DOUBLY DIAGONALLY DOMINANT MATRICES

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Abstract

We consider the class of doubly diagonally dominant matrices ($A = [a_{ij}] \in \mathbf{C}^{n,n},\ |a_{ii}||a_{jj}| \geq \sum_{k \neq i} |a_{ik}| \sum_{k \neq j} |a_{jk}|$, $i \neq j$) and its subclasses. We give necessary and sufficient conditions in terms of the directed graph so that an irreducibly doubly diagonally dominant matrix be a singular matrix or be an H–matrix. As in the case of diagonal dominance, we show that the Schur complements of doubly diagonally dominant matrices inherit this property. Moreover, we describe when a Schur complement of a strictly doubly diagonally dominant matrix is strictly diagonally dominant.

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1 Preliminaries

The theorem of Geršgorin and the theorem of Brauer are two classical results about regions in the complex plane that include the spectrum of a matrix (see e.g., Horn and Johnson [4]). They, respectively, locate the eigenvalues of an $n \times n$ complex matrix $A = [a_{ij}]$ in the union of n closed discs (known as the Geršgorin discs),

$$\{z \in \mathbf{C} : \ |z - a_{ii}| \ \leq \ \sum_{k \neq i} |a_{ik}|\} \ \ (i = 1, 2, \ldots, n),$$

or in the union of $n(n-1)/2$ ovals (known as the ovals of Cassini),

$$\{z \in \mathbf{C} : \ |z - a_{ii}| \, |z - a_{jj}| \ \leq \ (\sum_{k \neq i} |a_{ik}|) \, (\sum_{k \neq j} |a_{jk}|)\} \ \ (i, j = 1, 2, \ldots, n; \ i \neq j).$$

As a consequence of either of these theorems, but more precisely as a consequence of Geršgorin's theorem, every strictly diagonally dominant matrix is invertible. In geometric terms, strict diagonal dominance means that the origin does not belong to the union of the Geršgorin discs and hence it cannot be an eigenvalue. In this article we will consider a condition weaker than diagonal dominance, whose geometric interpretation regards the location of the origin relative to the ovals of Cassini. This condition gives rise to the class of doubly diagonally dominant matrices and its subclasses, whose precise definitions are found later in this section.

We continue with definitions, notation, and some background results.

Given a positive integer n let $\langle n \rangle = \{1, 2, \ldots, n\}$. Let $\mathbf{C}^{n,n}$ denote the collection of all $n \times n$ complex matrices and let $\mathbf{Z}^{n,n}$ denote the collection of all $n \times n$ real matrices $A = [a_{ij}]$ with $a_{ij} \leq 0$ for all distinct $i, j \in \langle n \rangle$. Throughout this section $A = [a_{ij}] \in \mathbf{C}^{n,n}$.

With A we associate its (loopless) *directed graph*, $D(A)$, defined as follows. The vertices of $D(A)$ are $1, 2, \ldots, n$. There is an arc (i, j) from i to j when $a_{ij} \neq 0$ and $i \neq j$. A *path (of length p) from i to j* is a sequence of distinct vertices $i = i_0, i_1, \ldots, i_p = j$ such that $(i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p)$ are arcs of $D(A)$. We denote such a path by $P_{ij} = (i_0, i_1, \ldots, i_p)$. A *circuit* γ of $D(A)$ consists of the distinct vertices $i_0, i_1, \cdots, i_p$, $p \geq 1$, provided that $(i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p)$, and (i_p, i_0) are arcs of $D(A)$. We write $\gamma = (i_0, i_1, \ldots, i_p, i_0)$ and denote the set of all circuits of $D(A)$ by $\mathcal{E}(A)$.

The matrix A is called *irreducible* if its directed graph is strongly connected, i.e., for every pair of distinct vertices i, j, there is a path P_{ij} in $D(A)$.

A particular directed graph which will arise in our subsequent discussion is the directed graph of a matrix $A \in \mathbf{C}^{n,n}$ whose diagonal entries are nonzero, the entries of the i_0–th row and column (for some $i_0 \in \langle n \rangle$) are nonzero, and all other entries are zero. Prompted by its shape, we refer to $D(A)$ as a *star centered at* i_0.

