# Extremal Properties of Ray–nonsingular Matrices

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# Abstract

A ray–nonsingular matrix is a square complex matrix, $A$, such that each complex matrix whose entries have the same arguments as the corresponding entries of $A$ is nonsingular. Extremal properties of ray–nonsingular matrices are studied in this paper. Combinatorial and probabilistic arguments are used to prove that if the order of a ray–nonsingular matrix is at least 6, then it must contain a zero entry, and that if each of its rows and columns have an equal number, $k$, of nonzeros, then $k \leq 13$.

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# 1 Introduction

A complex matrix is a *ray–pattern* matrix if each of its nonzero entries has modulus 1. A ray–pattern matrix is *full* if each of its entries is nonzero. An $n$ by $n$ complex matrix $A = [a_{j,k}]$ is a *ray–nonsingular* matrix provided $A \circ X$ is nonsingular for each real, entrywise positive matrix $X$, where $A \circ X$ denotes the Hadamard (entrywise) product of $A$ and $X$. Some general properties of ray–nonsingular matrices are proven in [MOTV]. Ray–nonsingular matrices whose entries are real are precisely the *sign–nonsingular* matrices, and these have been extensively studied (see [BS]).

Note that the matrix $A = [a_{j,k}]$ is ray–nonsingular if and only if the ray–pattern matrix obtained from $A$ by replacing each of its nonzero entries $a_{j,k}$ by $\frac{a_{j,k}}{|a_{j,k}|}$ is ray–nonsingular. Thus in discussing ray–nonsingular matrices we may assume without loss of generality that $A$ is a ray–pattern matrix. If $X$ is an entrywise positive matrix, we say that $A \circ X$ has ray–pattern $A$.

In this paper we address the following question, which is posed in [MOTV]:

For which $n$ does there exist a full $n$ by $n$ ray–nonsingular matrix?

The corresponding problem for sign–nonsingular matrices was originally posed by Pólya [P], and there are numerous ways to show that full $n$ by $n$ sign–nonsingular matrices exist only for $n = 1$ and $n = 2$. Examples of full $n$ by $n$ ray–nonsingular matrices for $n = 2, 3, 4$ are given in [MOTV]. In Section 2 we provide some interesting examples. In particular, we give an example of a full ray–nonsingular 4 by 4 pattern that has zero in the relative interior of the convex hull of its signed transversal products, answering negatively a question raised in [MOTV].

In Section 3, we use elementary probabilistic (counting) methods to show that full ray–nonsingular matrices do not exist for $n \geq 6$. In Section 4, the remaining case of $n = 5$ is discussed, but not resolved. In Section 5, the Lovász Local Lemma is used to show that if $A$ is a ray–nonsingular matrix with exactly $k$ nonzeros in each row and column, then $k \leq 13$.

We conclude this introductory section with some necessary technical definitions. We denote the set $\{1, 2, 3, \ldots, n\}$ by $\langle n \rangle$. Let $A$ be an $m$ by $n$ matrix, and let $\alpha \subseteq \langle m \rangle$, $\beta \subseteq \langle n \rangle$. Then $A[\alpha, \beta]$ denotes the submatrix of $A$ determined by the rows whose indices are in $\alpha$, and the columns whose indices are in $\beta$. We denote $A[\langle m \rangle \setminus \alpha, \beta]$ by $A(\alpha, \beta]$. The submatrices $A[\alpha, \beta)$ and $A(\alpha, \beta)$ are defined analogously. If $x$ is an $m$ by 1 column vector, then we write $x[\alpha]$ instead of $x[\alpha, \{1\}]$.

## 2 Examples with $n = 4$

In this section we present examples of full 4 by 4 ray–nonsingular patterns. The techniques used to prove that they are ray–nonsingular vary among the examples, illustrating some of the properties inherent to full 4 by 4 ray–nonsingular matrices, as well as differences between ray–nonsingularity and sign–nonsingularity.

Recall that the signed transversal products of a ray–pattern matrix are, by definition, the summands in its standard determinantal expansion. A necessary and sufficient condition for a real ray-pattern to be sign–nonsingular is that the nonzero signed transversal products, of which there is at least one, all have the same sign (see [BS]). It is shown in [MOTV] that a sufficient (but not necessary) condition for ray–nonsingularity is that zero is not in the relative interior of the convex hull of the signed transversal products.

