Inverse M–matrix Inequalities and Generalized Ultrametric Matrices

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Abstract

We use weighted directed graphs to introduce a class of nonnegative matrices which, under a simple condition, are inverse M–matrices. We call our class the generalized ultrametic matrices since it contains the class of (symmetric) ultrametric matrices and some unsymmetric matrices. We show that a generalized ultrametric matrix is the inverse of a row and column diagonally dominant M–matrix if and only if it contains no zero row and no two of its rows are identical. This theorem generalizes the known result that a (symmetric) strictly ultrametric matrix is the inverse of a strictly diagonally dominant M–matrix. We also present inequalities and conditions for equality among the entries of the inverse of a row diagonally dominant M–matrix. Some of these inequalities and conditions for equality generalize results of Stieltjes on inverses of symmetric diagonally dominant M–matrices.

1 Introduction

The difficulty of characterizing all nonnegative matrices whose inverses are M–matrices has led to the study of the general properties of inverse M–matrices and to the identification of particular classes of such matrices. Recently Martinez, Michon, and San Martin [8] showed that a certain class of nonnegative symmetric matrices, namely the *strictly ultrametric matrices*, are inverses of symmetric strictly diagonally dominant M–matrices. A matrix theoretic proof of this result was given by Nabben and Varga [9] who analyzed this class further in [13] and broadened it in [14]. In this paper we introduce a class of matrices which we call the *generalized ultrametric matrices*. Our class is defined in terms of triangles in the weighted graph of the matrix and it contains the ultrametric matrices as well as some unsymmetric matrices. We show that a generalized ultrametric matrix is the inverse of a row diagonally dominant M-matrix if and only if it contains no zero row and no two of its rows are identical, see Section 4. The symmetric case of this theorem extends the known results referred to above. We develop proof techniques in terms of graphs which we call *isosceles graphs* and the corresponding form of the matrix under permutation similarity which we call the *nested block form*.

In 1887, in one of the earliest papers on M–matrices, Stieltjes [11] showed that the inverse of a nonsingular symmetric diagonally dominant M–matrix is a nonnegative matrix whose diagonal entries are greater than or equal to the off–diagonal entries in the corresponding column, and he gave necessary and sufficient conditions for the equality to hold. In this paper we generalized Stieltjes' inequality to row diagonally dominant M–matrices that may not be symmetric. We also generalize his conditions for equality to possibly unsymmetric row diagonally dominant M–matrices in terms of certain access relations of their graphs, see Section 3.

After we obtained some of the results in this paper, we learned that Nabben and Varga have considered similar classes of matrices and that they have obtained some overlapping results.

2 Definitions

We begin with some standard notation and definitions.

Let $X = [x_{ij}] \in \mathbb{R}^{nn}$. Let $\langle n \rangle = \{1, \ldots, n\}$.

We let e denote the $n \times 1$ all ones vector, so that Xe is the vector of row sums of X. We also let $\rho(X)$ denote the *spectral radius* of X.

X is called:
positive ($X \gg 0$) if $x_{ij} > 0$, for all $i, j \in \langle n \rangle$;
semipositive ($X > 0$) if $x_{ij} \geq 0$, for all $i, j \in \langle n \rangle$ and $X \neq 0$; and
nonnegative ($X \geq 0$) if $x_{ij} \geq 0$, for all $i, j \in \langle n \rangle$.

We will write X^T to represent the transpose of X.

We will write $min(X) = min\{x_{ij} \mid i, j \in \langle n \rangle\}$ and $max(X) = max\{x_{ij} \mid i, j \in \langle n \rangle\}$.

We call X a *Z–matrix* if $X = aI - P$ for some $a \in \mathbb{R}$ with P nonnegative. If in addition $a \geq \rho(P)$, then we say X is an *M–matrix.* We say an M–matrix X is *row diagonally dominant* if $Xe \geq 0$. We say X is *strictly* row diagonally dominant if $Xe \gg 0$. Similarly, we say X is *(strictly) column diagonally dominant* if X^T is (strictly) row diagonally dominant.

Let $\alpha, \beta \subseteq \langle n \rangle$. We will write $X_{\alpha\beta}$ to represent the submatrix of X whose rows are indexed by the elements of α and whose columns are indexed by the elements of β. The set α' will be $\langle n \rangle \setminus \alpha$.

Let $\Gamma = (V, E)$ be a digraph, where V is a finite vertex set and E is an edge set. Let $U \subseteq V$ and $F = \{(i, j) \mid i, j \in U, (i, j) \in E\}$. Then we call (U, F) the *subgraph of* Γ *induced by* U. A *path* from j to k in Γ is a sequence of vertices $j = r_1, r_2, ..., r_t = k$, with $(r_i, r_{i+1}) \in E$, for $i = 1, ..., t-1$. A path for which the vertices are pairwise distinct is called a *simple path.* The empty path will be considered a simple path linking every vertex to itself. If there is a path from j to k, we say that j has *access* to k. If j has access to k and k has access to j, we say j and k *communicate.* The communication relation is an equivalence relation, hence we may partition V into equivalence classes, which we will refer to as the *classes* of Γ. If $\alpha \subseteq V$ and $i,\ j \in V$, we say i *has access to* j *through* α if there is a path from i to j in Γ such that all intermediate vertices (if any) belong to α.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs. If $V_1 = V_2$, then we define the *union* of the graphs to be $\Gamma_1 \cup \Gamma_2 = (V_1, E_1 \cup E_2)$. We define the *product* of the graphs to be $\Gamma_1\Gamma_2 = (V, E)$ where $V = V_1 \cup V_2$ and $E = \{(i, j) \mid i \in V_1, j \in V_2$, and there exists $k \in V_1 \cap V_2$ such that $(i, k) \in E_1$ and $(k, j) \in E_2\}$. We say $\Gamma_1 \subseteq \Gamma_2$ if there exists a bijection φ such that $\varphi(V_1) = V_2$ and $(\varphi \times \varphi)(E_1) \subseteq E_2$. We say $\Gamma_1 = \Gamma_2$ if there exists a bijection φ such that $\varphi(V_1) = V_2$ and $(\varphi \times \varphi)(E_1) = E_2$.