The *comparison matrix* of A, denoted by $\mathcal{M}(A) = [\alpha_{ij}] \in \mathbf{C}^{n,n}$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ii}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

If $A \in \mathbf{Z}^{n,n}$ then A is called an *M–matrix* provided that it can be expressed in the form $A = sI - B$, where B is an (entrywise) nonnegative matrix and $s \geq \rho(B)$ (where $\rho(B)$ denotes the spectral radius of B). The matrix A is called an *H–matrix* if $\mathcal{M}(A)$ is a nonsingular M–matrix. It is well known that H–matrices are nonsingular. For the properties of M–matrices and H–matrices and related material the reader is referred to Berman and Plemmons [1] and Horn and Johnson [5].

We will use the notation $R_i(A) = \sum_{k \neq i} |a_{ik}| \quad (i \in \langle n \rangle)$. Recall that A is called (row) *diagonally dominant* if

$$|a_{ii}| \geq R_i(A) \quad (i \in \langle n \rangle). \tag{1.1}$$

If the inequality in (1.1) is strict for all $i \in \langle n \rangle$ we say that A is *strictly diagonally dominant.* We say that A is *irreducibly diagonally dominant* if A is irreducible and at least one of the inequalities in (1.1) holds strictly. We now formally introduce the definitions and the notation pertaining to double diagonal dominance.

Definition 1.1 The matrix $A \in \mathbf{C}^{n,n}$ is *doubly diagonally dominant* ($A \in \mathbf{G}^{n,n}$) if

$$|a_{ii}||a_{jj}| \geq R_i(A)R_j(A), \quad i, j \in \langle n \rangle, \ i \neq j. \tag{1.2}$$

If the inequality in (1.2) is strict for all distinct $i, j \in \langle n \rangle$, we call A *strictly doubly diagonally dominant* ($A \in \mathbf{G}_1^{n,n}$). If A is an irreducible matrix that satisfies (1.2) and if at least one of the inequalities in (1.2) holds strictly, we call A *irreducibly doubly diagonally dominant* ($A \in \mathbf{G}_2^{n,n}$).

Notice that the diagonal entries of every matrix in $\mathbf{G}_1^{n,n}$ or $\mathbf{G}_2^{n,n}$ are nonzero.

Let us now review some classical results and note some similarities and differences between diagonal dominance and double diagonal dominance:

(1) If A is strictly diagonally dominant then $\det A \neq 0$ (Lévy–Desplanques theorem). If $A \in \mathbf{G}_1^{n,n}$ then $\det A \neq 0$ (by Brauer's theorem).

(2) If A is irreducibly diagonally dominant then $\det A \neq 0$ (see Taussky [8] and [9]). However, a matrix in $\mathbf{G}_2^{n,n}$ is not necessarily nonsingular as the following example shows:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

If $A \in \mathbf{G}_2^{n,n}$ and if (1.2) holds strictly for at least one pair of the vertices of some circuit $\gamma \in \mathcal{E}(A)$, we can conclude that $\det A \neq 0$ (see Zhang and Gu [11, Theorem 1]).

(3) If A is strictly diagonally dominant or irreducibly diagonally dominant then A is an H–matrix (see e.g., Varga [10]). More precisely, A is an H–matrix if and only if there exists a positive diagonal matrix D such that AD is strictly diagonally dominant. In the literature the latter property is referred to as 'generalized diagonal dominance' (see e.g., [1]), because it reduces to diagonal dominance when D is the identity. The example in (2) above also shows that not every matrix in $\mathbf{G}_2^{n,n}$ is an H–matrix.

(4) When A is irreducible, a form of diagonal dominance based on the circuits of $D(A)$, introduced by Brualdi in [2], implies the invertibility of A:

Theorem 1.2 ([2, Theorem 2.9]) *Let $A = [a_{ij}] \in \mathbf{C}^{n,n}$ be irreducible. Suppose*

$$\prod_{i\in\gamma} |a_{ii}| \;\geq\; \prod_{i\in\gamma} R_i(A) \quad (\gamma \in \mathcal{E}(A)),$$

with strict inequality holding for at least one circuit γ. Then $detA \neq 0$.

In what follows we will characterize H–matrices in $\mathbf{G}^{n,n}$ and $\mathbf{G}_2^{n,n}$, and will describe the singular matrices in $\mathbf{G}_2^{n,n}$ (section 2). In section 3 we will prove several results regarding the Schur complements of doubly diagonally dominant matrices, leading up to the fundamental result that the Schur complements of matrices in $\mathbf{G}^{n,n}$ are also doubly diagonally dominant.

2 Double Diagonal Dominance, Singularity and H–Matrices

We begin with some basic observations regarding matrices in $\mathbf{G}^{n,n}$.