In [MOTV] the following example of a full 4 by 4 ray–nonsingular pattern is given:

$$\begin{bmatrix} i & 1 & 1 & 1 \\ 1 & i & 1 & 1 \\ 1 & 1 & i & 1 \\ 1 & 1 & 1 & i \end{bmatrix}. \tag{2.1}$$

This ray–pattern matrix is ray–nonsingular because zero is not in the relative interior of the convex hull of its signed transversal products.

Also in [MOTV], a reducible 4 by 4 ray–nonsingular pattern is presented, where zero is in the relative interior of the convex hull of the signed transversal products. It is natural to ask whether there exists a full (or fully indecomposable, see [BR]) ray–nonsingular pattern that has zero in the relative interior of the convex hull of its signed transversal products. The answer is yes, as the following example shows:

$$\begin{bmatrix} 1 & 1 & 1 & i \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}. \tag{2.2}$$

The signed transversal products of the ray–pattern matrix in (2.2) are $1, -1, i$, and $-i$; however the ray–pattern matrix in (2.2) is ray–nonsingular. For suppose it is not, namely, there is a singular matrix $B$ with ray–pattern (2.2). Notice that the 3 by 4 matrix $C$ formed by deleting the first row of $B$ is an L-matrix (see [BS]) and thus has nullity one. Since $C$ is real, its nullspace is spanned by a real nullvector, $x$. Since any nullvector of $B$ is also an nullvector of $C$, it must be that $B$ also has nullvector $x$. Considering the first row of $B$, it follows that the last entry of $x$ must be zero. Thus the first three rows of $x$ are a nonzero nullvector of the 4 by 3 matrix $F$ formed by deleting the last column of $B$. But $F^T$ is an L-matrix, and hence has rank 3. This is a contradiction, and thus the ray–pattern matrix in (2.2) is indeed ray-nonsingular.

Next consider the ray–pattern matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ i & -i & 1 & -1 \\ i & i & 1 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ iA_{21} & A_{22} \end{bmatrix}, \tag{2.3}$$

where the $A_{jk}$ are all 2 by 2 blocks. Since $A_{11}$ is sign–nonsingular, we can consider the Schur complement $A/A_{11} = A_{22} - iA_{21}(A_{11})^{-1}A_{12}$. Notice that if a 2 by 2 complex matrix is of the form $B + iC$, where $B$ and $C$ are real, then Re(det($B + iC$))= det($B$)−det($C$). Since each of the $A_{jk}$ is sign–nonsingular, it follows that Re(det($A/A_{11}$))> 0 and hence $A$ is ray-nonsingular.

## 3 Nonexistence for $n \geq 6$

We begin with some definitions and notation, and some results from [MOTV]. A *complex signing*, $D$, of order $n$ is a nonzero $n$ by $n$ diagonal ray–pattern matrix. If each diagonal entry of $D$ is nonzero, then $D$ is a *strict* complex signing. A *strict* $(1, -1)$*-signing* is a strict complex signing each of whose diagonal entries is contained in the set $\{1, -1\}$.

An $m$ by 1 ray–pattern vector $x$ is *balanced* if $x$ is the zero vector or the origin is in the relative interior of the convex hull of the nonzero entries of $x$. The balanced vector $x$ is *strongly balanced* if there are at least 3 distinct values among its nonzero entries, and *weakly balanced* if it is the zero vector or there are exactly 2 distinct values among its nonzero entries. Note that if $x$ is strongly balanced, then so is each vector $y$ obtained from $x$ by appending on a new coordinate. The following gives another geometric condition, which is equivalent to $x$ being balanced and is easily proven.

**Lemma 3.1** *Let $x$ be an $m$ by 1 ray–pattern vector. Then the vector $x$ is*

(a) *weakly balanced if and only if $x$ is the zero vector or the nonzero entries of $x$ separate the unit circle into exactly 2 arcs each of length $\pi$, and*

(b) *strongly balanced if and only if the nonzero entries of $x$ separate the unit circle into arcs none of which has length $\pi$ or greater.*

The next result is Theorem 3.5 of [MOTV] and gives necessary and sufficient conditions for a ray–pattern matrix to be ray-nonsingular.