We define the *digraph* of X by $G(X) = (V, E)$, where $V = \langle n \rangle$ and $E = \{(i, j) \mid X_{ij} \neq 0\}$.

We define the *transitive closure of* $G(X)$ by $\overline{G(X)} = (V, E)$, where $V = \langle n \rangle$ and $E = \{(i, j) \mid i$ has access to j in $G(X)\}$

Let $\alpha \subseteq \langle n \rangle$. Let $\beta = \alpha'$. We write $G_\alpha(X) = G(X_{\beta\beta})$.

It is well known that X is permutation similar to a matrix in block lower triangular Frobenius normal form, with each diagonal block irreducible. The irreducible blocks of X correspond to the classes of $G(X)$. If an irreducible block is singular, we call the corresponding class a *singular class.* Similarly if an irreducible block is nonsingular, we call the corresponding class a *nonsingular class.*

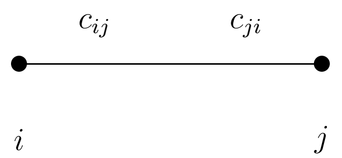
We call $D_n = (V, E)$, where $V = \langle n \rangle$ and $E = (\langle n \rangle \times \langle n \rangle) \setminus \{(i, i) \mid i \in \langle n \rangle\}$, the *complete (loopless) digraph* on n vertices.

Let $\alpha \subseteq \langle n \rangle$ and $\beta = \alpha'$. Then $C/C_{\alpha\alpha} = C_{\beta\beta} - C_{\beta\alpha}(C_{\alpha\alpha})^{-1}C_{\alpha\beta}$ is referred to as the *Schur complement* of C with respect to $C_{\alpha\alpha}$.

We say X is a *ultrametric matrix* if
(i) X is symmetric with nonnegative entries,
(ii) $x_{ij} \geq min\{x_{ik}, x_{kj}\}$, for all $i, j, k \in \langle n \rangle$,
(iii) $x_{ii} \geq max\{x_{ik} : k \in \langle n \rangle \setminus i\}$, for all $i \in \langle n \rangle$.
We say X is a *strictly ultrametric matrix* if the inequality in (iii) is strict for all $i \in \langle n \rangle$. If $n = 1$ and $X > 0$, then X is considered to be strictly ultrametric. We note that ultrametric matrices are called *pre–ultrametric* matrices in [14]. In that paper matrices called ultrametric are required to be nonsigular.

We introduce the following definitions.

Definition 2.1 Let $C \in \mathbb{R}^{\mathrm{nn}}$. We define $\Omega(C)$ to be the complete digraph D_n, with each directed edge (i,j) weighted by the value c_{ij}. For ease of representation, we represent a pair of directed weighted edges pictorially by:



where c_{ij} is always written closer to i and c_{ji} is written closer to j.

Definition 2.2 Let $C \in \mathbb{R}^{\mathrm{nn}}$. Let $\{i,j,k\} \subseteq \langle n \rangle$ be distinct. We call the subgraph of $\Omega(C)$ induced by $\{i,j,k\}$ a *triangle* and denote it by Δ_{ijk}.

Definition 2.3 Let $C \in \mathbb{R}^{\mathrm{nn}}$. Let $\{i,j,k\} \subseteq \langle n \rangle$ be distinct. We say i is a *preferred element* of $\{i,j,k\}$ or a *preferred vertex* of Δ_{ijk} if

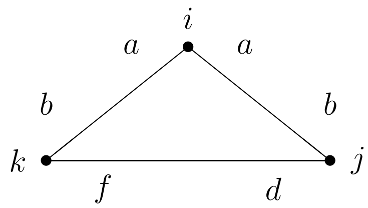
(i) $c_{ij} = c_{ik}$,

(ii) $c_{ji} = c_{ki}$,

(iii) $min\{c_{jk}, c_{kj}\} \geq min\{c_{ji}, c_{ij}\}$,

(iv) $max\{c_{jk}, c_{kj}\} \geq max\{c_{ji}, c_{ij}\}$.

Notice that if i is a preferred vertex of Δ_{ijk}, then Δ_{ijk} has the following labeled pattern:



where $a = c_{ij} = c_{ik}$, $b = c_{ji} = c_{ki}$, $f = c_{kj}$, $d = c_{jk}$, and $min\{f,d\} \geq min\{a,b\}$ and $max\{f,d\} \geq max\{a,b\}$.

Definition 2.4 Let $C \in \mathbb{R}^{\mathrm{nn}}$. We call C a *generalized ultrametric matrix* if

(i) C is nonnegative,

(ii) $c_{ii} \geq max\{c_{ij}, c_{ji}\}$, for all $i, j \in \langle n \rangle$,

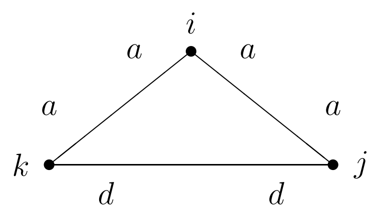
(iii) $n \leq 2$ or $n > 2$ and every subset of $\langle n \rangle$ with three distinct elements has a preferred element.

Definition 2.5 Let $C \in \mathbb{R}^{nn}$ be nonnegative. We call $\Omega(C)$ an *isosceles graph* if $n \leq 2$, or $n \geq 3$ and every triangle in $\Omega(C)$ has a preferred vertex.

Remark 2.6 Observe that an isosceles graph must be transitive. This is easily proved by considering each of the following possibilities in the diagram preceeding Definition 2.4: $a = 0$, or $b = 0$, or $d = 0$, or $f = 0$.

Clearly C is a generalized ultrametric matrix if and only if $\Omega(C)$ is an isosceles graph and $c_{ii} \geq max\{c_{ij}, c_{ji}\}$, for all $i, j \in \langle n \rangle$.