Theorem 2.1 *Let $A \in \mathbf{G}^{n,n}$. Then the following hold.*

(i) *$\mathcal{M}(A)$ is an M–matrix.*

(ii) *A is an H–matrix if and only if $\mathcal{M}(A)$ is nonsingular.*

(iii) *If $A \in \mathbf{G}_1^{n,n}$, then A is an H–matrix.*

(iv) *If $A \in \mathbf{G}_2^{n,n}$ is such that (1.2) holds strictly for at least one pair of vertices i, j that lie on a common circuit of $D(A)$, then A is an H–matrix.*

Proof. To show (i), for $\epsilon > 0$, let $B_\epsilon = \mathcal{M}(A) + \epsilon I = [b_{ij}]$. Since $|b_{ii}||b_{jj}| > R_i(B_\epsilon)R_j(B_\epsilon)$ for all i, j, $i \neq j$, it follows from Brauer's theorem that $B_\epsilon \in \mathbf{Z}^{n,n}$ is nonsingular for every $\epsilon > 0$, which implies that $\mathcal{M}(A)$ is an M–matrix (see e.g., condition (C_9) of Theorem 4.6 in [1, Chapter 6]). Parts (ii) and (iii) are immediate consequences of part (i) and Brauer's theorem. Part (iv) follows from part (ii) and Theorem 1.2 applied to $\mathcal{M}(A)$. ∎

Some results related to Theorem 2.1 appear in [6]. There it is claimed that matrices in $\mathbf{G}_2^{n,n}$ are H–matrices, which is false as we have seen by an example in section 2.

Next we will characterize the singular matrices in $\mathbf{G}_2^{n,n}$. First we need the following lemma.

Lemma 2.2 *Consider $A \in \mathbf{C}^{n,n}$ such that $D(A)$ is a star centered at $i_0 \in \langle n \rangle$. Then*

$$\det A = \prod_{j\neq i_0} a_{jj} \left[a_{i_0 i_0} - \sum_{k\neq i_0} \frac{a_{k i_0} a_{i_0 k}}{a_{kk}} \right].$$

Proof. The terms in the expansion of the determinant of a matrix A as prescribed are

$$\prod_{j=1}^{n} a_{jj} \quad \text{and} \quad -(a_{ki_0}a_{i_0k}\prod_{m\neq k,i_0} a_{mm}) \quad (k \in \langle n\rangle \setminus \{i_0\})$$

and the formula for the determinant follows readily. ∎

Theorem 2.3 *Let $A = [a_{ij}] \in \mathbf{G}_2^{n,n}$. Then A is singular if and only if $D(A)$ is a star centered at some $i_0 \in \langle n\rangle$ and the following hold:*

$$|a_{i_0i_0}| < R_{i_0}(A), \quad |a_{i_0i_0}||a_{jj}| = R_{i_0}(A)R_j(A) \quad (j \in \langle n\rangle \setminus \{i_0\}) \tag{2.3}$$

and

$$a_{i_0i_0} - \sum_{k\neq i_0} \frac{a_{ki_0}a_{i_0k}}{a_{kk}} = 0. \tag{2.4}$$

Proof.

Sufficiency: If $D(A)$ is a star centered at $i_0 \in \langle n\rangle$ and (2.4) holds, then by Lemma 2.2, A is singular.

Necessity: Assume that $A = [a_{ij}] \in \mathbf{G}_2^{n,n}$ is singular. Since $A \in \mathbf{G}_2^{n,n}$, one of the following two cases must occur. Either $|a_{ii}| \geq R_i(A)$ for all $i \in \langle n\rangle$ with at least one strict inequality holding, or there exists one and only one $i_0 \in \langle n\rangle$ such that

$$|a_{i_0i_0}| < R_{i_0} \;\text{ and }\; |a_{jj}| > R_j(A) \quad (j \in \langle n\rangle \setminus \{i_0\}). \tag{2.5}$$

In the former case A is an irreducibly diagonally dominant matrix and hence nonsingular, contradicting our assumption. Therefore (2.5) holds. It also follows from the definition of $\mathbf{G}_2^{n,n}$ that

$$\prod_{i\in\gamma} |a_{ii}| \;\geq\; \prod_{i\in\gamma} R_i(A) \quad (\gamma \in \mathcal{E}(A)). \tag{2.6}$$

If $\gamma \in \mathcal{E}(A)$ and $i_0 \notin \gamma$, it follows by (2.5) that

$$\prod_{i\in\gamma} |a_{ii}| > \prod_{i\in\gamma} R_i(A). \tag{2.7}$$

Then Theorem 1.2, (2.6) and (2.7) imply that $\det A \neq 0$, contradicting our assumption. Hence for every $\gamma \in \mathcal{E}(A),\ i_0 \in \gamma$.

We now claim that every $\gamma \in \mathcal{E}(A)$ is of the form $\gamma = (i_0, j, i_0)$ for some $j \in \langle n\rangle \setminus \{i_0\}$. Indeed if $\gamma = (i_0, i_1, \cdots, i_p, i_0)$ with $p \geq 2$, then

$$\begin{aligned}\prod_{i\in\gamma} |a_{ii}| &= |a_{i_0i_0}||a_{i_1i_1}| \prod_{i\in\gamma\setminus\{i_0,i_1\}} |a_{ii}| \\ &> |a_{i_0i_0}||a_{i_1i_1}| \prod_{i\in\gamma\setminus\{i_0,i_1\}} R_i(A) \\ &\geq \prod_{i\in\gamma} R_i(A),\end{aligned}$$

so, by Theorem 1.2, $\det A \neq 0$, contradicting again our assumption that A is singular.

As is well known, since $D(A)$ is by assumption strongly connected, every vertex i lies on some circuit $\gamma \in \mathcal{E}(A)$. Therefore we deduce that

$$\mathcal{E}(A) = \{\gamma_j : \gamma_j = (i_0, j, i_0),\ j \in \langle n \rangle \setminus \{i_0\}\}\,. \tag{2.8}$$

In particular, it follows that there are no arcs (i_1, i_2) in $D(A)$ with $i_1 \neq i_0$ and $i_2 \neq i_0$, otherwise $\gamma = (i_0, i_1, i_2, i_0) \in \mathcal{E}(A)$, contradicting (2.8). Thus $D(A)$ is a star centered at i_0. If for some j, $|a_{i_0 i_0}||a_{jj}| > R_{i_0}(A) R_j(A)$, then, by Theorem 1.2, we are led to the contradiction that $\det A \neq 0$. Thus for each $j \in \langle n \rangle \setminus \{i_0\}$, we have

$$|a_{i_0 i_0}||a_{jj}| = R_{i_0}(A) R_j(A).$$

Finally, by Lemma 2.2, we can now assert that A satisfies (2.4). ■

We note that the necessity part of Theorem 2.3 also follows from the results in Tam, Yang, and Zhang [7]. The next theorem offers a characterization of the H–matrices in $\mathbf{G}_2^{n,n}$.

Theorem 2.4 *Let $A = [a_{ij}] \in \mathbf{G}_2^{n,n}$. Then A is not an H–matrix if and only if $D(A)$ is a star centered at some $i_0 \in \langle n \rangle$ and*

$$|a_{i_0 i_0}||a_{jj}| = R_{i_0}(A) R_j(A) \quad (j \in \langle n \rangle \setminus \{i_0\}). \tag{2.9}$$

Proof.

Necessity: Suppose A is not an H–matrix. Note that if $A \in \mathbf{G}_2^{n,n}$, then $\mathcal{M}(A) \in \mathbf{G}_2^{n,n}$. The result follows by Theorem 2.1 part (ii) and Theorem 2.3 applied to $\mathcal{M}(A)$.

Sufficiency: By assumption, $D(\mathcal{M}(A))$ is a star centered at some $i_0 \in \langle n \rangle$ and (2.9) holds. Consider the vector $x = [x_1, x_2, \ldots, x_n]^T$, where $x_{i_0} = R_{i_0}$ and $x_i = |a_{i_0 i_0}|$ for all $i \neq i_0$. Then $\mathcal{M}(A)x = 0$, $x \neq 0$, and thus, by Theorem 2.1 part (ii), A is not an H–matrix. ■

If $A \in \mathbf{G}^{n,n}$ is singular, by Theorem 2.1 part (ii), $\mathcal{M}(A)$ is singular. The converse of this statement is not necessarily true. More specifically, $A \in \mathbf{G}_2^{n,n}$ being nonsingular does not in general imply that A is an H–matrix (i.e., that $\mathcal{M}(A)$ is nonsingular). This situation occurs in the next example.

Example 2.5 The following matrices illustrate the use of Theorems 2.3 and 2.4 in checking whether an irreducibly doubly diagonally dominant matrix is an H–matrix or not. Consider the following matrices in $\mathbf{G}_2^{3,3}$:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 0 \\ -1 & 0 & 3 \end{bmatrix},\ E = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 0 \\ 1 & 1 & 4 \end{bmatrix}.$$

The directed graph of A is a star centered at $i_0 = 1$ and A satisfies (2.4). From Lemma 2.2, A is singular. Since $\mathcal{M}(B) = A$, B is not an H–matrix (even though B is nonsingular). Note that $D(C)$ is a star centered at $i_0 = 1$ but $|c_{11}||c_{33}| = 3 > 2 = R_1(C)R_3(C)$. Hence, by Theorem 2.4, C is an H–matrix. Finally, $D(E)$ is not a star centered at any $i_0 \in \{1, 2, 3\}$ and so E must be an H–matrix.

3 Schur Complements

Let $A = [a_{ij}] \in \mathbf{C}^{n,n}$ be partitioned as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{3.10}$$

where A_{11} is the leading $k \times k$ principal submatrix of A, for some $k \in \langle n \rangle$. Assuming that A_{11} is invertible we can reduce A (using elementary row operations) to the matrix

$$\begin{bmatrix} U_k & * \\ \mathbf{0} & A/A_{11} \end{bmatrix}, \tag{3.11}$$

where $U_k \in \mathbf{C}^{k,k}$ is upper triangular and A/A_{11}, known as the *Schur complement of A relative to A_{11}*, is given by $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. In particular, if $a_{11} \neq 0$, we can reduce A to the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}, \tag{3.12}$$

where $b_{ij} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}$, $2 \leq i, j \leq n$. The trailing $(n-1) \times (n-1)$ submatrix of the matrix above is the Schur complement of A relative to $A_{11} = [a_{11}]$, which we will subsequently denote by $B = [b_{ij}]$, and index its entries by $2 \leq i, j \leq n$.

In this section, we shall prove that if A belongs to $\mathbf{G}^{n,n}$ and $\det A_{11} \neq 0$, then A/A_{11} belongs to $\mathbf{G}^{n-k,n-k}$. We will first consider the Schur complements of matrices in $\mathbf{G}_1^{n,n}$. We note that our proofs rely on the fact that if $A \in \mathbf{G}_1^{n,n}$, then all principal submatrices of A are invertible and so the associated Schur complements are well defined. The following is a well known fact in numerical linear algebra.

Lemma 3.1 *If $A \in \mathbf{C}^{n,n}$ is strictly diagonally dominant and partitioned as in (3.10), then $\det A_{11} \neq 0$ and A/A_{11} is also strictly diagonally dominant.*

Lemma 3.2 *Let $A = [a_{ij}] \in \mathbf{G}_1^{3,3}$. Then*

$$\left| a_{22} - \frac{a_{21}a_{12}}{a_{11}} \right| \left| a_{33} - \frac{a_{31}a_{13}}{a_{11}} \right| > \left| a_{23} - \frac{a_{21}a_{13}}{a_{11}} \right| \left| a_{32} - \frac{a_{31}a_{12}}{a_{11}} \right|. \tag{3.13}$$

Proof. Since $A = [a_{ij}] \in \mathbf{G}_1^{3,3}$, from Theorem 2.1 part (iii), A is an H–matrix. Hence there is a positive diagonal matrix $D = \text{diag}(d_1, d_2, d_3)$ such that AD is a strictly diagonally dominant matrix. Since $d_1 a_{11} \neq 0$, we can reduce AD to the matrix

$$\begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ 0 & d_2a_{22} - \frac{d_2a_{21}a_{12}}{a_{11}} & d_3a_{23} - \frac{d_3a_{21}a_{13}}{a_{11}} \\ 0 & d_2a_{32} - \frac{d_2a_{31}a_{12}}{a_{11}} & d_3a_{33} - \frac{d_3a_{31}a_{13}}{a_{11}} \end{bmatrix},$$

which, by Lemma 3.1, is also strictly diagonally dominant and (3.13) follows. ■

Theorem 3.3 *Let $A \in \mathbf{G}_1^{n,n}$ and let $B \in \mathbf{C}^{n-1,n-1}$ as in* (3.12). *Then $B \in \mathbf{G}_1^{n-1,n-1}$.*

Proof. Since $A = [a_{ij}] \in \mathbf{G}_1^{n,n}$, one of the following two cases must occur. Either there exists $i \in \langle n \rangle$ such that $|a_{ii}| \leq R_i(A)$, $|a_{jj}| > R_j(A)$, and $|a_{ii}||a_{jj}| > R_i(A)R_j(A) \quad (j \in \langle n \rangle \setminus \{i\})$, or $|a_{ii}| > R_i(A) \quad (i \in \langle n \rangle)$. In the latter case, by Lemma 3.1, B is strictly diagonally dominant and hence $B \in \mathbf{G}_1^{n-1,n-1}$. We now consider the former case in two subcases:

(i) $i = 1$.

In this case, we shall also prove that B is strictly diagonally dominant (and hence in $\mathbf{G}_1^{n-1,n-1}$). It suffices to prove that

$$|b_{22}| > \sum_{j=3}^{n} |b_{2j}|, \tag{3.14}$$

where $b_{22} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$, and $b_{2j} = a_{2j} - \frac{a_{21}a_{1j}}{a_{11}}$, with $j \geq 3$.

Since

$$\begin{aligned}|a_{11}||a_{22}| > \sum_{j=2}^{n} |a_{1j}| \sum_{j \neq 2} |a_{2j}| &= \sum_{j=2}^{n} |a_{1j}| \sum_{j=3}^{n} |a_{2j}| + |a_{21}| \sum_{j=2}^{n} |a_{1j}| \\ &\geq |a_{11}| \sum_{j=3}^{n} |a_{2j}| + |a_{21}||a_{12}| + |a_{21}| \sum_{j=3}^{n} |a_{1j}|,\end{aligned}$$

(where we used the assumption $|a_{11}| \leq R_1(A)$ for the last inequality), we have

$$|a_{11}a_{22}| - |a_{12}a_{21}| > |a_{11}| \sum_{j=3}^{n} \left(|a_{2j}| + \frac{|a_{21}a_{1j}|}{|a_{11}|} \right) \geq |a_{11}| \sum_{j=3}^{n} \left| a_{2j} - \frac{a_{21}a_{1j}}{a_{11}} \right|.$$

That is,

$$\left| a_{22} - \frac{a_{21}a_{12}}{a_{11}} \right| \geq |a_{22}| - \frac{|a_{21}a_{12}|}{|a_{11}|} > \sum_{j=3}^{n} \left| a_{2j} - \frac{a_{21}a_{1j}}{a_{11}} \right|,$$

which is equivalent to (3.14).

(ii) $i \geq 2$.

In this case we will see that B belongs to $\mathbf{G}_1^{n-1,n-1}$. Without loss of generality, we can assume that $i = 2$. Set

$$A_1 = \begin{bmatrix} |a_{11}| & -\sum_{j \neq 1,3} |a_{1j}| & -|a_{13}| \\ -|a_{21}| & |a_{22}| & -\sum_{j=3}^{n} |a_{2j}| \\ -|a_{31}| & -\sum_{j \neq 1,3} |a_{3j}| & |a_{33}| \end{bmatrix}.$$

Since $A \in \mathbf{G}_1^{n,n}$ it follows that $A_1 \in \mathbf{G}_1^{3,3} \cap \mathbf{Z}^{3,3}$ and that A_1 has positive diagonal entries. Applying Lemma 3.2 to A_1 we obtain

$$\left[|a_{22}| - \frac{|a_{21}|}{|a_{11}|} \sum_{j \neq 1,3} |a_{1j}| \right] \left[|a_{33}| - \frac{|a_{13}a_{31}|}{|a_{11}|} \right] >$$

$$\left[\sum_{j=3}^{n}|a_{2j}|+\frac{|a_{21}a_{13}|}{|a_{11}|}\right]\left[\sum_{j\neq 1,3}|a_{3j}|+\frac{|a_{31}|}{|a_{11}|}\sum_{j\neq 1,3}|a_{1j}|\right]. \tag{3.15}$$

Setting

$$\gamma_1=\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|\left[|a_{33}|-\frac{|a_{31}a_{13}|}{|a_{11}|}\right],\ \gamma_2=\sum_{j=3}^{n}|a_{2j}|+\frac{|a_{21}a_{13}|}{|a_{11}|},$$

$$\gamma_3=\sum_{j\neq 1,3}|a_{3j}|+\frac{|a_{31}|}{|a_{11}|}\sum_{j\neq 1,3}|a_{1j}|,$$

we see that (3.15) is equivalent to

$$\left[|a_{22}|-\frac{|a_{21}a_{12}|}{|a_{11}|}\right]\left[|a_{33}|-\frac{|a_{31}a_{13}|}{|a_{11}|}\right]>\gamma_1+\gamma_2\gamma_3. \tag{3.16}$$

For γ_1 we have

$$\begin{aligned}\gamma_1&=\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|\left||a_{33}|-\frac{|a_{31}a_{13}|}{|a_{11}|}\right|\\
&\overset{|a_{33}|>R_3(A)}{\geq}\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|\left[\sum_{j\neq 1,3}|a_{3j}|+\frac{|a_{31}|}{|a_{11}|}(|a_{11}|-|a_{13}|)\right]\\
&\overset{|a_{11}|>R_1(A)}{\geq}\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|\left[\sum_{j\neq 1,3}|a_{3j}|+\frac{|a_{31}|}{|a_{11}|}\sum_{j\neq 1,3}|a_{1j}|\right]=\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|\gamma_3.\end{aligned} \tag{3.17}$$

From (3.16) and (3.17), it follows that

$$\begin{aligned}\left|a_{22}-\frac{a_{21}a_{12}}{a_{11}}\right|\left|a_{33}-\frac{a_{31}a_{13}}{a_{11}}\right|&\geq\left[|a_{22}|-\frac{|a_{21}a_{12}|}{|a_{11}|}\right]\left[|a_{33}|-\frac{|a_{31}a_{13}|}{|a_{11}|}\right]>\gamma_1+\gamma_2\gamma_3\\
&\geq\left[\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|+\sum_{j=3}^{n}|a_{2j}|+\frac{|a_{21}a_{13}|}{|a_{11}|}\right]\gamma_3\\
&=\left[\frac{|a_{21}|}{|a_{11}|}\sum_{j=3}^{n}|a_{1j}|+\sum_{j=3}^{n}|a_{2j}|\right]\left[\sum_{j\neq 1,3}|a_{3j}|+\frac{|a_{31}|}{|a_{11}|}\sum_{j\neq 1,3}|a_{1j}|\right]\\
&\geq\sum_{j=3}^{n}\left|a_{2j}-\frac{a_{21}a_{1j}}{a_{11}}\right|\sum_{j\neq 1,3}\left|a_{3j}-\frac{a_{31}a_{1j}}{a_{11}}\right|,\end{aligned}$$

or equivalently $|b_{22}||b_{33}|>R_2(B)R_3(B)$. Similarly, $|b_{22}||b_{jj}|>R_2(B)R_j(B)$ for $j=4,5,\ldots,n$. In general, since row reduction with respect to a strictly diagonally dominant row preserves strict diagonal dominance, we have that $|b_{ii}||b_{jj}|>R_i(B)R_j(B)$ for $i,j=3,4,\ldots,n$ and $i\neq j$. Hence, $B\in\mathbf{G}_1^{n-1,n-1}$. ■

Corollary 3.4 *If $A=[a_{ij}]\in\mathbf{G}_1^{n,n}$ and $|a_{11}|\leq R_1(A)$, then B, as in (3.12), is strictly diagonally dominant.*

Proof. This is subcase (i) in the proof of the previous theorem. ■

We continue now with general Schur complements of matrices in $\mathbf{G}_1^{n,n}$.

Theorem 3.5 *Let $J = \{i \in \langle n \rangle : |a_{ii}| \leq R_i(A)\}$, where $A = [a_{ij}] \in \mathbf{G}_1^{n,n}$ is partitioned as in* (3.10). *Then*

(i) *A/A_{11} is strictly diagonally dominant if $J \subset \{1, 2, \cdots, k\}$.*

(ii) *$A/A_{11} \in \mathbf{G}_1^{n-k,n-k}$ if $\emptyset \neq J \subset \{k+1, \ldots, n\}$.*

Proof.

(i) If $J = \emptyset$, then A is strictly diagonally dominant and hence the result follows by Lemma 3.1. If $J \neq \emptyset$, then J can only contain one element. Without loss of generality, assume that $i = 1 \in J$ (otherwise we can symmetrically permute the first k rows and columns of A, an operation that leaves the Schur complement in question unaffected.) From Corollary 3.4, B (as defined in (3.12)) is strictly diagonally dominant. The result follows by noting that A/A_{11} is equal to a Schur complement of B (see e.g., Fiedler [3, Theorem 1.25]) and by applying Lemma 3.1 to B.

(ii) From Theorem 3.3 we have that $B \in \mathbf{G}_1^{n-1,n-1}$. Inductively, since A/A_{11} is equal to a Schur complement of B, it follows that if $\emptyset \neq J \subset \{k+1, \ldots, n\}$ then $A/A_{11} \in \mathbf{G}_1^{n-k,n-k}$.

■

Remark 3.6 If $\emptyset \neq J \subset \{k+1, \ldots, n\}$, A/A_{11} is not necessarily strictly diagonally dominant. For example, consider

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1.1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \in \mathbf{G}_1^{3,3}$$

Taking $A_{11} = [2]$ with $J = \{2\}$ we have that $A/A_{11} = \begin{bmatrix} 0.6 & -1 \\ 0 & 2 \end{bmatrix}$ which is not strictly diagonally dominant.

We can now turn our attention to Schur complements of matrices in $\mathbf{G}^{n,n}$.

Theorem 3.7 *If $A \in \mathbf{G}^{n,n}$ is partitioned as in* (3.10) *with* $\det A_{11} \neq 0$*, then $A/A_{11} \in \mathbf{G}^{n-k,n-k}$.*

Proof. Let $A = [a_{ij}]$ be as prescribed above. We first observe that $a_{ii} \neq 0$ for $i \in \{1, 2, \ldots, k\}$. Indeed, if $a_{ii} = 0$ for some $i \in \{1, 2, \ldots, k\}$, then $0 \geq R_i(A)R_j(A)$ for all $j \in \langle n \rangle \setminus \{i\}$. Also $R_i(A) \neq 0$ since $\det A_{11} \neq 0$ and hence $R_j(A) = 0$ for all $j \in \langle n \rangle \setminus \{i\}$. Thus the i–th column of A_{11} is zero, a contradiction.

Set now $D = \mathrm{diag}(e^{i\arg a_{11}}, \ldots, e^{i\arg a_{kk}}, \delta_{k+1}, \ldots, \delta_n)$, where, for $j \in \{k+1, k+2, \ldots, n\}$,

$$\delta_j = \begin{cases} e^{i\arg a_{jj}} & \text{if } a_{jj} \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Note that $A + \epsilon D \in G_1^{n,n}$, for every $\epsilon > 0$. Suppose that we row reduce $A + \epsilon D$ and obtain the matrix

$$\begin{bmatrix} (|a_{11}| + \epsilon)e^{i\arg a_{11}} & a_{12} & \cdots & a_{1n} \\ 0 & b_{22}(\epsilon) & \cdots & b_{2n}(\epsilon) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b_{n2}(\epsilon) & \cdots & b_{nn}(\epsilon) \end{bmatrix}.$$

Set $B(\epsilon) = [b_{ij}(\epsilon)]$. For $2 \le i \le k$ we have

$$b_{ii}(\epsilon) = (|a_{ii}| + \epsilon)e^{i\arg a_{ii}} - \frac{a_{i1}a_{1i}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}}, \tag{3.18}$$

$$b_{ij}(\epsilon) = a_{ij} - \frac{a_{i1}a_{1j}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}} \quad (j \neq i;\ j \geq 2). \tag{3.19}$$

For $k + 1 \le i \le n$,

$$b_{ii}(\epsilon) = a_{ii} + \epsilon\delta_i - \frac{a_{i1}a_{1i}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}}, \tag{3.20}$$

$$b_{ij}(\epsilon) = a_{ij} - \frac{a_{i1}a_{1j}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}} \quad (j \neq i;\ j \geq 2). \tag{3.21}$$

From Theorem 3.5 we obtain

$$|b_{ii}(\epsilon)||b_{jj}(\epsilon)| > R_i(B(\epsilon))R_j(B(\epsilon)) \quad (i \neq j;\ i, j \geq 2). \tag{3.22}$$

The combination of (3.18)–(3.21) gives

$$\lim_{\epsilon \to 0} |b_{ii}(\epsilon)| = \left| a_{ii} - \frac{a_{i1}a_{1i}}{a_{11}} \right| = |b_{ii}|, \ i \geq 2$$

and

$$\lim_{\epsilon \to 0} |b_{ij}(\epsilon)| = \left| a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}} \right| = |b_{ij}| \quad (i \neq j;\ i, j \geq 2),$$

(recalling B from (3.12)). Hence, by taking the limit in (3.22) as $\epsilon \to 0$, we have

$$|b_{ii}||b_{jj}| \geq R_i(B)R_j(B) \quad (i \neq j).$$

Thus $B \in \mathbf{G}^{n-1,n-1}$. The theorem follows by noting that A/A_{11} is equal to a Schur complement of B, and by applying the above argument inductively. ■

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