**Lemma 3.2** *Let $A$ be an $n$ by $n$ ray–pattern matrix. Then $A$ is a ray-nonsingular matrix if and only if for each complex signing $D$ there exists a column of $DA$ that is not balanced.*

We define an $m$ by $n$ ray–pattern matrix to be *generic* if no two nonzero entries in the same column are equal or opposites of each other. The next lemma asserts that for each ray–pattern matrix $A$ there is a strict complex signing $D$ such that $DA$ is generic.

**Lemma 3.3** *Let $A$ be an $m$ by $n$ ray–pattern matrix, and let $\epsilon$ be a real, positive number. Then there is a strict complex signing $D = \mathrm{diag}(d_1, d_2, \ldots, d_m)$ such that $DA$ is generic, and $0 < \arg(d_j) < \epsilon$ for all $j \in \langle m \rangle$.*

**Proof.** If for each column of $A$ the nonzero entries are equal, set $\theta = \pi$. Otherwise, define $\theta$ to be the minimum of

$$|\arg(a_{p,\ell}) - \arg(a_{q,\ell})|, \quad |\pi - |\arg(a_{p,\ell}) - \arg(a_{q,\ell})||,$$

over all pairs $a_{p,\ell}$ and $a_{q,\ell}$ such that $a_{p,\ell}$ and $a_{q,\ell}$ are nonzero and $a_{p,\ell} \neq \pm a_{q,\ell}$. Thus, $\theta$ measures the closest to 0 or $\pi$ that the difference between the arguments of two nonzero entries of the same column of $A$ can be. Note in particular that $0 < \theta \leq \pi$.

Set $\hat{\theta} = \min\{\theta, 4m\epsilon\}$, and

$$D = \mathrm{diag}(e^{i\hat{\theta}/(4m)}, e^{2i\hat{\theta}/(4m)}, e^{3i\hat{\theta}/(4m)}, \ldots, e^{mi\hat{\theta}/(4m)}).$$

Note that the argument of each diagonal entry of $D$ is positive and less than $\epsilon$. Consider two nonzero entries, $a_{p,\ell}$ and $a_{q,\ell}$ with $q < p$, from the same column of $A$. Then

$$\arg(e^{pi\hat{\theta}/(4m)}a_{p,\ell}) - \arg(e^{qi\hat{\theta}/(4m)}a_{q,\ell})$$

is congruent modulo $2\pi$ to

$$\arg(a_{p,\ell}) - \arg(a_{q,\ell}) + \frac{(p-q)\hat{\theta}}{4m}.$$

The definition of $\hat{\theta}$ and the facts that $0 < \hat{\theta} \leq \pi$, and $0 < p - q < m$ now imply that

$$\arg(e^{pi\hat{\theta}/(4m)}a_{p,\ell}) - \arg(e^{qi\hat{\theta}/(4m)}a_{q,\ell})$$

is not an integer multiple of $\pi$ and hence $e^{pi\hat{\theta}/(4m)}a_{p,\ell} \neq \pm e^{qi\hat{\theta}/(4m)}a_{q,\ell}$. Therefore $D$ is a strict complex signing with the desired property. ■

If $x$ is an $n$ by 1 ray–pattern vector, then the number of strict $(1,-1)$-signings $D$ such that $Dx$ is not balanced is not completely determined by the number of nonzero entries in $x$. For example, there are exactly 2 strict $(1,-1)$-signings $D$ such that

$$D\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is not balanced, but there are 8 strict $(1,-1)$-signings $E$ such that

$$E\begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix}$$

is not balanced. The following lemma shows that this type of discrepancy does not occur for generic vectors.

**Lemma 3.4** *Let $x = [x_j]$ be an $n$ by $1$ generic, ray–pattern vector with exactly $k$ nonzero entries. Then the number of strict $(1,-1)$-signings $D$ such that $Dx$ is not balanced equals $k2^{n-k+1}$.*

**Proof.** Clearly, if $E$ is a strict $(1,-1)$-signing, then the number of strict $(1,-1)$-signings $D$ such that $DEx$ is not balanced equals the number of $D$ such that $Dx$ is not balanced. Thus, since $x$ is generic, we may without loss of generality assume that

$$x_{k+1} = x_{k+2} = \cdots = x_n = 0$$

and that

$$0 \leq \arg(x_1) < \arg(x_2) < \cdots < \arg(x_k) < \pi.$$

It is easy to verify that each of the $k2^{n-k+1}$ strict $(1,-1)$-signings $D$ with at most one sign-change in the first $k$ diagonal entries has the property that $Dx$ is not balanced.

To complete the proof, we assume that $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is a strict $(1, -1)$-signing whose first $k$ diagonal entries have at least two sign changes, and show that $Dx$ is balanced. Since $Dx$ is not balanced if and only if $(-D)x$ is not balanced, we may assume that the first entry of $D$ is $+1$.

Since the first $k$ diagonal entries of $D$ have at least 2 sign changes, there exist $p$ and $q$ with $1 < p < q \leq k$ such that $d_p = -1$ and $d_q = 1$. The points $x_1, -x_p, x_q$ separate the unit circle into 3 arcs: the arcs from $x_1$ to $x_q$, from $x_q$ to $-x_p$, and from $-x_p$ to $x_1$. We claim that each of these arcs has length less than $\pi$. Since $\arg(x_1) < \arg(x_q) < \pi$, the arc from $x_1$ to $x_q$ has length less than $\pi$. Similarly the arc from $-x_p$ to $x_1$ has length less than $\pi$. Since $\arg(x_p) < \arg(x_q) < \pi \leq \arg(x_p) + \pi$, the arc from $x_q$ to $-x_p$ has length less than $\pi$. Hence by Lemma 3.1, $Dx$ is balanced. ■

The following result is analogous to the result in [T], which asserts that every sign-nonsingular matrix of order $n$ has a column with at most $\lfloor \lg n \rfloor + 1$ nonzero entries.

**Theorem 3.5** *Let $A$ be an $n$ by $n$ ray-nonsingular matrix. Then there exists a column of $A$ with at most $\lfloor 2 \lg n \rfloor + 1$ nonzero entries.*

**Proof.** Without loss of generality, by Lemma 3.3, we may assume that $A$ is a generic, ray-nonsingular matrix. Let $k_j$ be the number of nonzeros in column $j$ of $A$.

By Lemma 3.2, for each strict $(1, -1)$-signing $D$ some column of $DA$ is not balanced. By Lemma 3.4, there are exactly $k_j 2^{n-k_j+1}$ strict $(1, -1)$-signings $D$ such that column $j$ of $A$ is not balanced. Since there are exactly $2^n$ strict $(1, -1)$-signings, we conclude that

$$\sum_{j=1}^{n} k_j 2^{n-k_j+1} \geq 2^n.$$

Setting $\ell$ to be the minimum of the $k_j$'s, we obtain

$$2^{n-\ell+1} n^2 \geq 2^{n-\ell+1} \sum_{j=1}^{n} k_j \geq \sum_{j=1}^{n} k_j 2^{n-k_j+1} \geq 2^n.$$

Hence

$$(n - \ell + 1) + 2 \lg n \geq n,$$

which implies that $2 \lg n + 1 \geq \ell$. The theorem follows by noting that $\ell$ is an integer. ■

**Corollary 3.6** *Let $A$ be a full $n$ by $n$ ray-nonsingular matrix. Then $n \leq 6$.*

**Proof.** By Theorem 3.5, $n \leq 2 \lg n + 1$, or equivalently $2^{n-1} \leq n^2$. This only holds for $n \leq 6$. ■

**Lemma 3.7** *Let $A$ be a full $n$ by $n$ ray–pattern matrix, and let $B$ be an $m$ by $m$ submatrix of $A$. If there is a strict complex signing $E$ of order $m$ with the property that each column of $EB$ is strongly balanced, then $A$ is not ray-nonsingular.*

**Proof.** The proof is by induction on $n - m$. If $n = m$, then by Lemma 3.2 $A$ is not ray-nonsingular. Assume that $m < n$ and that the result holds for submatrices of order $m + 1$. Without loss of generality we may assume that $B = A[\langle m\rangle, \langle m\rangle]$ and $E$ is as claimed. We show that there is a strict complex signing $F$ such that each column of $FA[\langle m{+}1\rangle, \langle m{+}1\rangle]$ is strongly balanced. It follows from Lemma 3.3 that there exists a strict complex signing $D$ such that $DEA[\langle m\rangle, \langle n\rangle]$ is generic and each column of $DEB$ is still strongly balanced. Let $\theta$ be an arbitrary angle and let $F_\theta$ be the strict complex signing defined by $F_\theta = DE \oplus [e^{i\theta}]$. Each of columns 1 through $m$ of $F_\theta A[\langle m + 1\rangle, \langle m + 1\rangle]$ is balanced. It is easy to verify that there exists a choice of $\theta$ such that the $(m + 1)$th column of $F_\theta A[\langle m + 1\rangle, \langle m + 1\rangle$ is balanced. Since the columns of $DEA[\langle m\rangle, \langle n\rangle]$ are generic, it follows that for this choice of $\theta$ each column of $F_\theta A[\langle m + 1\rangle, \langle m + 1\rangle]$ is strongly balanced. Hence by Lemma 3.2, $A$ is not ray-nonsingular. ■

We note that if a full $m$ by 1 vector is strongly balanced, then $m \geq 3$. Hence, Lemma 3.7 only applies to $n$ by $n$ matrices $A$ with $n \geq 4$.

**Corollary 3.8** *Let $A$ be a full 6 by 6 ray–pattern matrix. Then $A$ is not ray-nonsingular.*

**Proof.** By Lemma 3.3 we may assume that $A$ a generic, ray–pattern matrix. Let $B$ be the 5 by 6 matrix obtained from $A$ by deleting its first row. For $j \in \langle 6\rangle$, Lemma 3.4 implies that there are exactly 10 strict $(1, -1)$-signings $D$ such that column $j$ of $DB$ is not balanced. Hence there are 60 columns among all the $DB$ that are not balanced. Since there are 32 strict $(1, -1)$-signings $D$ of order 5, the average number of columns not balanced among the $DB$'s equals $60/32 < 2$. Therefore, there exists a strict $(1,-1)$-signing $E$ such that at most one column of $EB$ is not balanced and thus $EB$ contains a 5 by 5 submatrix with every column strongly balanced. The result now follows from Lemma 3.7. ■

Note that Theorem 3.5 and Corollaries 3.6 and 3.8 are essentially probabilistic arguments. Let $X$ be the probability space consisting of all strict $(1,-1)$-signings of order $n$, each with probability $1/2^n$. Lemma 3.4 implies that if $x$ is a generic vector with $k$ nonzero entries, then the probability of the event

$$\{D \in X : Dx \text{ is not balanced}\}$$

is $k/2^{k-1}$. The proof of Theorem 3.5 shows that if each column of $A$ has more than $\lfloor 2 \lg n \rfloor + 1$ nonzero entries, then the event

$$\{D \in X : \text{each column of } DA \text{ is balanced}\}$$

has nonzero probability, and hence $A$ is not ray-nonsingular. These results are based on the discrete probability space $X$.

One can also consider the continuous probability space $Y$ consisting of all strict complex signings of order $n$ with the uniform distribution. It follows from basic properties of order statistics (see Theorem 2 on page 28 of [Fe]), that for a (not necessarily generic) $n$ by 1 vector $x$ with $k$ nonzero entries the event

$$\{D \in Y : \ Dx \text{ is strongly balanced }\}$$

has probability $k/2^{k-1}$. This is the same probability as in the discrete case. Using this one can easily adapt the proofs of Theorem 3.5 and Corollaries 3.6 and 3.8. These new arguments avoid the use of generic matrices.

We chose to emphasize the discrete argument because it is closer to the arguments used for sign-nonsingular matrices, it is conducive to further combinatorial analysis, and because we obtain a slightly stronger result. Namely, if $A$ is a full $n$ by $n$ generic ray-pattern matrix with $n \geq 7$, then there exists a matrix $\widetilde{A}$ each of whose entries has the same argument as the corresponding entry of $A$ and a $\pm 1$ vector $x$ such that $\widetilde{A}x = 0$.

# 4 The case $n = 5$

In the previous section the question of existence of a full $n$ by $n$ ray-nonsingular matrix is settled except for the case of $n = 5$. In this section, we describe some structural properties that a full 5 by 5 ray-nonsingular matrix, should one exist, must possess.

Let $A$ be a full 5 by 5 ray–pattern matrix, and assume that $A$ is generic. Consider the 4 by 5 submatrix $B = [\langle 4\rangle, \langle 5\rangle]$. Lemma 3.4 asserts that for every $j \in \langle 5\rangle$, there exist exactly 8 strict $(1, -1)$-signings $D$ such that column $j$ of $DB$ is not balanced. This gives a total of 40 columns among the $DB$'s that are not balanced. As there are only 16 strict $(1, -1)$-signings of order 4, the method used to prove Corollary 3.8 cannot be used to show that $A$ is not ray-nonsingular.

Indeed, let $w_{j,k}$ $(j, k \in \langle 5\rangle)$ be complex numbers of modulus 1 such that

$$0 \le \arg(w_{1,k}) < \arg(w_{2,k}) < \arg(w_{3,k}) < \arg(w_{4,k}) < \arg(w_{5,k}) < \pi,$$

and consider the matrix

$$A = \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & -w_{1,4} & w_{2,5} \\ w_{2,1} & w_{5,2} & w_{2,3} & w_{2,4} & -w_{5,5} \\ w_{3,1} & -w_{4,2} & w_{5,3} & w_{3,4} & w_{3,5} \\ w_{4,1} & w_{2,2} & -w_{4,3} & w_{4,4} & -w_{1,5} \\ w_{5,1} & w_{3,2} & w_{3,3} & -w_{5,4} & w_{4,5} \end{bmatrix}.$$

It can be verified that $A$ is a generic ray–pattern matrix satisfying

> **Property 1:** For each strict $(1, -1)$-signing $D$, there exists a column of $DA$ that is not balanced, and
>
> **Property 2:** For each 4 by 4 submatrix $B$ of $A$, and each strict $(1, -1)$-signing $E$ of order 4, there exists a column of $EB$ that is not strongly balanced.

However, each column of the $A[\{2, 3, 4\}, \{2, 3, 5\}]$ is strongly balanced. Applying Lemma 3.7, we see that a matrix of this type cannot be ray–nonsingular.

More precisely, if a full 5 by 5 ray-nonsingular matrix $A$ exists, Lemma 3.7 implies that it must satisfy Properties 1, 2, and 3, where

> **Property 3:** For each 3 by 3 submatrix $B$ of $A$ and each strict $(1, -1)$-signing $F$ of order 3, there exists a column of $FB$ that is not strongly balanced.

We note that if $A$ is the matrix in (4), then $A$ satisfies Properties 1 and 2, and nearly satisfies Property 3 ($A[\{2, 3, 4\}, \{2, 3, 5\}]$ is the only exception).

We do not know if there exists a full 5 by 5 ray–pattern matrix that satisfies all three of the properties.

As a corollary to Lemma 3.7 we show that a necessary condition for a full 5 by 5 matrix to be ray-nonsingular is that each of its 4 by 5 submatrices has a column that either is not generic or is balanced.

**Corollary 4.1** *Let $A$ be a full 5 by 5 ray–pattern matrix. If $A$ contains a 4 by 5 submatrix each of whose columns is generic and not balanced, then $A$ is not ray-nonsingular.*

**Proof.** Without loss of generality, we may assume that the each of the columns of the matrix obtained from $A$ by deleting its last row is generic and not balanced. Then there exists a strict complex signing $E$ such that each of the entries in the first four rows of $EA$ has positive imaginary part. It follows from Lemma 3.3 that there exists a strict complex signing $D$ such that $DEA$ is generic and each of the entries in the first four rows of $DEA$ has positive imaginary part. Let $B$ be the 4 by 5 matrix obtained from $DEA$ by deleting its last row. Let $S$ be the set of all strict $(1,-1)$-signings of order 4 that have exactly two entries equal to $-1$. Thus, $S$ has 6 elements. Since column $j$ of $B$ is generic and not balanced, there are exactly two $F \in S$ such column $j$ of $FB$ is not balanced ($j \in \langle 5 \rangle$). Thus, $10/6$ is the average number, over all $F \in S$, of columns of $FB$ that are not balanced. It follows that there exists a $F \in S$ such that $FB$ has at most 1 column that is not balanced. Hence $FB$ contains a 4 by 4 submatrix each of whose columns is strongly balanced. Thus, by Lemma 3.7, $A$ is not ray-nonsingular. ■

The 4 by 4 full ray–nonsingular patterns shown in Section 2 consist only of entries from $\pm 1, \pm i$. Next we show that there is no full 5 by 5 ray–nonsingular pattern of this type. To simplify our proof we first define the spread of a vector and provide a lemma.

Let $x$ be a full $m$ by 1 ray–pattern vector. The *spread of* $x$ is defined to be $2\pi$ if $x$ is balanced, and the length (in radians) of the smallest arc of the unit circle that contains the entries of $x$, otherwise.

Given a matrix $A$ each of whose entries is in the set $\{\pm 1, \pm i\}$, we say that $A$ *can be signed* to be the matrix $B$ if there exist strict complex signings $D$ and $E$ each of whose diagonal entries are in $\{\pm 1, \pm i\}$ such that $DAE = B$.

**Lemma 4.2** *Let $A$ be a full 5 by 5 ray–pattern matrix such that the first two columns of $A[\langle 3\rangle, \langle 5\rangle]$ are strongly balanced, and the sum of the spreads of columns 3, 4, and 5 of $A[\langle 3\rangle, \langle 5\rangle]$ is greater than $2\pi$. Then $A$ is not ray-nonsingular.*

**Proof.** Let $D_\theta = \mathrm{diag}(1, 1, 1, e^{i\theta})$ and consider $D_\theta A[\langle 4\rangle, \langle 5\rangle]$. First let $\theta$ range through the angles which strongly balance column 3. If either column 4 or column 5 is also strongly balanced by one of these $\theta$, then by Lemma 3.7 $A$ is not ray-nonsingular. Otherwise, since the sum of the spreads of columns 3, 4, and 5 of $A[\langle 3\rangle, \langle 5\rangle]$ is greater than $2\pi$, there exists a $\theta$ such that $D_\theta A[\langle 4\rangle, \langle 5\rangle]$ has columns 4 and 5 strongly balanced and hence by Lemma 3.7, $A$ is not ray-nonsingular. ■

**Theorem 4.3** *Let $A$ be a full 5 by 5 ray–pattern matrix whose entries consist only of $\pm 1, \pm i$. Then $A$ is not ray–nonsingular.*

**Proof.** Assume to the contrary that $A$ is as prescribed and ray-nonsingular. Thus $A$ satisfies Properties 1, 2 and 3.

First suppose that $A$ has a 2 by 3 submatrix that can be signed to be all 1's. Without loss of generality assume that $A[\langle 2\rangle, \langle 3\rangle]$ is this all 1's submatrix. It is easy to check that if $j = 3, 4, 5$ and $A[\{j\}, \langle 3\rangle]$ contains both a real and an imaginary entry, then the submatrix $A[\{1, 2, j\}, \langle 3\rangle]$ violates Property 3. Thus, each row of $A[\langle 5\rangle, \langle 3\rangle]$ has either all real or all imaginary entries. Without loss of generality, we may assume that $A[\langle 5\rangle, \langle 3\rangle]$ is a real matrix. But then for $j = 3, 4, 5$, $A[\{1, 2, j\}, \langle 3\rangle]$ has a 3 by 2 submatrix that can be signed to have all 1's. Hence applying the previous argument to $A^T$, we can also assume that each column of $A[\{1, 2, j\}, \langle 5\rangle]$ has either all real entries or all imaginary entries, and so $A$ can be signed to be a real matrix. This contradicts the fact that there are no full 5 by 5 sign-nonsingular matrices. Thus we conclude that no submatrix of $A$ can be signed to be the 2 by 3 (or a 3 by 2) matrix of 1's.

Observe that over all strict $(\pm 1, \pm i)$-signings of the first two rows of $A$, there are on average $\frac{2^2 \cdot 2 \cdot 5}{4^2} > 2$ purely real or purely imaginary columns. Thus we may assume, without loss of generality, that

$$A[\langle 2\rangle, \langle 5\rangle] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & c_4 & c_5 \end{bmatrix}.$$

Since no submatrix of $A$ can be signed to be the 3 by 2 submatrix of all 1's, some row, say row 3, of $A[\{3, 4, 5\}, \langle 2\rangle]$ has both a real and an imaginary entry.

Thus, by possibly signing row 3 and conjugating, we may assume that

$$B = A[\langle 3\rangle, \langle 5\rangle] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & c_4 & c_5 \\ 1 & i & d_3 & d_4 & d_5 \end{bmatrix}.$$

Suppose now that $c_4 = c_5$. They cannot both be equal to $-1$, otherwise we could sign the first row of $B$ so that it had a 2 by 3 submatrix of 1's. Suppose $c_4 = c_5 = i$. Letting $D = \text{diag}(1, e^{\frac{5\pi i}{8}}, e^{\frac{17\pi i}{16}})$, columns 1 and 2 of $DB$ are balanced and the spread of the remaining columns is at least $2\pi$; so by Lemma 4.2, $A$ is not ray-nonsingular, contradicting our assumption. This rules out the case $c_4 = c_5 = i$. An analogous argument rules out the case $c_4 = c_5 = -i$. We thus conclude that $c_4 \neq c_5$.

Suppose next that $c_4 = -1$. Then $d_3 \neq d_4$ and $c_5 = \pm i$, otherwise we could sign $B$ so that it had a 3 by 2 or a 2 by 3 submatrix of 1's. Suppose that $c_5 = i$. Letting $D = \text{diag}(1, e^{\frac{5\pi i}{8}}, e^{\frac{17\pi i}{16}})$, columns 1 and 2 of $DB$ are balanced and the spread of the remaining columns is greater than $2\pi$, unless $d_3 = i$ and $d_4 = -1$ (or vice versa) and $d_5 \neq -i$. If $d_5 = i$, then we can sign $B$ so that it has a 2 by 3 submatrix of 1's. Hence $d_5 = \pm 1$. But now, by interchanging rows two and three, rearranging the columns, and possibly signing $B$, we get the case with $c_4 = c_5 = \pm i$, which was ruled out above. The case $c_5 = -i$ can be ruled out similarly.

Thus, the only case left to consider is $c_4 = i$ and $c_5 = -i$. Let $D_1 = \text{diag}(1, e^{\frac{3\pi i}{4}}, e^{\frac{9\pi i}{8}})$ and $D_2 = \text{diag}(1, e^{\frac{5\pi i}{4}}, e^{\frac{3\pi i}{8}})$. Then $D_1B$ and $D_2B$ both have columns 1 and 2 strongly balanced. If $d_4 = \pm i$, then column 4 is also strongly balanced in either $D_1B$ or $D_2B$ and hence these two cases for $d_4$ are ruled out. If $d_5 = \pm 1$, then column 5 is strongly balanced in either $D_1B$ or $D_2B$ and hence we can also rule out these two cases for $d_5$. We can therefore conclude that $d_4 = \pm 1$ and $d_5 = \pm i$. Now consider column 3. If $d_3 \in \{1, -i\}$, the spread of columns 3, 4 and 5 in $D_1B$ is greater than $2\pi$; by Lemma 4.2 this case is ruled out. If $d_3 \notin \{1, -i\}$, the spread of columns 3, 4 and 5 in $D_2B$ is greater than $2\pi$; by Lemma 4.2 this last case is ruled out.

We have ruled out every possible case, and thus we conclude that $A$ is not ray–nonsingular. ■

In our next proposition, we show that if a ray–pattern matrix has one column consisting only of 1's, and the remaining columns are generic, then the matrix cannot be ray–nonsingular.

**Proposition 4.4** *Let $A$ be a full 5 by 5 ray–pattern matrix such that column 1 consists of all 1's and each of the remaining columns is generic. Then $A$ is not ray–nonsingular.*

**Proof:** There are exactly 2 strict $(1,-1)$-signings $E$ such that column 1 of $EA$ is not balanced, and for $k = 2,3,4,5$ exactly 10 strict $(1,-1)$-signings $E$ such that column $k$ of $EA$ is not balanced. Thus, among all $EA$'s there are exactly $2 + 4 \cdot 10 = 42$ columns that are not balanced. Since there are exactly 32 strict $(1,-1)$-signings $E$ of order 5, and for each such signing there exists a column of $EA$ that is not balanced, we conclude that there are at least 22 signings $E$ such that $EA$ has exactly 1 column that is not balanced. Possibly 2 of these signings are $\pm I$. Thus, there are at least 20 signings $E$ such that $EA$ has exactly 1 column that is not balanced and this column is not column 1. It follows that for some $k = 2,3,4,5$, there are at least 5 signings $E$ such that column $k$ of $EA$ is the only column that is not balanced. