partitioned conformally to A,

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{if and only if} \quad B \begin{bmatrix} y_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}.$$

The operation of obtaining B from A is encountered in several contexts, including mathematical programming, statistics and numerical analysis, and is known by several other names, e.g., sweep operator and exchange operator; see [11] for the history and a survey of properties of PPTs.

Our goal in this paper is to examine matrix properties preserved by principal pivot transforms, as well as to identify mapping properties of PPTs among certain matrix classes. More specifically, in Section 3 we consider, among others, the classes of (almost) P-matrices and (almost) N-matrices that play a key role in the solvability of the linear complementarity problem and the study of univalence for continuously differential maps from $\mathbb{R}^n$ to $\mathbb{R}^n$; see e.g., [5, 6, 10, 11]. These facts serve as our motivation in investigating and classifying the Schur complements and the principal pivot transforms of matrices in the matrix classes named above.

In Section 4, we examine preservation properties of PPTs regarding the ranks of the Hermitian and skew-Hermitian parts for general matrices. Finally, in Section 5, we determine conditions for the preservation of left eigenspaces by a PPT.

2. Global definitions, notation and preliminaries. This section contains material used throughout this paper. Let n be a positive integer and $A \in \mathcal{M}_n(\mathbb{C})$.

- $\langle n \rangle = \{1, 2, \ldots, n\}$. For any $\alpha \subseteq \langle n \rangle$ in ascending order, the cardinality of α is denoted by $|\alpha|$ and its complement in $\langle n \rangle$ by $\bar{\alpha} = \alpha \setminus \langle n \rangle$.
- Let $\gamma \subseteq \langle n \rangle$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_k\} \subseteq \langle |\gamma| \rangle$. Define the indexing operation $[\gamma]_\beta$ as

$$[\gamma]_\beta := \{\gamma_{\beta_1}, \gamma_{\beta_2}, \ldots, \gamma_{\beta_k}\} \subseteq \gamma.$$

- $A[\alpha, \beta]$ is the submatrix of A whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively; the elements of α, β are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1. We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$.
- $A/A[\alpha]$ denotes the *Schur complement* of an invertible principal submatrix $A[\alpha]$ in A, that is, $A/A[\alpha] = A[\bar{\alpha}] - A[\bar{\alpha}, \alpha]A[\alpha]^{-1}A[\alpha, \bar{\alpha}]$.

- $\sigma(A)$ denotes the spectrum of A.

The following fact about Schur complements is given in [1].

Lemma 2.1. *Let $A \in \mathcal{M}_n(\mathbb{C}), \alpha \subseteq \langle n\rangle$, $\beta \subseteq \langle|\bar{\alpha}|\rangle$ and $\beta' = [\bar{\alpha}]_\beta$. If both $A[\alpha]$ and $A[\alpha\cup\beta']$ are invertible, then*

$$A/A[\alpha\cup\beta'] = (A/A[\alpha])/((A/A[\alpha])[\beta]).$$

We shall also make use of the following (Schur-Banachiewicz) block representation of the inverse; see, e.g., [9, (1.9)] or [4, Chapter 0, 7.3].

Lemma 2.2. *Given an invertible $A \in \mathcal{M}_n(\mathbb{C})$ and $\alpha \subseteq \langle n\rangle$ such that $A[\alpha]$ and $A[\bar{\alpha}]$ are invertible, A^{-1} is obtained from A by replacing*

$$\begin{array}{l} A[\alpha] \ \ by \ \ (A/A[\bar{\alpha}])^{-1}, \quad A[\alpha,\bar{\alpha}] \ \ by \ \ -A[\alpha]^{-1}A[\alpha,\bar{\alpha}](A/A[\alpha])^{-1}, \\ A[\bar{\alpha},\alpha] \ \ by \ \ (A/A[\alpha])^{-1}A[\bar{\alpha},\alpha]A[\alpha]^{-1}, \quad and \ \ A[\bar{\alpha}] \ \ by \ \ (A/A[\alpha])^{-1}. \end{array}$$

Definition 2.3. Given $\alpha \subseteq \langle n\rangle$ and provided that $A[\alpha]$ is invertible, we define the *principal pivot transform* (PPT) *of* $A \in \mathcal{M}_n(\mathbb{C})$ *relative to* α as the matrix $\operatorname{ppt}(A,\alpha)$ obtained from A by replacing

$$\begin{array}{l} A[\alpha] \ \text{ by } \ A[\alpha]^{-1}, \quad A[\alpha,\bar{\alpha}] \ \text{ by } \ -A[\alpha]^{-1}A[\alpha,\bar{\alpha}], \\ A[\bar{\alpha},\alpha] \ \text{ by } \ A[\bar{\alpha},\alpha]A[\alpha]^{-1}, \quad \text{and} \ \ A[\bar{\alpha}] \ \text{ by } \ A/A[\alpha]. \end{array}$$

By convention, $\operatorname{ppt}(A,\emptyset) = A$ and $\operatorname{ppt}(A,\langle n\rangle) = A^{-1}$.

For convenience and without loss of generality, let us assume that $\alpha = \langle k\rangle$ for some positive integer $k \le n$, that is, $A[\alpha]$ is a leading principal submatrix of $A \in \mathcal{M}_n(\mathbb{C})$; otherwise our arguments apply to a matrix that is permutationally similar to A. In this regard, $A_{11} = A[\alpha]$ is an invertible leading principal submatrix of A as in (1.1). Let B then be the PPT of A relative to A_{11} given in (1.2). Considering the matrices

$$C_1 = \begin{bmatrix} I & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix}, \tag{2.1}$$

observe the following two basic decompositions of A and B; see [11, Lemma 3.4]:

$$B = C_1C_2^{-1} \quad \text{and} \quad A = C_1 + C_2 - I. \tag{2.2}$$

Finally, in Section 3 we shall use the following lemma.

Lemma 2.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$, α,β be proper subsets of $\langle n\rangle$ and $B = \operatorname{ppt}(A,\alpha)$. Then the principal minor* $\det B[\beta]$ *satisfies the following:*

(a) *If $\beta \subseteq \alpha$, then* $\det B[\beta] = \det A[\beta]^{-1}$.
(b) *If $\beta \subseteq \bar{\alpha}$, then* $\det B[\beta] = \det(A/A[\alpha])[\beta']$, *where $\beta' = [\bar{\alpha}]_{\langle|\beta|\rangle}$.*
(c) *If $\beta = \gamma\cup\delta$, where $\emptyset \ne \gamma \subseteq \alpha$ and $\emptyset \ne \delta \subseteq \bar{\alpha}$, then* $\det B[\beta] = \det A[\delta] \,/\, \det A[\gamma]$.

Proof. Cases (a) and (b) follow readily from the block form of the PPT in Definition 2.3. Suppose, without loss of generality, that $A[\alpha]$ is a leading submatrix of A and that β is as prescribed in case (c). Let

$$\widehat{B} := \begin{bmatrix} I & 0 \\ -A[\bar{\alpha},\alpha] & I \end{bmatrix} \underbrace{\begin{bmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A[\alpha,\bar{\alpha}] \\ A[\bar{\alpha},\alpha]A[\alpha]^{-1} & A/A[\alpha] \end{bmatrix}}_{B} = \begin{bmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A[\alpha,\bar{\alpha}] \\ 0 & A[\bar{\alpha}] \end{bmatrix}.$$

Notice that $\widehat{B}$ is the outcome of adding multiples of the rows of A indexed by α to the rows of A indexed by $\bar{\alpha}$. As a consequence of determinantal properties and the block triangular form of $\widehat{B}$, we have that

$$\det B[\beta] = \det \widehat{B}[\beta] = \frac{\det A[\delta]}{\det A[\gamma]}. \qquad \square$$

3. Schur complements and PPTs of positivity classes. We begin by defining the matrix classes to be considered in this section:

- $A \in \mathcal{M}_n(\mathbb{C})$ is a *P-matrix* if all of the principal minors of A are positive.
- $A \in \mathcal{M}_n(\mathbb{C})$ is an *almost P-matrix* if all of the proper principal minors of A are positive and $\det A < 0$.
- $A \in \mathcal{M}_n(\mathbb{C})$ is an *N-matrix* if all of the principal minors of A are negative.
- $A \in \mathcal{M}_n(\mathbb{C})$ is an *almost N-matrix* if all of the proper principal minors of A are negative and $\det A > 0$.
- $A \in \mathcal{M}_n(\mathbb{C})$ is an *α-P-matrix*, where α is a nonempty subset of $\langle n \rangle$, if all of the principal minors of A are positive, except $\det A[\alpha] < 0$.
- $A \in \mathcal{M}_n(\mathbb{C})$ is an *$\alpha/\bar{\alpha}$-P-matrix*, where α is a nonempty proper subset of $\langle n \rangle$, if all of the principal minors of A are positive, except $\det A[\alpha] < 0$ and $\det A[\bar{\alpha}] < 0$.

We note in passing the availability of the algorithm MAT2PM[1] developed in [3], which allows one to detect matrices in the above classes by computing and indexing (as efficiently as possible) all the principal minors of a given matrix.

Next, note that as follows readily from definitions and determinantal properties, each one of the classes of N-, P-, almost P- and almost N-matrices is invariant under transposition, permutation similarity, signature similarity (i.e., similarity by a diagonal matrix all of whose nonzero entries are ± 1) and diagonal scaling by positive diagonal matrices.

We proceed by reviewing properties of the Schur complements of these matrix classes as needed.

THEOREM 3.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and $\alpha \subseteq \langle n \rangle$.*

(1) *If A is a P-matrix, then $A/A[\alpha]$ is a P-matrix.*
(2) *If A is an almost P-matrix, then $A/A[\alpha]$ is an almost P-matrix.*
(3) *If A is an N-matrix, then $A/A[\alpha]$ is a P-matrix for all nonempty $\alpha \subseteq \langle n \rangle$.*
(4) *If A is an α-P-matrix, then the following hold for $A/A[\delta]$:*
   (a) *If $\delta = \alpha$, then $A/A[\delta]$ is an N-matrix.*

[1] MATLAB® implementation linked at `http://www.math.wsu.edu/math/faculty/tsat/cv.html`

(b) *If δ is a proper subset of α , then $A/A[\delta]$ is an α-P-matrix.*
(c) *If $\delta \subseteq \bar{\alpha}$, then $A/A[\delta]$ is a P-matrix.*

(5) *If A is an almost N-matrix and α is a proper subset of $\langle n \rangle$, then $A/A[\alpha]$ is an almost P-matrix.*

Proof. Let $A \in \mathcal{M}_n(\mathbb{C})$, $\alpha \subseteq \langle n \rangle$, $\beta \subseteq \langle |\bar{\alpha}| \rangle$, $\beta' = [\bar{\alpha}]_\beta$ and denote $C = A/A[\alpha]$.

First, recall a well-known property of Schur complements; see, e.g., [2]:

$$\det A \;=\; \det A[\alpha]\; \det(A/A[\alpha]) \;=\; \det A[\alpha]\; \det C. \tag{3.1}$$

Also observe that from Lemma 2.1 and (3.1),

$$\det(A/A[\alpha]) \;=\; \det(A/A[\alpha \cup \beta'])\; \det((A/A[\alpha])[\beta]) \;=\; \det(A/A[\alpha \cup \beta'])\; \det C[\beta]. \tag{3.2}$$

The proofs of clauses (1)–(4) consist of repeated invocations of (3.1) and (3.2) relative to appropriate index sets outlined below.

(1) If A is a P-matrix, then by (3.1) applied to α and $\alpha \cup \beta$, $\det C > 0$ and $\det(A/A[\alpha \cup \beta']) > 0$, respectively. Thus, by (3.2), $\det C[\beta] > 0$. Hence $C = A/A[\alpha]$ is a P-matrix.

(2) If A is an almost P-matrix, then as above, (3.1) implies that $\det C < 0$ and $\det(A/A[\alpha \cup \beta']) < 0$. Thus, by (3.2), $\det C[\beta] > 0$. Hence $A/A[\alpha]$ is an almost P-matrix.

(3) If A is a N-matrix, then (3.1) implies that $\det C > 0$ and $\det(A/A[\alpha \cup \beta']) > 0$. Thus, by (3.2), $\det C[\beta] > 0$. Hence $A/A[\alpha]$ is a P-matrix.

(4) Let A be an α-P-matrix, $\delta \subseteq \langle n \rangle$, $\gamma \subseteq \langle |\bar{\delta}| \rangle$ and $\gamma' = [\bar{\delta}]_\gamma$.

(a) If $\delta = \alpha$, then from (3.1), $\det(A/A[\delta]) < 0$. Also, (3.1) implies that $\det(A/A[\delta \cup \gamma']) > 0$, and (3.2) implies that $\det((A/A[\alpha])[\gamma]) < 0$. Thus, $A/A[\delta]$ is an N-matrix.
(b) If δ is a proper subset of α , then from (3.1), $\det(A/A[\delta]) > 0$. Notice that if $\delta \cup \gamma' = \alpha$, then from (3.2), $\det(A/A[\delta \cup \gamma']) < 0$ and $\det((A/A[\delta])[\gamma]) < 0$. If $\delta \cup \gamma' \neq \alpha$, then $\det((A/A[\delta])[\gamma]) > 0$. Therefore, $A/A[\delta]$ is an α-P-matrix.
(c) If $\delta \subseteq \bar{\alpha}$, then by (3.1), $\det(A/A[\delta]) > 0$ and $\det(A/A[\delta \cup \gamma]) > 0$. Thus from (3.2), $\det((A/A[\delta])[\gamma]) > 0$. Therefore, $A/A[\delta]$ is a P-matrix.

(5) If A is an almost N-matrix, then equation (3.1) implies that $\det C < 0$ and $\det(A/A[\alpha \cup \beta']) < 0$. Now from the equation (3.2) $\det C[\beta] > 0$. Thus $A/A[\alpha]$ is an almost P-matrix. ◻

The fact that PPTs preserve the class of P-matrices is a fundamental result which Tucker asserted in [12] for real matrices. A proof of this result valid for complex matrices is provided in [11] from where we quote the following theorem.

Theorem 3.2. [11] *Let $\alpha \subseteq \langle n \rangle$. Then $A \in \mathcal{M}_n(\mathbb{C})$ is a P-matrix if and only if* $\operatorname{ppt}(A, \alpha)$ *is a P-matrix.*

The next result is stated in [8]; we include a proof for completeness.

Theorem 3.3. [8] *A is an N-matrix if and only if A^{-1} is an almost P-matrix.*

Proof. Let A be an N-matrix. By Lemma 2.2, every proper principal submatrix of A^{-1} is the inverse of a Schur complement of A. Thus, by Theorem 3.1 part(3), every proper principal minor of A^{-1} is positive. Also $\det A^{-1} = (\det A)^{-1} < 0$. That is, A^{-1} is an almost P-matrix. The proof of the converse is similar. ∎

Our next theorem extends the above result.

THEOREM 3.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be an almost P-matrix, α be a proper subset of $\langle n \rangle$ and $B = \text{ppt}\,(A, \alpha)$. Then B is an $\bar{\alpha}$-P-matrix, namely, $\det(B[\bar{\alpha}]) < 0$ and all other principal minors of B are positive.*

Proof. If $\alpha = \emptyset$, we have that $B = A$ is an almost P-matrix and thus has exactly one negative principal minor, namely, $\det B < 0$. If $\alpha \neq \emptyset$, referring to the cases in Lemma 2.4 for $B[\beta]$, the following hold: In case (a), $\det B[\beta] > 0$ since, by Theorem 3.2, $A[\alpha]^{-1}$ is a P-matrix. In case (b), by Theorem 3.1 part (2), if $\beta = \bar{\alpha}$, then $\det B[\beta] = \det A/A[\alpha] < 0$; if β is a proper subset of $\bar{\alpha}$, then $\det B[\beta] < 0$. In case (c), $\det B[\beta] > 0$ because proper principal minors of A are by assumption positive. ∎

THEOREM 3.5. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be an N-matrix, $\alpha \neq \emptyset$ be a proper subset of $\langle n \rangle$ and $B = \text{ppt}\,(A, \alpha)$. Then B is an α-P-matrix, namely, $\det(B[\alpha]) < 0$ and all other principal minors of B are positive.*

Proof. Since $A[\alpha]$ is an N-matrix, $A[\alpha]^{-1}$ is an almost P-matrix by Theorem 3.3. Also, by Theorem 3.1 part (1), $A/A[\alpha]$ is a P-matrix. Consequently, referring to the cases in Lemma 2.4 for $B[\beta]$, the following hold: In case (a), $\det B[\beta] < 0$ if $\beta = \alpha$, and $\det B[\beta] > 0$ if β is a proper subset of α. In case (b), $\det B[\beta] > 0$. In case (c), $\det B[\beta] > 0$ because proper principal minors of A are by assumption negative. ∎

We proceed with the inverse and other PPTs of an almost N-matrix.

THEOREM 3.6. *$A \in \mathcal{M}_n(\mathbb{C})$ is an almost N-matrix if and only if A^{-1} is an almost N-matrix.*

Proof. If A is an almost N-matrix, $\det A^{-1} = (\det A)^{-1} > 0$. Let $\alpha \neq \emptyset$ be a proper subset of $\langle n \rangle$. By Lemma 2.2, assuming without loss of generality that $A[\alpha]$ is a leading principal submatrix of A, we have

$$A^{-1} = \begin{bmatrix} (A/A[\bar{\alpha}])^{-1} & -A[\alpha]^{-1}A[\alpha,\bar{\alpha}](A/A[\alpha])^{-1} \\ -(A/A[\alpha])^{-1}A[\bar{\alpha},\alpha]A[\alpha]^{-1} & (A/A[\alpha])^{-1} \end{bmatrix}.$$

By Theorem 3.1 part (5), $A/A[\bar{\alpha}]$ and $A/A[\alpha]$ are almost P-matrices and by Theorem 3.3, $(A/A[\bar{\alpha}])^{-1}$ and $(A/A[\alpha])^{-1}$ are N-matrices. Thus all principal minors of A^{-1} whose rows are indexed by a subset of α or a subset of $\bar{\alpha}$ are negative. Let now $\gamma \neq \emptyset$ be a proper subset of α, $\delta \neq \emptyset$ a proper subset of $\bar{\alpha}$, and $\beta = \gamma \cup \delta$. Consider

$$C = \begin{bmatrix} I & 0 \\ -(A/A[\alpha])^{-1}A[\bar{\alpha},\alpha]A[\alpha]^{-1}A/A[\alpha] & I \end{bmatrix} A^{-1} = \begin{bmatrix} (A/A[\alpha])^{-1} & -A[\alpha]^{-1}A[\alpha,\bar{\alpha}](A/A[\alpha])^{-1} \\ 0 & A^{-1}/(A/A[\bar{\alpha}])^{-1} \end{bmatrix},$$

whose principal minor $\det C[\beta]$ coincides with $\det A^{-1}[\beta]$ because C is obtained from A^{-1} by adding multiples of the rows indexed by α to the rows indexed by $\bar{\alpha}$. We need to show that $\det C[\beta] < 0$.

By Lemma 2.2 applied to A^{-1} as partitioned above, we have that $A^{-1}/(A/A[\bar{\alpha}])^{-1} = A[\alpha]^{-1}$ which is an almost P-matrix by Theorem 3.3. Also by Theorem 3.3, $(A/A[\alpha])^{-1}$ is an N-matrix. Finally notice that $\det C[\beta] = \det C[\gamma] \det C[\delta] < 0$, completing the proof. □

THEOREM 3.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be an almost N-matrix, α be a proper nonempty subset of $\langle n \rangle$ and $B = \text{ppt}\,(A, \alpha)$. Then B is an $\alpha/\bar{\alpha}$-P-matrix.*

Proof. Since A is an almost N-matrix, $A[\alpha]$ is an N-matrix and by Theorem 3.3, $A[\alpha]^{-1}$ is an almost P-matrix. Also, by Theorem 3.1 part (5), $A/A[\alpha]$ is an almost P-matrix. Referring to the cases in Lemma 2.4 for $B[\beta]$, the following hold: In case (a), $\det B[\beta] < 0$ if $\beta = \alpha$, and $\det B[\beta] > 0$ if β is a proper subset of α. In case (b), $\det B[\beta] < 0$ if $\beta = \bar{\alpha}$, and $\det B[\beta] > 0$ if β is a proper subset of $\bar{\alpha}$. In case (c), $\det B[\beta] > 0$ because proper principal minors of A are by assumption negative. □

Our findings in this section are summarized by the diagrams in Figure 3.1. Let $B = \text{ppt}\,(A, \alpha)$. Note that the PPT is an involution, namely, $A = \text{ppt}\,(B, \alpha)$. Also $\text{ppt}\,(B, \bar{\alpha}) = \text{ppt}\,(A, \alpha \cup \bar{\alpha}) = A^{-1}$, assuming all said PPTs and inverses exist; see [11]. As a consequence, Theorems 3.2, 3.4 and 3.5 provide, respectively, characterizations of P-, almost P- and N-matrices in terms of their principal pivot transformations. This is reflected by dual directions in the first diagram. Also, by Theorem 3.7, when α is a nonempty proper subset of $\langle n \rangle$ and A is an almost N-matrix, then B is an $\alpha/\bar{\alpha}$-P-matrix. Therefore, $\text{ppt}\,(B, \alpha)$ is an almost N-matrix and by Theorem 3.6, $\text{ppt}\,(B, \bar{\alpha}) = A^{-1}$ is also an almost N-matrix. This situation is depicted in the last diagram.

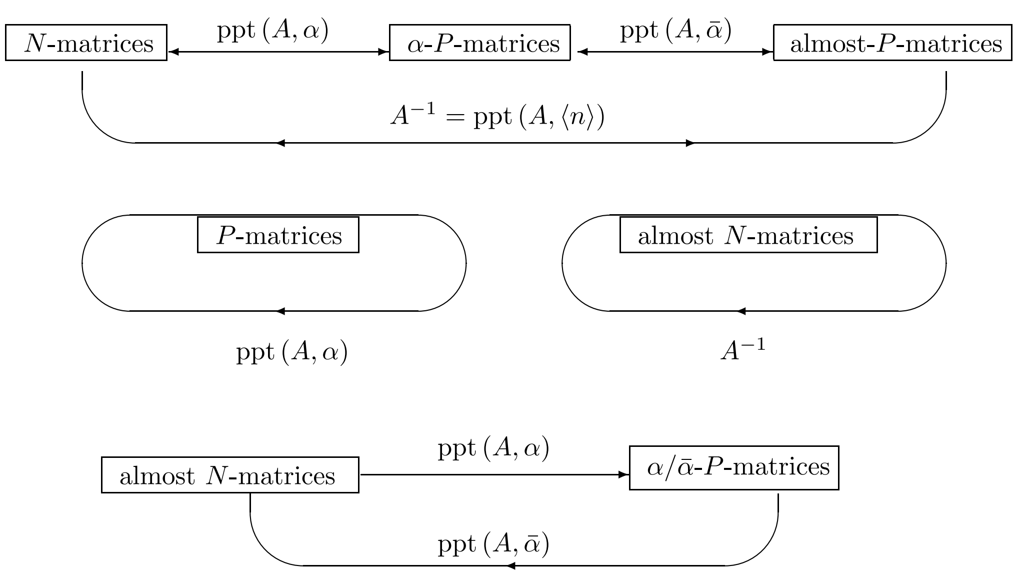

FIG. 3.1. Principal pivot transforms for $A \in \mathcal{M}_n(\mathbb{C})$, $\emptyset \neq \alpha \subseteq \langle n \rangle$.

We conclude this section with some illustrative examples.

EXAMPLE 3.8. Consider the N-matrix A and $B = \text{ppt}\,(A, \alpha)$, where $\alpha = \{1, 4\}$:

$$A = \begin{bmatrix} -1 & -2 & -2 & -3 \\ -3 & -3 & -3 & -3 \\ -5 & -3 & -1 & -2 \\ -1 & -2 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & -2 & -2 & -1.5 \\ 0 & 3 & 3 & 3 \\ -1.5 & 7 & 9 & 6.5 \\ -0.5 & 0 & 0 & 0.5 \end{bmatrix}.$$

By Theorem 3.5, B is an α-P-matrix. Indeed, $\det B[\alpha] = -0.5 < 0$ and all other principal minors are positive. Next consider the almost P-matrix C and $F = \text{ppt}\,(C, \beta)$, where $\beta = \{1, 2\}$:

$$C = \begin{bmatrix} 1/2 & -2/3 & 0 & 1/2 \\ -3/4 & 3/2 & -1/2 & -5/4 \\ 3/4 & -7/6 & 1/2 & 1/4 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix}, \quad F = \begin{bmatrix} 6 & 8/3 & 4/3 & 1/3 \\ 3 & 2 & 1 & 1 \\ 1 & -1/3 & 1/3 & -2/3 \\ -3 & -1/3 & -2/3 & 1/3 \end{bmatrix}.$$

By Theorem 3.4, F is an $\bar{\alpha}$-P-matrix. Indeed, $\det F[\bar{\alpha}] = -.3333 < 0$ and all other principal minors are positive. Finally, consider the almost N-matrix X, as well as X^{-1} and $Y = \text{ppt}\,(X, \gamma)$, where $\gamma = \{2, 3\}$:

$$X = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -2 & -2 \\ -2 & -4 & -3 \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} -0.0256 & 0.3077 & -0.1795 \\ 0.1282 & -0.0385 & -0.1026 \\ -0.1538 & -0.1538 & -0.0769 \end{bmatrix}, \quad Y = \begin{bmatrix} -39 & 12 & -7 \\ -5 & 1.5 & -1 \\ 6 & -2 & 1 \end{bmatrix}.$$

By Theorem 3.6, X^{-1} is also an almost N-matrix. Indeed, $\det X^{-1} = 0.0128 > 0$ and all other principal minors are negative. By Theorem 3.7, Y is a $\gamma/\bar{\gamma}$-P-matrix. Indeed, $\det Y[\gamma] = -.5 < 0$, $\det Y[\bar{\gamma}] = -39 < 0$ and all other principal minors are positive.

4. Rank of the Hermitian part of a PPT. The observations in this section are a generalization of results presented in [7] regarding almost skew-symmetric matrices. For any $A \in \mathcal{M}_n(\mathbb{C})$, write $A = H(A) + K(A)$, where

$$H(A) = \frac{A + A^*}{2} \quad \text{and} \quad K(A) = \frac{A - A^*}{2}$$

are the *Hermitian part* and the *skew-Hermitian part* of A, respectively.

THEOREM 4.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ such that $A[\alpha]$ is invertible for some $\alpha \subseteq \langle n \rangle$. Denote $B = \text{ppt}\,(A, \alpha)$ and $\hat{B} = \text{ppt}\,(iA, \alpha)$. Then $\text{rank}\, H(B) = \text{rank}\, H(A)$ and $\text{rank}\, H(\hat{B}) = \text{rank}\, K(A)$.*

Proof. Without loss of generality, let $A[\alpha] = A_{11}$ and A be partitioned as in (1.1). Consider the multiplicative decomposition of B in (2.2) and the *congruence of B given by

$$C_2^*\, B\, C_2 \;=\; C_2^*\, C_1\, C_2^{-1}\, C_2 \;=\; C_2^*\, C_1. \tag{4.1}$$

Observe also that

$$2H(C_2^*\, C_1) \;=\; C_2^*\, C_1 + C_1^* C_2 \;=\; \begin{bmatrix} A_{11} + A_{11}^* & A_{12} + A_{21}^* \\ A_{12}^* + A_{21} & A_{22} + A_{22}^* \end{bmatrix} \;=\; 2H(A). \tag{4.2}$$

By Sylvester's law of inertia (see e.g., [4, Theorem 4.5.8]) the rank of a Hermitian matrix is preserved by *congruence via an invertible matrix. Thus, since C_2 is by assumption invertible, applying (4.1) and (4.2), we obtain that

$$\operatorname{rank} H(B) \;=\; \operatorname{rank} H(C_2^*\, C_1) \;=\; \operatorname{rank} H(A).$$

Finally, by the above proven result and since $H(iA) = i\, K(A)$,

$$\operatorname{rank} H(\hat{B}) = \operatorname{rank} H(iA) = \operatorname{rank} K(A). \;\square$$

EXAMPLE 4.2. The principal pivot transform is not a rank preserving transformation. By the above theorem, however, the rank of the Hermitian part is always preserved. To illustrate these facts, consider $\alpha = \{1, 2\}$ as well as

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \operatorname{ppt}(A, \alpha) = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We compute $\operatorname{rank} A = \operatorname{rank} H(A) = \operatorname{rank} H(B) = 2$ and $\operatorname{rank} B = 3$. Note that the PPT is an involution and so $A = \operatorname{ppt}(B, \alpha)$. Thus B provides a full rank example whose PPT and Hermitian parts have lower ranks. Next note that

$$\hat{B} = \operatorname{ppt}(iA, \alpha) = \begin{bmatrix} i & -i & -1 \\ -2i & i & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

so that $\operatorname{rank} H(\hat{B}) = \operatorname{rank} K(B) = 2$. Finally, note that the rank of the skew-Hermitian part is not generally preserved by a PPT. This is easily illustrated by applying a PPT to a Hermitian matrix; e.g., if

$$C = \operatorname{ppt}(H(A), \alpha) = \begin{bmatrix} -0.8 & 1.2 & -1 \\ 1.2 & -0.8 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then clearly, $\operatorname{rank} K(H(A)) = 0$ while $\operatorname{rank} K(C) = 2$.

5. Preservation of left eigenspaces by a PPT. Generally, the spectrum and eigenspaces of the principal pivot transforms of $A \in \mathcal{M}_n(\mathbb{C})$ are hard to track in relation to those of A; see [11]. However, the nature of a PPT as a transformation of (right) domain-range pairs allows the possibility that the transformation of the left eigenspaces be tractable. This possibility is examined in this section.

Suppose that A is partitioned as in (1.1), where $A_{11} = A[\alpha]$ is invertible and $B = \operatorname{ppt}(A, \alpha)$. Consider also the decompositions of A and B given in (2.1) and (2.2). We aim to explore when do A and B share a left eigenvector, i.e., explore when there exist $\lambda, \mu \in \mathbb{C}$ and nonzero $x \in \mathbb{C}^n$ such that

$$x^T A = \lambda x^T \quad \text{and} \quad x^T B = \mu x^T \quad (x \neq 0). \tag{5.1}$$

First, the situation regarding the eigenvalue -1 is straightforward.

OBSERVATION 5.1. *$-1 \in \sigma(A)$ if and only if $-1 \in \sigma(B)$; the corresponding left eigenvectors are also necessarily common to A and B.*

Proof. The following equivalences ensue:

$$\begin{aligned} x^T A = -x^T &\iff x^T(C_1 + C_2 - I) = -x^T \\ &\iff x^T(C_1 + C_2) = 0 \\ &\iff x^T C_1 = -x^T C_2 \\ &\iff x^T C_1 C_2^{-1} = -x^T \\ &\iff x^T B = -x^T. \quad \square \end{aligned}$$

Notice now that the occurrence of a common left eigenvector in (5.1) is equivalent to the equations

$$x^T(C_1 + C_2 - I) = \lambda x^T \quad \text{and} \quad x^T C_1 = \mu x^T C_2. \tag{5.2}$$

Eliminating $x^T C_1$ in the first equation of (5.2), we get

$$x^T \left[\, (1+\mu)C_2 - (1+\lambda)I \,\right] = 0.$$

That is, every common left eigenvector of A and B must be a nonzero left nullvector of

$$F := (1+\mu)\, C_2 - (1+\lambda)\, I. \tag{5.3}$$

Similarly, by eliminating $x^T C_2$ from the second equation in (5.2), every common left eigenvector of A and B must also be a nonzero left nullvector of

$$G := (1+\mu)\, C_1 - \mu(1+\lambda)\, I. \tag{5.4}$$

Next we see that the existence of common left nullvectors of A and B force relations among the corresponding eigenvalues, as well as among the spectra of the principal submatrices.

THEOREM 5.2. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be partitioned as in (1.1) with $A_{11} = A[\alpha]$ invertible and let $B = \mathrm{ppt}\,(A,\alpha)$. Suppose that $x^T A = \lambda x^T$ and $x^T B = \mu x^T$ for some nonzero $x \in \mathbb{C}^n$. Then both of the following statements hold:*

$$\text{(i)} \quad [\,\lambda = \mu\,] \qquad \textit{or} \qquad \left[\, \frac{1+\lambda}{1+\mu} \in \sigma(A_{11}) \,\right]$$

$$\text{(ii)} \quad \left[\, 0 \neq \lambda = \frac{1}{\mu} \,\right] \qquad \textit{or} \qquad \left[\, \frac{\mu(1+\lambda)}{1+\mu} \in \sigma(A_{22}) \,\right].$$

Proof. (i) As argued above, if (5.1) and thus (5.2) hold, then x is a left nullvector of F in (5.3). It follows that

$$F = \begin{bmatrix} (1+\mu)A_{11} - (1+\lambda)I & (1+\mu)A_{12} \\ 0 & (\mu-\lambda)I \end{bmatrix}$$

is singular. If $\lambda \neq \mu$, then $1+\mu \neq 0$; otherwise, by Observation 5.1, $\lambda = -1 = \mu$. Hence,

$$\frac{1+\lambda}{1+\mu} \in \sigma(A_{11}).$$

(ii) Similarly to part (i), x must also be a left nullvector of the matrix in (5.4),

$$G = \begin{bmatrix} (1-\mu\lambda)I & 0 \\ (1+\mu)A_{21} & (1+\mu)A_{22} - \mu(1+\lambda)I \end{bmatrix}.$$

So, either $1-\mu\lambda = 0$ or $1+\mu \neq 0$; in the latter case,

$$\frac{\mu(1+\lambda)}{1+\mu} \in \sigma(A_{22}). \ \square$$

Next we describe the common left eigenvectors of A and B. To do so, by Theorem 5.2 part (i), we only need to consider the following cases in (5.1):

Case (i) $\lambda = \mu$.

If $\lambda = \mu = -1$, by Observation 5.1, the left eigenvectors corresponding to -1 coincide. So suppose $\lambda \neq -1$. Let $x^T = (\, x_1^T \ x_2^T \,)$ be partitioned conformally to A. As x is a left nullvector of

$$\frac{1}{1+\lambda} F = C_2 - I$$

(see (5.3)), we have that

$$\begin{cases} x_1^T A_{11} = x_1^T \quad (\Rightarrow 1 \in \sigma(A_{11}\,)) \\ x_1^T A_{12} = 0. \end{cases} \tag{5.5}$$

Similarly, as x is a left nullvector of

$$\frac{1}{1+\lambda} G = C_1 - \mu I,$$

we have

$$\begin{cases} x_2^T (A_{22} - \lambda I) = 0 \quad (\Rightarrow \lambda = \mu \in \sigma(A_{22}\,)) \\ (1-\lambda) x_1^T + x_2^T A_{21} = 0. \end{cases} \tag{5.6}$$

Case (ii) $\dfrac{1+\lambda}{1+\mu} \in \sigma(A_{11})$.

By (5.3) we have

$$(x_1^T \ x_2^T) \begin{bmatrix} (1+\mu)A_{11} - (1+\lambda)I & (1+\mu)A_{12} \\ 0 & (\mu-\lambda)I \end{bmatrix} = (\,0 \ \ 0\,),$$

which implies that

$$\begin{cases} x_1^T A_{11} = \frac{1+\lambda}{1+\mu} x_1^T \\ \\ x_1^T A_{12} + \frac{\mu-\lambda}{1+\mu} x_2^T = 0. \end{cases} \tag{5.7}$$

Similarly, from (5.4) it follows that

$$(x_1^T\ x_2^T) \begin{bmatrix} (1+\mu)A_{11} - \mu(1+\lambda)I & 0 \\ (1+\mu)A_{21} & (1+\mu)A_{22} - \mu(1+\lambda)I \end{bmatrix} = (0\ \ 0),$$

which implies

$$\begin{cases} x_2^T A_{22} = \frac{\mu\,(1+\lambda)}{1+\mu} x_2^T \\ \\ x_1^T(1-\mu\lambda) + (1+\mu)x_2^T A_{21} = 0. \end{cases} \tag{5.8}$$

In fact, the analysis of the above two cases characterizes the common left eigenvectors of A and its principal pivot transform as follows.

