THE REACHABILITY CONES OF ESSENTIALLY NONNEGATIVE MATRICES

by
Michael Neumann *
Department of Mathematics,
University of Connecticut, Storrs, CT 06269–3009
and

Ronald J. Stern †
Department of Mathematics,
Concordia University, Montreal, Canada H4X 1J7
and

Michael J. Tsatsomeros ‡
Department of Mathematics,
University of Connecticut, Storrs, CT 06269–3009

*Research supported in part by US Air Force Research Grant No AFOSR-88-0047 and by NSF Grant No DMS-8901860. This author would also like to thank NSERC for making it possible for him to visit Ronald J. Stern in Montreal.

†Research supported by the Natural Sciences and Engineering Council of Canada, grant No A4641.

‡Research supported in part by US Air Force Research Grant No AFOSR-88-0047 and by NSF Grant No DMS-8901860.

Abstract

Let A be an $n \times n$ essentially nonnegative matrix and consider the linear differential system $\dot{x}(t) = Ax(t)$, $t \geq 0$. We show that there exists a constant $h(A) > 0$ such that the trajectory emanating from x_o reaches R^n_+ at a finite time $t_o = t(x_o) \geq 0$ if and only if the sequence of points generated by a finite differences approximation from x_o, with time–step $0 < h < h(A)$, reaches R^n_+ at a finite index $k_o = k(x_o) \geq 0$. This generalizes and strengthens earlier results of two of the authors, where some additional spectral restrictions were imposed on A. Our proof makes use of the existence of a basis of nonnegative vectors to the Perron eigenspace.

1 Introduction

Let A be an $n \times n$ matrix and consider the linear differential system

$$\dot{x}(t) = Ax(t), \quad t \geq 0. \tag{1.1}$$

The solution to (1), given by

$$x(t) = e^{tA}x_o, \quad \forall t \ \geq \ 0, \tag{1.2}$$

will be referred to as *the trajectory of the differential system emanating from* x_o. A set $\Gamma \subset R^n$ is called *positively invariant* with respect to A if

$$e^{tA}(\Gamma) \subseteq \Gamma, \quad \forall t \geq 0.$$

This condition has the implication that once a trajectory emanating from x_o reaches Γ in some finite time, it remains in Γ thereafter. The set of all initial points of trajectories which reach Γ is called the *reachability set for* Γ under A and is denoted by $X_A(\Gamma)$. Clearly,

$$X_A(\Gamma) = \{x_o \in R^n \mid (\exists t_o = t(x_o) \geq 0)(\forall t \geq t_o)[e^{tA}x_o \in \Gamma]\} = \bigcup_{t \geq 0} e^{-tA}\Gamma. \tag{1.3}$$

In the case when Γ is a proper cone, namely, a closed, convex, pointed and solid cone, then it was shown in Neumann and Stern [**?**] that $X_A(\Gamma)$ is a convex cone itself, which contains Γ, but which is not necessarily closed or pointed. We shall denote the closure of $X_A(\Gamma)$ by $\overline{X}_A(\Gamma)$.

In this paper $A = (a_{ij})$ will always denote an *essentially nonnegative* matrix, that is, a matrix whose off–diagonal entries are nonnegative. This condition is known to be equivalent to the positive invariance of the *nonnegative orthant* R^n_+ with respect to A (see Bellman [**?**] and Birkhoff and Varga [**?**]).

In Neumann and Stern [**?**] and in Berman, Neumann, and Stern [**?**], explicit formulas were derived for $\overline{X}_A(R^n_+)$ under additional spectral assumptions on A. In Neumann and Stern [**?**], a numerical characterization of $\overline{X}_A(R^n_+)$ based on a Cauchy–Euler approximation to the solution to (1.1) was developed. However, there, in addition to A being essentially nonnegative, it was assumed that A has a *real spectrum* or that it possesses a *strictly positive generalized eigenvector* corresponding to its Perron root. For further description of these reachability results the reader is referred to

Chapter 6 of Berman, Neumann, and Stern [**?**]. *In the present work we pose the questions of generalizing and strengthening the aforementioned numerical characterization of* $\overline{X}_A(R^n_+)$ *by assuming that* A *is only essentially nonnegative and by numerically characterizing* $X_A(R^n_+)$ *and not just its closure.*

For the sake of completeness we first give one of the main results in Neumann and Stern [**?**]. Define

$$h(A) = \sup\{h \mid \min_{1 \le i \le n}(1 + ha_{ii}) > 0\}. \tag{1.4}$$

It is clear that $h(A)$ can be infinite and that for any $0 < h < h(A)$ the matrix $I + hA$ is nonnegative. For any such h we define the *discrete reachability set* of R^n_+ to be

$$X_{A,h}(R^n_+) = \{x_o \in R^n \mid (\exists k_o = k(x_o) \ge 0)(\forall k \ge k_o)[(I + hA)^k x_o \ge 0]\} \tag{1.5}$$

which can be shown to be an invariant cone under $I + hA$. The sequence of vectors $x_k = (I + hA)^k x_o$, $k = 0, 1, \ldots,$ will be referred to as the *discrete trajectory emanating from* x_o. This sequence represents an approximation to the solution of (1.1), in the sense of Cauchy–Euler, at a discrete set of times. Next, it is known by the classical Perron–Frobenius theory that the Perron root

$$\lambda_1 := \max\{Re\mu \mid \mu \in \sigma(A)\} \ \in \ \sigma(A), \tag{1.6}$$

where $\sigma(A)$ denotes the spectrum of A. Recall now that the *Perron (generalized) eigenspace* of A is defined to be the A–invariant subspace

$$W_A = N((\lambda_1 I - A)^p), \tag{1.7}$$

where p is the multiplicity of λ_1 in the minimal polynomial of A. The result of Neumann and Stern which we wish to quote here can now be stated as follows.

Theorem 1.1 ([?], Theorem 2.2) *Suppose* A *is an* $n \times n$ *essentially nonnegative matrix. If*

$$W_A \cap int(R^n_+) \ne \emptyset, \tag{1.8}$$

then for any $0 < h < h(A)$ *which satisfies*

$$1 + h\lambda_1 > |1 + h\mu|, \quad \forall \mu \in \sigma(A) \setminus \{\lambda_1\}, \tag{1.9}$$

we have that

$$\overline{X}_{A,h}(R^n_+) = \overline{X}_A(R^n_+). \tag{1.10}$$

We should emphasize the fact that the time–step h in (1.9) depends only on A and is not necessarily small. In other words, Theorem 1.1 provides us with a way of testing whether a point lies in $\overline{X}_A(R^n_+)$ regardless of how much the continuous and discrete trajectories emanating from x_o diverge from each other.

The main objectives of this paper are to extend and improve the result of Theorem 1.1. This will be achieved in two principal stages. In the first stage, Theorem 3.1, we shall show that the conclusion of Theorem 1.1 holds for *any* essentially nonnegative matrix. Our proof of this result will be aided by a block triangular form, due to Hartwig, Neumann, and Rose [**?**], to which, actually, any nonnegative matrix can be symmetrically permuted. For the sake of completeness, this form is displayed in Lemma 2.1. In the second stage, Theorem 3.3, we shall show that with a possible slight additional restriction on the supremum of h, the equality in (1.10) holds *even* when the closure signs are removed. The proof of Theorem 3.3 requires that we further expose geometrical properties of the discrete reachability cone $X_{A,h}(R^n_+)$. We do this in Lemma 3.2.

2 Preliminaries

Suppose that B is an $n \times n$ nonnegative matrix and let

$$W_B = N((\rho(B)I - B)^p)$$

be the generalized Perron eigenspace of B corresponding to its spectral radius $\rho(B)$, where p is the degree of $\rho(B)$ in the minimal polynomial of B. The lemma to follow assigns to B a certain block upper–triangular form which will be essential to the generalization of Theorem 1.1. It is based on Lemma 3.1 and Corollaries 3.1 – 3.2 of Hartwig, Neumann, and Rose [**?**]. We comment that this form could also be derived from proofs of the existence of a nonnegative basis for W_B, which are based on the Frobenius normal form for B, as given in Rothblum [**?**] and Richman and Schneider [**?**]. However, the proof of our main result in Section 3 does not require that we consider the Frobenius normal form of a matrix.

For a nonnegative vector w, we shall let $\nu(w)$ denote the number of positive entries in w.

Lemma 2.1 *Any* $n \times n$ *nonnegative matrix* B *is permutationally similar* *to a block upper–triangular matrix*

$$\begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \dots & \tilde{B}_{1p} \\ & \tilde{B}_{22} & \dots & \tilde{B}_{2p} \\ & & \ddots & \vdots \\ & 0 & & \tilde{B}_{pp} \end{bmatrix} \tag{2.1}$$

with the following properties. If for each $j \in \{1, 2, \dots, p\}$ *we set*

$$\tilde{B}_j = \begin{bmatrix} \tilde{B}_{jj} & \dots & \tilde{B}_{jp} \\ & \ddots & \vdots \\ 0 & & \tilde{B}_{pp} \end{bmatrix} \in R_+^{q,q}, \tag{2.2}$$

then each diagonal block $\tilde{B}_{jj}$ *in (2.1) is of size* $k_j \times k_j$, *where*

$$k_j = \max\{\nu(w) \mid w \in W_{\tilde{B}_j} \cap R_+^q\}, \quad q = n - k_1 - \dots - k_{j-1}$$

and it possesses a strictly positive generalized eigenvector

$$u_j \ \in \ W_{\tilde{B}_{jj}} \ \cap \ int R_+^{k_j} \tag{2.3}$$

corresponding to $\rho(\tilde{B}_{jj})$. *Moreover,*

$$\rho(\tilde{B}_j) = \rho(\tilde{B}_{jj}), \quad \rho(\tilde{B}_j) > \rho(\tilde{B}_{j+1}), \ \forall\ 1 \le j \le p-1, \tag{2.4}$$

$$W_{\tilde{B}_j} = \{u \in R^q \mid u = [\bar{u}^T, 0]^T, \ \bar{u} \in W_{\tilde{B}_{jj}}\} \tag{2.5}$$

and

$$(\tilde{B}_{jj} - \rho(\tilde{B}_{jj})I)^{m_j - 1} u_j \in R_+^{k_j} \setminus \{0\}, \tag{2.6}$$

where $m_j = index_{\rho(\tilde{B}_{jj})} \tilde{B}_{jj}$.

We conclude this section by recalling that R_+^n admits an alternative representation as the intersection of n closed halfspaces (e.g. Rockafellar [**?**]) as follows : Let $\langle\ .\ ,\ .\ \rangle$ denote the usual inner product in R^n and suppose that $\{\nu_i \mid i = 1, 2, \dots, n\}$ is the set of all outward unit normals to R_+^n (that is, $\nu_i = [0, \dots, 0, -1, 0, \dots, 0]^T$, where the nonzero entry occurs in the i-th position). Then,

$$R_+^n = \bigcap_{i=1}^{n} \{z \in R^n \mid \langle \nu_i, z \rangle \le 0\}. \tag{2.7}$$

3 The Main Results

Let A be an $n \times n$ essentially nonnegative matrix and let $h(A)$ be as defined in (1.4). In what follows we shall assume, without loss of generality, that the matrix $B = B(h) = I + hA$ is already in the upper triangular block form given in Lemma 2.1. Otherwise, our considerations apply to a permutation similarity of A. We are now ready to state the first main result of this paper.

Theorem 3.1 *Let A be an $n \times n$ essentially nonnegative matrix. Then for all $0 < h < h(A)$,*

$$\overline{X}_{A,h}(R^n_+) \quad = \quad \overline{X}_A(R^n_+). \tag{3.1}$$

Proof
We will first show that $\overline{X}_A(R^n_+) \subseteq \overline{X}_{A,h}(R^n_+)$. Suppose that there exists $z \in \overline{X}_A(R^n_+)$ such that

$$z \notin \overline{X}_{A,h}(R^n_+). \tag{3.2}$$

Define the vector

$$u = [u_1^T, u_2^T, \ldots, u_p^T]^T \in int R^n_+, \tag{3.3}$$

where the vectors u_j, $j = 1, 2, \ldots, p$ are defined as in Lemma 2.1. Then there exists a strictly decreasing sequence of positive numbers $\{\epsilon_m\}_{m=1}^\infty$, with $\epsilon_m \to 0$, such that

$$z + \epsilon_m u \notin \overline{X}_{A,h}(R^n_+), \quad \forall m \geq 1. \tag{3.4}$$

For an arbitrary but fixed m, the exclusion in (3.4) means by (2.7) that for each $i \geq 1$ there exists an outward unit normal $\nu^{(m,i)}$ to R^n_+ such that

$$\langle \nu^{(m,i)}, \; B^i(z + \epsilon_m u)\rangle > 0. \tag{3.5}$$

Since there are only a finite number of such normals there exists a normal, say $\nu^{(m)}$, and a sequence $\{i_k^{(m)}\}_{k=1}^\infty$ such that

$$\langle \nu^{(m)}, \; B^{i_k^{(m)}}(z + \epsilon_m u)\rangle > 0, \; \forall k \geq 1. \tag{3.6}$$

Suppose that z and $\nu^{(m)}$ are partitioned in conformity with (2.1). Assume further that the nonzero entry of $\nu^{(m)}$ occurs in the l^{th} block, namely,

$$z = [z_1^T, \ldots, z_l^T, \ldots, z_p^T]^T \quad and \quad \nu^{(m)} = [0, \ldots, 0, (\nu_l^{(m)})^T, 0, \ldots, 0]^T. \tag{3.7}$$

Consider now the trailing submatrix (viz. Lemma 2.1)

$$B_l = I + hA_l \in R^{q,q}, \quad q = n - k_1 - \ldots - k_{l-1} \tag{3.8}$$

and construct the following q–vectors:

$$\bar{\nu}^{(m)} := [(\nu_l^{(m)})^T, 0, \ldots, 0]^T, \quad \bar{z}^{(m)} := [z_l^T, \ldots, z_p^T]^T,$$

$$\bar{u}^{(m)} := [u_l^T, 0, \ldots, 0]^T \in R_+^q, \quad \hat{u}^{(m)} := [0, u_{l+1}^T, \ldots, u_p^T]^T \in R_+^q. \tag{3.9}$$

Then, $\forall k \geq 1$,

$$\begin{aligned}
\langle \bar{\nu}^{(m)}, B_l^{i_k^{(m)}}(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle &\geq \langle \bar{\nu}^{(m)}, B_l^{i_k^{(m)}}(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle \ + \\
&\quad + \langle \bar{\nu}^{(m)}, \epsilon_m B_l^{i_k^{(m)}} \hat{u}^{(m)}\rangle \\
&= \langle \bar{\nu}^{(m)}, B_l^{i_k^{(m)}}(\bar{z}^{(m)} + \epsilon_m(\bar{u}^{(m)} + \hat{u}^{(m)}))\rangle \\
&= \langle \nu^{(m)}, B^{i_k^{(m)}}(z + \epsilon_m u)\rangle.
\end{aligned} \tag{3.10}$$

Thus by (3.5) and (3.10),

$$\langle \bar{\nu}^{(m)}, B_l^{i_k^{(m)}}(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle > 0, \quad \forall k \geq 1. \tag{3.11}$$

Next, as e^{tA} is upper triangular $\forall t \geq 0$, a necessary condition for the assumption $z \in \overline{X}_A(R_+^n)$ to hold true is that

$$\bar{z} \in \overline{X}_{A_l}(R_+^q). \tag{3.12}$$

Recall now that $u_l \in intR_+^{k_l}$. It was shown in Neumann and Stern [**?**] that $intX_{A_{ll}}(R_+^{k_l})$=$X_{A_{ll}}(intR_+^{k_l})$ and, furthermore, once a trajectory has entered $intR_+^{k_l}$ it cannot in finite time reach the boundary of $R_+^{k_l}$. Consequently

$$\langle \bar{\nu}^{(m)}, \epsilon_m e^{tA_l} \bar{u}^{(m)}\rangle < 0, \quad \forall t \geq 0. \tag{3.13}$$

Then, (3.12) and (3.13) have the implication that there exists a sufficiently large $t_m \geq 0$ such that

$$\langle \bar{\nu}^{(m)}, e^{tA_l}(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle < 0, \quad \forall t \geq t_m. \tag{3.14}$$

We shall next show that (3.11) and (3.14) are incompatible as we *vary* m. But first, let

$$\lambda_l := \max\{Re\mu \mid \mu \in \sigma(A_l)\} \tag{3.15}$$

and write

$$B_l = (1 + h\lambda_l)I \ + \ h(A_l - \lambda_l I). \tag{3.16}$$

Observe then that, by Lemma 2.1, $\bar{u}^{(m)} \in W_{A_l} \cap R^q_{\neq}$ and consider the resolution of $\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}$ into

$$\bar{z}^{(m)} \ + \ \epsilon_m \bar{u}^{(m)} \ = \ (\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) + \bar{r}^{(m)}, \tag{3.17}$$

where $\bar{r}^{(m)}$ is the projection of $\bar{z}^{(m)}$ (and hence of $\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}$) onto U_{A_l}, the join of all eigenspaces of A_l corresponding to eigenvalues $\mu \neq \lambda_l$ along W_{A_l}. We claim that, independently of m,

$$\langle \bar{\nu}^{(m)}, \bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)} \rangle = 0. \tag{3.18}$$

Suppose to the contrary. As $A_l - \lambda_l I$ is nilpotent on W_{A_l}, let $p_1 \geq 0$ be the largest integer such that

$$\langle \bar{\nu}^{(m)}, (A_l - \lambda_l I)^{p_1} (\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle \neq 0 \tag{3.19}$$

and

$$\langle \bar{\nu}^{(m)}, (A_l - \lambda_l I)^{j} (\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle = 0, \ \forall j > p_1. \tag{3.20}$$

If A is a nonnegative matrix, so that $h(A) = \infty$, then for any $h \in (0, \infty)$ the eigenvalues of $I + hA_l$ satisfy

$$1 \ + \ h\lambda_l > |1 + h\mu|, \ \forall \mu \in \sigma(A_l) \setminus \{\lambda_l\} \tag{3.21}$$

or else (3.15) is violated. Similarly, if $h(A) < \infty$, (3.21) holds true for all $0 < h < h(A)$. Consequently, the restriction of $B_l/(1 + h\lambda_l)$ to U_{A_l} is a convergent matrix, namely,

$$\lim_{k \to \infty} \left[\frac{B_l}{1 + h\lambda_l} \right]^{i_k^{(m)}} \bar{r}^{(m)} \ = \ 0. \tag{3.22}$$

Then for $i_k^{(m)} > p_1$ by (3.19) and (3.20) we can write that

$$\begin{aligned}
& \langle \bar{\nu}^{(m)}, \ B_l^{i_k^{(m)}} (\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle = \\
= \ & \binom{i_k^{(m)}}{p_1} (1 + h\lambda_l)^{i_k^{(m)} - p_1} \ h^{p_1} \ \langle \bar{\nu}^{(m)}, \ (A_l - \lambda_l I)^{p_1} \ (\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle + \\
+ \ & \sum_{j=0}^{p_1 - 1} \binom{i_k^{(m)}}{j} (1 + h\lambda_l)^{i_k^{(m)} - j} \ h^j \ \langle \bar{\nu}^{(m)}, \ (A_l - \lambda_l I)^j \ (\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle + \\
+ \ & \langle \bar{\nu}^{(m)}, \ B_l^{i_k^{(m)}} \ \bar{r}^{(m)} \rangle.
\end{aligned} \tag{3.23}$$

Note that as $k \to \infty$ and for all $0 \le j < p_1$,

$$\binom{i_k^{(m)}}{j} \Big/ \binom{i_k^{(m)}}{p_1} \ \to\ 0. \tag{3.24}$$

Also, by (3.22),

$$\lim_{k\to\infty} \frac{(1+h\lambda_l)^{p_1}}{\binom{i_k^{(m)}}{p_1} h^{p_1}} \ \langle \bar{\nu}^{(m)},\ \left[\frac{B_l}{1+h\lambda_l}\right]^{i_k^{(m)}} \bar{r}^{(m)}\rangle = 0. \tag{3.25}$$

Now, since $1 + h\lambda_l > 0$, upon taking k sufficiently large, relations (3.11) and (3.23)–(3.25) have the implication that

$$\langle \bar{\nu}^{(m)},\ (A_l - \lambda_l I)^{p_1}\ (\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle > 0. \tag{3.26}$$

On the other hand, the restriction of $A_l - \lambda_l I$ to U_{A_l} is a stability matrix and so

$$\lim_{t\to\infty} e^{t\ (A_l - \lambda_l I)} \bar{r}^{(m)}\ =\ 0. \tag{3.27}$$

Now, by (3.19) and (3.20) one obtains

$$\begin{aligned}
&\langle \bar{\nu}^{(m)},\ e^{tA_l}(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}\rangle\ = \\
=\ & e^{t\lambda_l}\langle \bar{\nu}^{(m)},\ e^{t(A_l-\lambda_l I)}(\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle + e^{t\lambda_l}\langle \bar{\nu}^{(m)}, e^{t(A_l-\lambda_l I)}\bar{r}^{(m)}\rangle \\
=\ & \frac{t^{p_1} e^{t\lambda_l}}{p_1!}\langle \bar{\nu}^{(m)},\ (A_l - \lambda_l I)^{p_1}(\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle \\
+\ & e^{t\lambda_l}\sum_{j=0}^{p_1-1}\frac{t^j}{j!}\langle \bar{\nu}^{(m)},\ (A_l - \lambda_l I)^{l}(\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle \\
+\ & e^{t\lambda_l}\langle \bar{\nu}^{(m)},\ e^{t(A_l-\lambda_l I)}\bar{r}^{(m)}\rangle.
\end{aligned} \tag{3.28}$$

But then, for t sufficiently large, (3.14), (3.27) and (3.28) give us that

$$\langle \bar{\nu}^{(m)},\ (A_l - \lambda_l I)^{p_1}(\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)})\rangle\ \le\ 0, \tag{3.29}$$

a contradiction to (3.26) showing that (3.18) is valid. On setting $w^{(m)} := [0, \bar{w}^{(m)}]^T \in R^n$, by (3.17) we obtain that

$$\langle \nu^{(m)}, w^{(m)}\rangle + \epsilon_m \langle \nu^{(m)}, u\rangle = 0. \tag{3.30}$$

Now vary m. Then, since the number of outward normals is finite, there must exist indices $m_2 > m_1 \geq 1$ such that $\nu := \nu^{(m_1)} = \nu^{(m_2)}$, and hence $w := w^{(m_1)} = w^{(m_2)}$, so that

$$\langle \nu, w \rangle + \epsilon_{m_i} \langle \nu, u \rangle = 0, \quad i = 1, 2 \tag{3.31}$$

which, as $\epsilon_{m_1} < \epsilon_{m_2}$, is only possible if

$$\langle \nu, u \rangle = 0, \tag{3.32}$$

which is not possible as $u \in intR^n_+$. This shows that

$$\overline{X}_A(R^n_+) \subseteq \overline{X}_{A,h}(R^n_+) \tag{3.33}$$

and completes the first part of the theorem.

We shall next prove the reverse containment, namely, that

$$\overline{X}_{A,h}(R^n_+) \ \subseteq \ \overline{X}_A(R^n_+). \tag{3.34}$$

Let $z \in \overline{X}_{A,h}(R^n_+)$ such that

$$z \notin \overline{X}_A(R^n_+). \tag{3.35}$$

Suppose that u and $\{\epsilon_m\}_{m=1}^{\infty}$ are chosen as before so that

$$z + \epsilon_m u \notin \overline{X}_A(R^n_+), \quad \forall m \geq 1. \tag{3.36}$$

Consider m arbitrary but fixed. The exclusion in (3.36) means that for any unbounded sequence of strictly increasing positive times $\{t_j^{(m)}\}_{j=1}^{\infty}$, there exist $\nu^{(m)}, \bar{\nu}^{(m)}, \bar{z}^{(m)}, \bar{u}^{(m)}$, defined in a manner similar to the first part of the theorem (viz. equations (3.3)–(3.11)), so that

$$\langle \bar{\nu}^{(m)}, \ e^{t_j^{(m)} A_l} (\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)} \rangle > 0, \ \forall j \geq 1. \tag{3.37}$$

Next, by equations (2.4) and (2.6) of Lemma 2.1 we can deduce that

$$B_{ll}^j u_l \in intR^{k_l}_+, \quad \forall j \geq 0.$$

Observe now that the assumption $z \in \overline{X}_{A,h}(R^n_+)$ has the implication that there exists a sufficiently large exponent j_m so that

$$\langle \bar{\nu}^{(m)}, B_l^j (\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle < 0, \quad \forall j \geq j_m. \tag{3.38}$$

We shall now show that, as we vary m, equations (3.37) and (3.38) are incompatible. For this, let us resolve $\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}$ as in (3.17). Then, an analysis similar to the one for equations (2.23) and (2.28) shows that

$$\langle \bar{\nu}^{(m)}, \bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)} \rangle = 0 \tag{3.39}$$

or else, for some $p_1 \geq 0$,

$$sgn\langle \bar{\nu}^{(m)},\ e^{t_j^{(m)} A_l}(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle \ = \ sgn\langle \bar{\nu}^{(m)},\ (A_l - \lambda_l I)^{p_1}(\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle \tag{3.40}$$

and

$$sgn\langle \bar{\nu}^{(m)},\ B_l^j(\bar{z}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle \ = \ sgn\langle \bar{\nu}^{(m)},\ (A_l - \lambda_l I)^{p_1}(\bar{w}^{(m)} + \epsilon_m \bar{u}^{(m)}) \rangle, \tag{3.41}$$

which contradict (3.37) and (3.38). Then, in a similar fashion to (3.30) and (3.31), upon varying m, it follows from (3.39) that for some outward normal ν to R^n_+, $\langle \nu, u \rangle = 0$. This is a contradiction to (3.3) which completes the proof of the theorem $\square$

As a consequence of the fact that in general the reachability cone is not closed, in order to achieve our goal of providing a numerical characterization for the elements of $X_A(R^n_+)$ we must improve the result of Theorem 3.1 by distinguishing between those boundary points which reach R^n_+ and those which do not. The set of all such points was termed in [?] the *effective part of the boundary* of $X_A(R^n_+)$. In the remainder of this section we shall utilize some geometrical properties of the continuous and discrete reachability cones to show that for all but a finite number of values of h in $(0, h(A))$ we can strengthen the result of Theorem 3.1, given in equation (3.1), by proving that

$$X_A(R^n_+) = X_{A,h}(R^n_+). \tag{3.42}$$

For this purpose we require the following lemma, some of whose clauses have already been established in the literature.

For any set $\Gamma \subseteq R^n$ and $A \in R^{n,n}$ we define $core_A(\Gamma)$ to be the *maximal subset of* Γ *which is positively invariant with respect to* A and $core_{A,h}(\Gamma)$ to be the *maximal subset of* Γ *which is invariant with respect to* $B = B(h) = I + hA$. As usual, we denote the boundary of Γ by $\partial\Gamma$.

Lemma 3.2 *Let* A *be an* $n \times n$ *essentially nonnegative matrix and suppose that* $0 < h < h(A)$ *is chosen so that* $B = B(h) = I + hA$ *is invertible. Then the following hold.*

(i) $X_A(intR^n_+) = intX_A(R^n_+)$.

(ii) $\partial X_A(R^n_+)$ is positively invariant with respect to A.

(iii) $X_{A,h}(intR^n_+) = intX_{A,h}(R^n_+)$.

(iv) $\partial X_{A,h}(R^n_+)$ is invariant with respect to B.

(v) $X_A(R^n_+) \cap \partial X_A(R^n_+) = X_A(core_A(\partial R^n_+))$.

(vi) $X_{A,h}(R^n_+) \cap \partial X_{A,h}(R^n_+) = X_{A,h}(core_{A,h}(\partial R^n_+))$.

(vii) $core_A(\partial R^n_+)$ is the union of all the positively invariant faces of R^n_+.

(viii) $core_{A,h}(\partial R^n_+) = core_A(\partial R^n_+)$.

Proof

Claims (i), (ii), (v) and (vii) are due to Neumann and Stern [**?**] (see also Berman, Neumann, and Stern [**?**]). We comment that it is evident from the proof of (i) in [**?**] that once a trajectory emanating from a point in $intX_A(R^n_+)$ has entered $intR^n_+$, it remains in $intR^n_+$ for all finite time. Now let h be as prescribed. Then the proofs of (iii), (iv), and (vi) are very similar to their respective counterparts in the continuous case and therefore they are omitted. To show (viii) consider a point $z \in core_A(\partial R^n_+)$. By (vii) z belongs to some positively invariant face F of R^n_+. Thus, since F is a polyhedral proper cone in the span of F, the restriction of A to F must be essentially F–nonnegative, namely for some $\alpha \geq 0$,

$$(A + \alpha I)F \subseteq F \tag{3.43}$$

(see Schneider and Vidyasagar [**?**] and Stern [**?**]). Then, for any $x{\in}F{\subseteq}R^n_+$ we have that $y{:=}(A + h^{-1}I)x \in R^n_+$ and by (3.43) there exists some $\beta{\geq}0$ such that

$$y + \beta x \in F$$

and so, by the definition of a face, we have the implication

$$0 \leq y \leq y + \beta x \in F \;\Rightarrow\; y \in F. \tag{3.44}$$

Thus, $hy = Bx \in F$, that is, for any choice of $h \in (0, h(A))$, F is $B{=}B(h)$–invariant. This means that $z \in core_{A,h}(\partial R^n_+)$ and hence

$$core_A(\partial R^n_+) \subseteq core_{A,h}(\partial R^n_+). \tag{3.45}$$

Conversely, suppose that $z \in core_{A,h}(\partial R^n_+)$ and note that, in particular, $z \in \partial R^n_+$. Thus if $z \notin core_A(\partial R^n_+)$, then we must have that $z \in X_A(intR^n_+)$. It follows from (3.1) that

$$intX_A(R^n_+) = intX_{A,h}(R^n_+) \tag{3.46}$$

which, combined with claims (i) and (iii), has the implication that

$$X_A(intR^n_+) = X_{A,h}(intR^n_+). \tag{3.47}$$

But then $z \in X_{A,h}(intR^n_+)$ contradicting the fact that $z \in core_{A,h}(\partial R^n_+)$. This completes the proof of the lemma □

We can now prove our second main result.

Theorem 3.3 *Let A be an $n \times n$ essentially nonnegative matrix. Then for all $0 < h < h(A)$ such that $B = B(h) = I + hA$ is invertible we have*

$$X_A(R^n_+) = X_{A,h}(R^n_+). \tag{3.48}$$

Proof

Let h be as prescribed in the statement of the theorem. First we will show that

$$X_A(R^n_+) \subseteq X_{A,h}(R^n_+). \tag{3.49}$$

By Theorem 3.1 the interior points of the two cones coincide so it suffices to show that

$$X_A(R^n_+) \cap \partial X_A(R^n_+) \subseteq X_{A,h}(R^n_+). \tag{3.50}$$

Suppose that $z \in X_A(R^n_+) \cap \partial X_A(R^n_+)$. Then, by (v) and (vii) of Lemma 3.2, $z \in X_A(F_1)$, where F_1 is some positively invariant and hence, according to the explanation involving (3.43) and (3.44), a B–invariant face of R^n_+. Without loss of generality assume that

$$F_1 = \{x = (x_1, x_2, \ldots, x_n)^T \mid x_j = 0, \ \forall j > k \ \text{ for some } k \geq 1\} \tag{3.51}$$

in which case, as $BF_1 \subseteq F_1$, B has necessarily the upper triangular form

$$\begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} = I + h \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{3.52}$$

where $A_{11} \in R^{n-k,n-k}$ and $A_{22} \in R^{k,k}$. Otherwise, our considerations apply to a permutation similarity of A. Then z has the form

$$z = [z_1^T, 0]^T, \quad z_1 \in X_{A_{11}}(R^{n-k}_+). \tag{3.53}$$

Thus, since $h(A) \leq h(A_{11})$, Theorem 3.1 applied to A_{11} yields that

$$z_1 \in \overline{X}_{A_{11},h}(R_+^{n-k}) = \overline{X}_{A_{11}}(R_+^{n-k}) \tag{3.54}$$

and so, by (3.53),

$$z \in \overline{X}_{A,h}(F_1) = \overline{X}_A(F_1). \tag{3.55}$$

If now $z_1 \in intX_{A_{11}}(R_+^{n-k}) = intX_{A_{11},h}(R_+^{n-k})$, then $z \in X_{A,h}(F_1)$ which implies that

$$z \in X_{A,h}(R_+^n) \tag{3.56}$$

and we are done. Suppose then that

$$z_1 \in X_{A_{11}}(R_+^{n-k}) \cap \partial X_{A_{11}}(R_+^{n-k}). \tag{3.57}$$

We may then apply a similar analysis to z_1 to show that there exists a B–invariant face F_2 of F_1 such that

$$z \in \overline{X}_{A,h}(F_2) = \overline{X}_A(F_2). \tag{3.58}$$

Then, either $z \in X_{A,h}(F_2)$ or we continue the reduction. The process must terminate when the face under consideration is of dimension 0 (the origin) or of dimension 1 (the span of a Perron eigenvector). In either case the problem is trivial and (3.49) is shown. The reverse containment follows similarly by applying (vi) and (viii) of Lemma 3.2 $\square$