Remark 2.7 Notice that if C is an ultrametric matrix and $\{i, j, k\} \subseteq \langle n \rangle$ are distinct, then Δ_{ijk} has the following labelled pattern:



where $d \geq a$. This shows that the ultrametric matrices are just the generalized ultrametric matrices which are symmetric.

We give an inductive definition of a matrix in nested block form.

Definition 2.8 Let $C \in \mathbb{R}^{nn}$ be nonnegative. We define inductively what it means for C to be in *nested block form.*

(i) If $n = 1$, then C is in nested block form.

(ii) If $n > 1$ and nested block form has been defined for all $k \times k$ nonnegative matrices with $k < n$, then C is in nested block form if

$$C = \begin{bmatrix} C_{11} & b_{12}E_{12} \\ b_{21}E_{21} & C_{22} \end{bmatrix},$$

where C_{11} and C_{22} are square matrices in nested block form, E_{12} and E_{21} are all ones matrices of the appropriate sizes, $b_{12} \geq b_{21}$, $min\{c_{ij}, c_{ji}\} \geq b_{21}$, for all $i, j \in \langle n \rangle$, and $max\{c_{ij}, c_{ji}\} \geq b_{12}$, for all $i, j \in \langle n \rangle$.

In Section 5 we give an example of a generalized ultrametric matrix in nested block form and illustrate its isosceles graph.

Remark 2.9 Let $i, j \in \langle n \rangle$, $i < j$. If C is in nested block form then c_{ij} occurs in an off–diagonal block of a submatrix of C in nested block form, and hence $c_{ij} \geq c_{ji}$. Consequently, if $i < j$ then $c_{ij} \geq b_{12}$ and $c_{ji} \geq b_{21}$.

Remark 2.10 It is readily seen that a matrix in nested block form can be decomposed into the sum of an ultrametric matrix and an upper triangular nilpotent nonnegative matrix.

3 Inverse M–matrix Inequalities

In this section we discuss inequalities for inverses of M–matrices. An example illustrating the results of this section will be given in Section 5.

We begin with a lemma and a theorem in which we partially reprove known results, and then we provide additional information.

Let A be a nonsingular M–matrix, $\alpha \subseteq \langle n \rangle$, and $B = A/A_{\alpha\alpha}$. It is well known that B is again an M–matrix (implied by [4, Lemma 1] or see e.g. [1, Exercise 5.8, p. 159]). Further, if A is a nonsingular row diagonally dominant M–matrix then it is easy to see that B is again row diagonally dominant (stated in part (ii) of our lemma). In fact, in part (ii) we prove a more precise version of this result. Part (iii) can be derived from [12, Lemma 2], [7, Lemma 2], or [10, Lemma 2.2].

Lemma 3.1 *Let $A \in \mathbb{R}^{nn}$ be a nonsingular M–matrix with $Ae = s$. Let $\alpha \subseteq \langle n \rangle$, $\beta = \alpha'$, and $B = A/A_{\alpha\alpha}$. Then:*

(i) *B is a nonsingular M–matrix.*

(ii) *If $s \geq 0$, then $Be_\beta = r \geq 0$ and for any $i \in \beta$, $r_i > 0$ if and only if either $s_i > 0$ or there exists $j \in \alpha$ such that $s_j > 0$ and i has access to j through α in $G(A)$.*

(iii) *For any $i, j \in \beta$, $i \neq j$, $b_{ij} < 0$ if and only if i has access to j through α in $G(A)$.*

Proof:

Recall that

$$B = A/A_{\alpha\alpha} = A_{\beta\beta} - A_{\beta\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\beta}.$$

(i) See the comments preceding the statement of the theorem.

(ii) Let $T = -A_{\beta\alpha}(A_{\alpha\alpha})^{-1} \geq 0$. Let $j \in \alpha$. Since by [10, Lemma 2.2], $G(T) = G(A_{\beta\alpha})\overline{G(A_{\alpha\alpha})}$, it follows that $t_{ij} > 0$ if and only if i has access to j through α in $G(A)$.

Since

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ 0 & B \end{bmatrix} e = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix} e = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} s_\alpha \\ s_\beta \end{bmatrix},$$

we see that

$$r = Ts_\alpha + s_\beta.$$

Thus (as is known) $r \geq 0$. Moreover, $r_i > 0$ if and only if either $s_i > 0$ or there is a vertex $j \in \alpha$ such that $s_j > 0$ and i has access to j through α in $G(A)$.

(iii) From [10, Lemma 2.2],

$$G(B) = G(A_{\beta\beta}) \cup G(A_{\beta\alpha})\overline{G(A_{\alpha\alpha})}G(A_{\alpha\beta}),$$

and the result follows. □

We now apply Lemma 3.1 to obtain inequalities and conditions for equality for the entries of the inverse of a row diagonally dominant M–matrix. Theorem 3.2 below was proven for symmetric matrices by Stieltjes [11, pp. 396–399] who, in this case, proved inequality (i) and also stated necessary and sufficient conditions for the equality in (ii) in terms of direct summands of $A_{i'i'}$. For general row diagonally dominant M–matrices the inequality (i) can be found, for example, in [1, Chapter 9, Lemma 3.14]. The equality in (ii) generalizes the results of Stieltjes to the unsymmetric case.

Theorem 3.2 *Let $A \in \mathbb{R}^{nn}$ be a nonsingular row diagonally dominant M–matrix. Let $C = A^{-1}$ and $Ae = s$. Fix $i \in \langle n \rangle$. Let*

$$\gamma_i = \{k \in \langle n \rangle \backslash \{i\} \mid k \text{ does not have access in } G_i(A) \text{ to any } j \text{ for which } s_j > 0\}.$$

Then:

(i) $c_{ii} \geq c_{ki}$ *for all* $k \in \langle n \rangle$.

(ii) $c_{ii} = c_{ki}$ *if and only if* $k \in \gamma_i$.