THEOREM 5.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be partitioned as in* (1.1) *with $A_{11} = A[\alpha]$ invertible and let $B = \mathrm{ppt}\,(A,\alpha)$. Recalling the two alternatives in* Theorem 5.2 part (i), *the following statements hold:*

(a) *When $\lambda = \mu \neq -1$, A and B have a common left eigenvector x corresponding to $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ if and only if* (5.5) *and* (5.6) *hold.*

(b) *When $\dfrac{1+\lambda}{1+\mu} \in \sigma(A_{11})$, A and B have a common left eigenvector x corresponding to $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ if and only if* (5.7) *and* (5.8) *hold.*

Proof. To prove part (a), first recall that if A and B have a common left eigenvector x corresponding to $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ with $\lambda = \mu$, then (5.5) and (5.6) hold by our analysis in Case (i) above. For the converse, if (5.5) and (5.6) hold and $\lambda = \mu$, then $x^T(C_2 - I) = 0$ and $x^T(C_1 - \mu I) = 0$. Therefore,

$$x^T A = x^T(C_1 + C_2 - I) = x^T C_1 = \mu x^T = \lambda x^T.$$

Also

$$x^T B = x^T C_1 C_2^{-1} = \mu x^T C_2^{-1} = \mu x^T$$

as claimed. The proof of part (b) is similar. □

The following is a consequence of our discussion, in particular a consequence of the two cases above.

COROLLARY 5.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be partitioned as in* (1.1) *with $A_{11} = A[\alpha]$ invertible and let $B = \mathrm{ppt}\,(A,\alpha)$. Suppose that $x^T A = \lambda x^T$ and $x^T B = \mu x^T$ for some nonzero $x \in \mathbb{C}^n$, where $\lambda \neq -1$ (and thus $\mu \neq -1$). Then there exist $\alpha \in \sigma(A_{11})$ and $\beta \in \sigma(A_{22})$ such that*

$$\lambda = \alpha + \beta - 1 \quad \textit{and} \quad \mu = \frac{\beta}{\alpha}.$$

Proof. Recall that $\mu \neq -1$ since $\lambda \neq -1$. By Theorem 5.2 part (i), we <u>either</u> have $\lambda = \mu$, in which case

$$\frac{1+\lambda}{1+\mu} = 1 \in \sigma(A_{11}) \quad \text{(cf. (5.5))} \quad \text{and} \quad \mu\frac{1+\lambda}{1+\mu} = \mu \in \sigma(A_{22}) \quad \text{(cf. (5.6))}$$

<u>or</u> $\dfrac{1+\lambda}{1+\mu} \in \sigma(A_{11})$, in which case

$$\mu\frac{1+\lambda}{1+\mu} = \mu \in \sigma(A_{22}) \quad \text{(cf. (5.8))}.$$

That is, in all cases,

$$\alpha := \frac{1+\lambda}{1+\mu} \in \sigma(A_{11}) \qquad \text{and} \qquad \beta := \mu\,\frac{1+\lambda}{1+\mu} \in \sigma(A_{22}),$$

from which the corollary follows readily. ∎

Corollary 5.4 does not hold for the eigenvalue $\lambda = \mu = -1$. As a counterexample consider

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \text{ppt}\,(A, \{1,2\}) = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Then -1 is an eigenvalue of A and B, sharing a common left eigenvector; however,

$$\sigma(A[\{1,2\}]) = \{-1, 1\} \quad \text{and} \quad \sigma(A[\{3\}]) = \{3\}$$

and so the conclusion of Corollary 5.4 is not valid.

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REFERENCES

[1] D.E. Crabtree and E.V. Haynsworth. An identity for the Schur complement of a matrix. *Proceedings of American Mathematical Society*, 22:364–366, 1969.
[2] M. Fiedler. *Special Matrices and their Applications in Numerical Mathematics.* Martinus Nijhoff, Dordrecht, 1986.
[3] K. Griffin and M.J. Tsatsomeros. Principal Minors, Part I: A method for computing all the principal minors of a matrix. To appear in *Linear Algebra and its Applications.*
[4] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, New York, 1990.
[5] G.A. Johnson. A generalization of N-matrices. *Linear Algebra and its Applications*, 48:201–217, 1982.
[6] J.S. Maybee. Some aspects of the theory of PN-matrices. *SIAM Journal on Applied Mathematics*, 312:397–410, 1976.
[7] J.J. McDonald, P.J. Psarrakos, M.J. Tsatsomeros. Almost skew-symmetric matrices. *Rocky Mountain Journal of Mathematics*, 34:269–288, 2004.
[8] C. Olech, T. Parthasarathy, G. Ravindran. Almost N-matrices and linear complementarity. *Linear Algebra and its Applications*, 145:107–125, 1991.

[9] D. Ouellette. Schur complements and statistics. *Linear Algebra and its Applications*, 36:187–295, 1981.
[10] T. Parthasarathy and G. Ravindran. N-matrices. *Linear Algebra and its Applications*, 139:89–102, 1990.
[11] M. Tsatsomeros. Principal pivot transforms: Properties and applications. *Linear Algebra and its Applications*, 307:151–165, 2000.
[12] A. W. Tucker. Principal pivotal transforms of square matrices. *SIAM Review*, 5:305, 1963.