Proof:

(i) Let $k \in \langle n \rangle \setminus \{i\}$, and let $\beta = \{k, i\}$, $\alpha = \beta'$, and $B = A/A_{\alpha\alpha}$. It is well known (see e.g. [15]) that $B = (C_{\beta\beta})^{-1}$ and using classical results about adjoints that

$$B = (1/det(C_{\beta\beta}))\ adj(C_{\beta\beta}) = det(B) \begin{bmatrix} c_{ii} & -c_{ki} \\ -c_{ik} & c_{kk} \end{bmatrix}.$$

Using Lemma 3.1, we obtain (i).

(ii) Observe that $c_{ii} = c_{ki}$ is equivalent to $r_k = 0$, where $r = Be_\beta$. Hence by Lemma 3.1, (ii) holds if and only if $s_k = 0$ and k does not have access in $G_i(A)$ to a vertex $j \in \alpha$ for which $s_j > 0$. But this condition is easily seen to be equivalent to $k \in \gamma_i$. $\square$.

Remark 3.3 Note that the proof of Theorem 3.2 immediately yields a strict inequality in (i) when A is a strictly row diagonally dominant M–matrix. This result can be found in Fiedler and Ptak [5] in the course of the proof of their result (3,5) on page 427, and is also true for all real strictly diagonally dominant matrices (see [6, Theorem 2.5.12]).

Example 3.4 The converse of Remark 3.3 need not hold. Consider

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix},$$

$$C = A^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 4 & 4 \\ 4 & 5 & 3 \\ 4 & 3 & 5 \end{bmatrix}.$$

Notice that A is not strictly row diagonally dominant, however $c_{ii} > c_{ki}$ for all $i \neq k$.

Corollary 3.5 *Let A be a nonsingular row diagonally dominant M–matrix. Let $C = A^{-1}$. Then the following are equivalent:*

(i) *Row i is the only row for which A has a nonzero row sum.*

(ii) *Column i of C has all of its entries equal to c_{ii}.*

Proof:

Follows from Theorem 3.2.

Next we look at a theorem on the relationship between the sum of the entries of a nonsingular row diagonally dominant M–matrix and the minimum entry of each row of its inverse.

Theorem 3.6 *Let $A \in \mathbb{R}^{\mathrm{nn}}$ be a nonsingular row diagonally dominant M–matrix. Let $t = \sum_{i,j=1}^{n} a_{ij}$, $C = A^{-1}$, $\mu_i = min\{c_{ij} \mid j \in \langle n \rangle\}$, and $\mu = min(C)$. Then:*

(i) *$\mu_i t \leq 1$, for all $i \in \langle n \rangle$,*

(ii) *$\mu t = 1$ if and only if C has a column whose entries are all equal to μ.*

Proof:

Let $Ae = r$ and $E = ee^T$. Then $r \geq 0$, $C \geq 0$, and

$$e = Cr \geq \begin{bmatrix} \mu_1 & \mu_1 & \dots & \mu_1 \\ \mu_2 & \mu_2 & \dots & \mu_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_n & \dots & \mu_n \end{bmatrix} r = \begin{bmatrix} \mu_1 t \\ \mu_2 t \\ \vdots \\ \mu_n t \end{bmatrix}.$$

Thus $1 \geq \mu_i t$, for all $i \in \langle n \rangle$ establishing (i).

If $\mu t = 1$, then

$$1 = \sum_{j=1}^{n} c_{ij} r_j \geq \sum_{j=1}^{n} \mu r_j = \mu t = 1,$$

and hence equality must hold throughout. Moreover, since $c_{ij} - \mu \geq 0$, it follows that $c_{ij} r_j = \mu r_j$, for all $i, j \in \langle n \rangle$. Since A is nonsingular there exists $i \in \langle n \rangle$ such that $r_i > 0$. Then $c_{ii} = \mu$ and by Theorem 3.2(i), $\mu \leq c_{ki} \leq c_{ii} = \mu$, for all $k \neq i$, and hence equality must hold.

If all the entries in column i of C are equal to μ, then by Corollary 3.5, $r_i > 0$ and $r_j = 0$ for all $j \neq i$. Then

$$1 = \sum_{j=1}^{n} c_{ij} r_j = c_{ii} r_i = \mu t.$$

Hence $\mu t = 1$. □

The next two examples illustrate the significance of $\mu = min(C)$ in Theorem 3.6 (ii).

Example 3.7 Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{then} \quad A^{-1} = C = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that if t and μ_i are as in Theorem 3.6, then $t\mu_1 = (2)(\frac{1}{2}) = 1$, but C does not contain a column for which all the entries are equal.

Example 3.8 Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}, \quad \text{then} \quad A^{-1} = C = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Notice that if t, μ_i, and μ are as in Theorem 3.6, then C contains a column of constants, but $t\mu = t\mu_i = (\frac{1}{2})(1) < 1$, for all $i \in \langle 3 \rangle$.

We conclude this section by examining some relationships between the entries of the inverse of a nonsingular M–matrix which need not be row diagonally dominant.

Theorem 3.9 *Let A be a nonsingular M–matrix. Let $C = A^{-1}$ and let $\{i, j, k\} \subseteq \langle n \rangle$ be distinct. Then:*

(i) *$c_{jk} = c_{ji}c_{ik}/c_{ii}$, whenever j does not have access to k in $G_i(A)$,*

(ii) *$c_{jk} > c_{ji}c_{ik}/c_{ii}$, whenever j has access to k in $G_i(A)$.*

Proof:

Let $\beta = \{i, j, k\}$ and $\alpha = \beta'$. Let $B = A/A_{\alpha\alpha}$. It is well known that (see e.g. [15]) $B = (C_{\beta\beta})^{-1}$ and B is an M–matrix. By Lemma 3.1, $b_{jk} = 0$ if and only if j does not have access to k in $G_i(A)$. Using classical theory about adjoints we see that

$$0 \geq b_{jk} = -det(B)\ det \begin{bmatrix} c_{ii} & c_{ik} \\ c_{ji} & c_{jk} \end{bmatrix} = -det(B)\ (c_{ii}c_{jk} - c_{ji}c_{ik}).$$

Since $det(B) > 0$, the result follows. □

It is known that $c_{jk} \geq c_{ji}c_{ik}/c_{ii}$, for all distinct i, j, k, is a necessary condition for a nonnegative matrix C to be the inverse of an M–matrix. (see for example Willoughby [16, Theorem 1]). Thus our results in Theorem 3.9 represent a sharpening and a graph–theoretical refinement of Willoughby's observation.

In the symmetric diagonally dominant case, the following corollary is essentially due to Stieltjes [11, p.399].

Corollary 3.10 *Let $A \in \mathbb{R}^{nn}$ be a nonsingular M–matrix and let $C = A^{-1}$. Let $\{i, j, k\} \subseteq \langle n \rangle$ be distinct.*

(i) *If $c_{ii} = c_{ji}$ then:*

- **(a)** *$c_{jk} = c_{ik}$, whenever j does not have access to k in $G_i(A)$,*
- **(b)** *$c_{jk} > c_{ik}$, whenever j has access to k in $G_i(A)$.*

(ii) *If $c_{ii} = c_{ik}$ then:*

- **(a)** *$c_{jk} = c_{ji}$, whenever j does not have access to k in $G_i(A)$,*
- **(b)** *$c_{jk} > c_{ji}$, whenever j has access to k in $G_i(A)$.*

Proof:

Follows from Theorem 3.9. □

4 Generalized Ultrametric Matrices

We begin this section by showing that a generalized ultrametric matrix can be represented by an isosceles graph and that it is permutation similar to a matrix in nested block form.

Lemma 4.1 *Let $C \in \mathbb{R}^{nn}$ be nonnegative. Then the following are equivalent:*

(i) *C is a generalized ultrametric matrix.*

(ii) *$\Omega(C)$ is an isosceles graph and $c_{ii} \geq max\{c_{ij}, c_{ji}\}$, for all $i, j \in \langle n \rangle$.*

(iii) *C is permutation similar to a matrix in nested block form.*

Proof:
(i) implies (ii):

Follows easily from the definitions.

(ii) implies (iii):

If $n \leq 2$, then the result follows from the definitions.

If $n \geq 3$, fix $i, j \in \langle n \rangle$ so that $c_{ij} \geq c_{ji}$ and $c_{ij}+c_{ji} \leq c_{k\ell}+c_{\ell k}$, for all $k, \ell \in \langle n \rangle$. Let $a = c_{ij}$ and $b = c_{ji}$. We now choose nonempty subsets $\alpha, \beta \subseteq \langle n \rangle$ so that $\alpha \cap \beta = \emptyset,\ \alpha \cup \beta = \langle n \rangle$, and moreover,

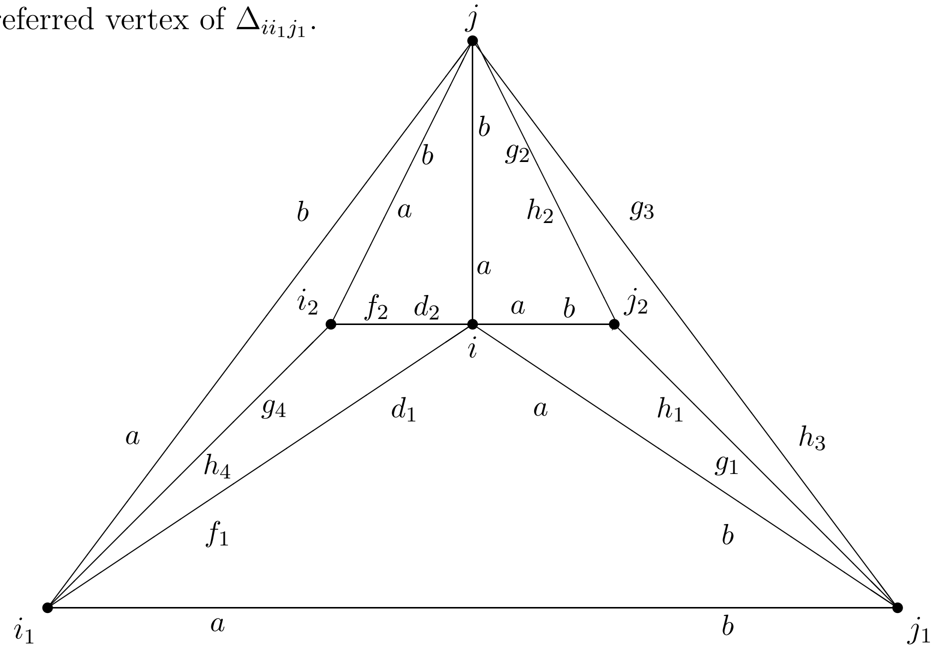
$$C_{\alpha\beta} = aE_{\alpha\beta},\ C_{\beta\alpha} = bE_{\beta\alpha}, \tag{4.2}$$

where $E_{\alpha\beta},\ E_{\beta\alpha}$ are all ones matrices of appropriate sizes, and

$$max\{c_{k\ell}, c_{\ell k}\} \geq a,\ min\{c_{k\ell}, c_{\ell k}\} \geq b,\ \text{for all } k, \ell \in \langle n \rangle. \tag{4.3}$$

Begin by putting $i \in \alpha$ and $j \in \beta$. For each $k \in \langle n \rangle$, if i is a preferred vertex of $\{i, j, k\}$ then put $k \in \beta$, otherwise put $k \in \alpha$. We now show that α and β are index sets for which (4.2) and (4.3) above are satisfied.

Let $i_1, i_2 \in \alpha \setminus \{i\}$ and $j_1, j_2 \in \beta \setminus \{j\}$. Since $\Omega(C)$ is an isosceles graph, every triangle has a preferred vertex. We now sketch the ideas used to deduce the weights for the triangles drawn in the following diagram. Note that this diagram only contains the edges which are used in the proof. The weights of Δ_{ijj_1} and Δ_{ijj_2} are determined by the choices of $j_1, j_2 \in \beta$, which imply that i is a preferred vertex in each case. The weights of Δ_{iji_1} and Δ_{iji_2} are determined by the choices of $i_1, i_2 \in \alpha$, which imply that i is not a preferred vertex of either triangle, and the minimality of $a + b$, which implies that j must be the only preferred vertex in each case. We can complete the diagram by observing that j is a preferred vertex of $\Delta_{ji_1i_2}$ and j_1 is the only possible preferred vertex of $\Delta_{ii_1j_1}$.



where $f_m \neq b$ or $d_m \neq a$, for $m = 1, 2$,
$max\{f_m, d_m\} \geq a$ and $min\{f_m, d_m\} \geq b$, for $m = 1, 2$, and
$max\{g_m, h_m\} \geq a$ and $min\{g_m, h_m\} \geq b$, for $m = 1, 2, 3, 4$.

Reading weights from the triangles above and using the condition that $c_{kk} \geq max\{c_{k\ell}, c_{\ell k}\}$, for all $k, \ell \in \langle n \rangle$, it is easy to verify that α and β are as desired.

Since $\Omega(C_{\alpha\alpha})$ and $\Omega(C_{\beta\beta})$ are also isosceles graphs, α and β can be partitioned likewise. Thus C is permutation similar to a matrix in nested block form.

(iii) implies (i):

Let C satisfy (iii). Choose $\alpha, \beta \subseteq \langle n \rangle$ so that

$$\begin{bmatrix} C_{\alpha\alpha} & aE_{\alpha\beta} \\ bE_{\beta\alpha} & C_{\beta\beta} \end{bmatrix}$$

is in nested block form. If $i \in \alpha$, $j \in \beta$, and $k \in \beta$, clearly i is a preferred element of $\{i, j, k\}$. If $i \in \alpha$, $j \in \alpha$, and $k \in \beta$, clearly k is a preferred element of $\{i, j, k\}$. Since $C_{\alpha\alpha}$ and $C_{\beta\beta}$ are also in nested block form, a simple induction shows that if $i, j, k \in \alpha$ or $i, j, k \in \beta$, then $\{i, j, k\}$ has a preferred element. Thus every three distinct element subset of $\langle n \rangle$ has a preferred element.

Since every off–diagonal entry of C must be in some off–diagonal block of a matrix in nested block form, it follows that $c_{ii} \geq max\{c_{ij}, c_{ji}\}$, for all $i, j \in \langle n \rangle$. $\square$

Notice that, by Remark 2.10 and Lemma 4.1, any generalized ultrametric matrix can be decomposed into the sum of an ultrametric matrix and a nilpotent nonnegative matrix.

Notice also that it follows from Definition 2.4 (ii) that a generalized ultrametric matrix C has a row all of whose entries are equal to $min(C)$ if and only if it has a column all of whose entries are equal to $min(C)$. Using the nested block form of C and induction, it can be shown that a generalized ultrametric matrix has two rows the same if and only if it has two columns the same.

In the following theorem, we show that if a generalized ultrametric matrix does not contain a row of zeros and does not have two rows which are equal, then it is nonsingular, and moreover, it is the inverse of a row and column diagonally dominant M–matrix. We could also state part (ii) of our theorem using columns instead of rows. This generalizes the results of [8] and [13] where the authors show that the inverse of a strictly ultrametric matrix is a strictly row and column diagonally dominant M–matrix. A necessary and sufficient condition for an ultrametric matrix to be nonsingular, stated in terms of a certain decomposition of the matrix, may be found in [14]. Recall that ultrametric and strictly ultrametric matrices are defined to be symmetric.

Theorem 4.4 *Let $C \in \mathbb{R}^{nn}$ be a generalized ultrametric matrix. Then the following are equivalent:*

(i) *C is nonsingular.*

(ii) *C does not contain a row of zeros, and no two rows of C are the same.*

(iii) *C is nonsingular and C^{-1} is a row and column diagonally dominant M-matrix.*

Proof:

(i) implies (ii):

If C is nonsingular then (ii) must hold.

(ii) implies (iii):

We proceed by induction on the size of C.

If $n = 1$, then (iii) holds trivially.

If $n \geq 2$, then assume (ii) implies (iii) is true for all generalized ultrametric matrices of size $k \times k$, where $k < n$.

By Lemma 4.1, we can assume without loss of generality that C is in nested block form, labelled as in Definition 2.8. Since (ii) holds for C, it is clear from the definition of nested block form that C_{11} and C_{22} also satisfy (ii), and so by the inductive hypothesis, C_{11} and C_{22} are nonsingular and their inverses are row and column diagonally dominant M–matrices.

If $b_{12} = b_{21} = 0$, then C is a block diagonal matrix and we are done. If not, then $b_{12} > 0$ (since in the definition of the nested block form we require $b_{12} \geq b_{21}$).

Notice also that

$$b_{21} \leq min(C_{mm}), \quad \text{for } m = 1, 2. \tag{4.5}$$

Let t_m be the sum of the entries in $(C_{mm})^{-1}$, $m = 1, 2$.

Claim 4.6 *$t_m b_{21} \leq t_m b_{12} \leq 1$, for $m = 1, 2$.*

The first inequality follows easily since $b_{21} \leq b_{12}$. Since C is in nested block form, by Remark 2.9, $b_{12} \leq c_{ij}$ for all $i < j$ and hence $b_{12} \leq$ the minimum entry in the first row of C_{mm}. The second inequality now follows from Theorem 3.6 (i) applied to $(C_{mm})^{-1}$.

Claim 4.7 *C/C_{mm} is a generalized ultrametric matrix satisfying condition (ii), for $m = 1, 2$.*

Proof:
Let $\ell = 3 - m$. Then

$$C/C_{mm} = C_{\ell\ell} - b_{\ell m} E_{\ell m} (C_{mm})^{-1} b_{m\ell} E_{m\ell} = C_{\ell\ell} - b_{\ell m} b_{m\ell} t_m E_{\ell\ell}. \tag{4.8}$$

By Claim 4.6, (4.5), and (4.8) it is easily checked that C/C_{mm} satisfies the definition of a generalized ultrametric matrix. Moreover C/C_{mm} does not have two equal rows, otherwise $C_{\ell\ell}$ and hence C would have two equal rows. It remains to show that C/C_{mm} does not contain a row of zeros. Suppose it does. Then by (4.8) $C_{\ell\ell}$ has a row all of whose entries equal $b_{21}b_{12}t_m$ and hence, by Claim 4.6 and (4.5)

$$b_{21}b_{12}t_m \leq b_{21} \leq min(C_{\ell\ell}) \leq b_{21}b_{12}t_m.$$

If $b_{21} = 0$, then $C_{\ell\ell}$ and hence C would have a row of zeros. Thus $b_{21} > 0$ and equality must hold throughout. Hence

$$b_{21}b_{12}t_m = b_{21} = min(C_{\ell\ell}), \tag{4.9}$$

and

$$t_m = 1/b_{12}. \tag{4.10}$$

Since $C_{\ell\ell}$ has a row of entries equal to b_{21}, it has a diagonal entry equal to b_{21} and since the diagonal entry is greater than or equal to b_{12} it follows that $b_{21} = b_{12}$. Hence by (4.10), (4.5) and Theorem 3.6 (i) applied to C_{mm}, $t_m = 1/min(C_{mm})$. By Theorem 3.6 (ii) applied to $((C_{mm})^{-1})^T$ we now have that C_{mm} also has a row all of whose entries equal b_{21}. Hence our assumption that C/C_{mm} contains a row of zeros implies that C contains two rows all of whose entries equal b_{21}, a contradiction. This establishes the claim.

By Claim 4.7 and our inductive assumption, C_{mm} and C/C_{mm}, $m = 1, 2$, are the inverses of row and column diagonally dominant M–matrices. Consequently, combining formulas from [2, (10), p. 773] (see also [3]) and [15, (4), p. 251], $A = C^{-1}$ can be written as

$$A = \begin{bmatrix} (C/C_{22})^{-1} & -(C_{11})^{-1} b_{12} E_{12} (C/C_{11})^{-1} \\ -(C_{22})^{-1} b_{21} E_{21} (C/C_{22})^{-1} & (C/C_{11})^{-1} \end{bmatrix}, \tag{4.11}$$

or

$$(4.12) \quad A = \begin{bmatrix} (C/C_{22})^{-1} & -(C/C_{22})^{-1} b_{12} E_{12} (C_{22})^{-1} \\ -(C/C_{11})^{-1} b_{21} E_{21} (C_{11})^{-1} & (C/C_{11})^{-1} \end{bmatrix}.$$

We now show that A is a Z–matrix. Since the diagonal blocks of A in (4.11) are M–matrices, we need only show that $A_{m\ell} \leq 0$, for $m = 1, 2,\ \ell = 3 - m$. Let $s_j \geq 0$ and $r_j \geq 0$ denote the sum of the entries in column j of $(C/C_{mm})^{-1}$ and in row j of $(C_{mm})^{-1}$, respectively. Then, by (4.11),

$$A_{m\ell} = -(C_{mm})^{-1} b_{m\ell} E_{m\ell} (C/C_{mm})^{-1} = -b_{m\ell} \begin{bmatrix} s_1 r_1 & s_2 r_1 & \dots & s_p r_1 \\ s_1 r_2 & s_2 r_2 & \dots & s_p r_2 \\ \vdots & \vdots & \ddots & \vdots \\ s_1 r_q & s_2 r_q & \dots & s_p r_q \end{bmatrix} \leq 0.$$

Next we show that A is row diagonally dominant. From (4.12) and Claim 4.6, if $m = 1, 2$ and $\ell = 3 - m$, then

$$A_{mm} e_m + A_{m\ell} e_\ell = (C/C_{\ell\ell})^{-1} e_m - (C/C_{\ell\ell})^{-1} b_{m\ell} E_{m\ell} (C_{\ell\ell})^{-1} e_\ell$$
$$= (C/C_{\ell\ell})^{-1} e_m - (C/C_{\ell\ell})^{-1} b_{m\ell} t_\ell e_m = (1 - b_{m\ell} t_\ell)(C/C_{\ell\ell})^{-1} e_m \geq 0.$$

Similarly, from (4.11) and Claim 4.6, A is column diagonally dominant because

$$(e_m)^T A_{mm} + (e_\ell)^T A_{\ell m} = (e_m)^T (C/C_{\ell\ell})^{-1} - (e_\ell)^T (C_{\ell\ell})^{-1} b_{\ell m} E_{\ell m} (C/C_{\ell\ell})^{-1}$$
$$= (e_m)^T (C/C_{\ell\ell})^{-1} - (e_m)^T t_\ell b_{\ell m} (C/C_{\ell\ell})^{-1} = (e_m)^T (C/C_{\ell\ell})^{-1} (1 - b_{\ell m} t_\ell) \geq 0.$$

(iii) implies (i):

Follows trivially. □

Corollary 4.13 *Let C be a nonsingular generalized ultrametric matrix which is in nested block form (labeled as in Definition 2.8). Then C_{11}, C_{22}, and the Schur complements C/C_{11} and C/C_{22} are nonsingular generalized ultrametric matrices.*

Proof:

Follows from Theorem 4.4 and Claim 4.7. □

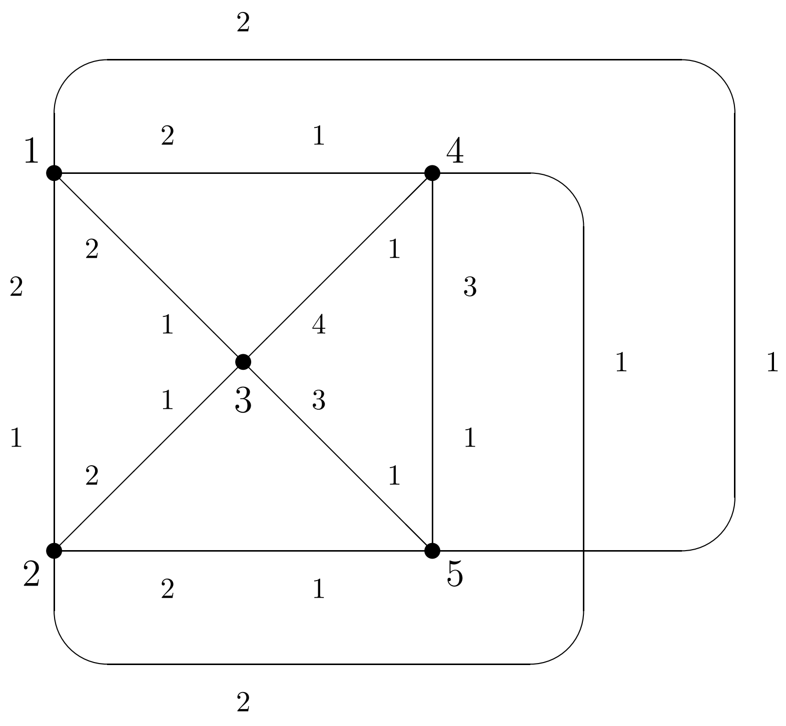
We remark here that not all Schur complements of generalized ultrametric matrices are generalized ultrametric matrices (see Section 5).

5 Illustrative Examples

Example 5.1 Consider the following generalized ultrametric matrix in nested block form:

$$C = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 2 & 2 & 2 \\ 1 & 1 & 5 & 4 & 3 \\ 1 & 1 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

Consider the representation $\Omega(C)$



Notice that every triangle has a preferred vertex, hence $\Omega(C)$ is an isosceles graph as required by Lemma 4.1.

Observe that we can write C as a sum of an ultrametric matrix and a nonnegative nilpotent matrix as follows:

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\alpha = \{1,2\}$ and $\beta = \alpha'$ and consider the Schur complements

$$C/C_{\alpha\alpha} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix},$$

$$C/C_{\beta\beta} = 1/3 \begin{bmatrix} 4 & 4 \\ 1 & 7 \end{bmatrix},$$

which are also generalized ultrametric matrices, as required by Corollary 4.13. However, observe that if $\alpha = \{1,3\}$, then

$$C/C_{\alpha\alpha} = 1/4 \begin{bmatrix} 8 & 1 & 2 \\ 0 & 16 & 8 \\ 0 & 0 & 8 \end{bmatrix}$$

is not a generalized ultrametric matrix. Hence a Schur complement of a generalized ultrametric matrix need not be a generalized ultrametric matrix.

Consider

$$A = C^{-1} = 1/32 \begin{bmatrix} 28 & -16 & -4 & -1 & -3 \\ -4 & 16 & -4 & -1 & -3 \\ 0 & 0 & 8 & -6 & -2 \\ 0 & 0 & 0 & 8 & -8 \\ -8 & 0 & 0 & 0 & 16 \end{bmatrix}$$

and let

$$e = [1 \quad 1 \quad 1 \quad 1 \quad 1]^T.$$

Then

$$r = Ae = 1/8\,[1 \quad 1 \quad 0 \quad 0 \quad 2]^T,$$

and

$$s = e^T A = 1/2\,[1 \quad 0 \quad 0 \quad 0 \quad 0\,]\,.$$

Thus A is an M–matrix which is both row and column diagonally dominant as predicted by Theorem 4.4. Moreover, the sum of the entries of A is $1/2 < 1 = min(C)$ as required by Theorem 3.6.

Note that row 1 of C has all its entries equal to c_{11}, and $s_i = 0$, for all $i \neq 1$, as required by Corollary 3.5 applied to A^T.

Let γ_i be defined as in Theorem 3.2. Notice that $r_3 = r_4 = 0$, while $r_1 > 0$ and $r_2 > 0$. The vertices 3 and 4 do not access vertices 1 and 2 in $G_5(A)$. Hence $\gamma_5 = \{3, 4\}$. We also note that $c_{55} = c_{45} = c_{35}$, as required by Theorem 3.2 applied to A. Further, as required by Corollary 3.10, since 4 does not have access to 3 in $G_5(A)$, $c_{43} = c_{53}$. Since 3 has access to 4 in $G_5(A)$, $c_{34} > c_{54}$. Since 3 and 4 do not have access to 1 and 2 in $G_5(A)$, it follows that $c_{31} = c_{41} = c_{51}$ and $c_{32} = c_{42} = c_{52}$.

Example 5.2 An example of a nonsingular (symmetric) ultrametric matrix which is not strictly diagonally dominant is given by

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Example 5.3 Note that Definition 2.3 allows $min\{f, d\} \leq max\{a, b\}$. For example the following matrix is a nonsingular generalized ultrametric matrix whose inverse is a strictly diagonally dominant M–matrix.

$$C = \begin{bmatrix} 4 & 3 & 3 & 3 \\ 1 & 4 & 3 & 3 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

However the matrix

$$B = \begin{bmatrix} 4 & 1 & 3 & 3 \\ 1 & 4 & 3 & 3 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix},$$

with

$$B^{-1} = \frac{1}{39} \begin{bmatrix} 14 & 1 & -9 & -9 \\ 1 & 14 & -9 & -9 \\ -3 & -3 & 14 & 1 \\ -3 & -3 & 1 & 14 \end{bmatrix},$$

shows that if we remove the condition that $max\{c, d\} \geq max\{a, b\}$ from Definition 2.3, the inverse need not be an M–matrix.

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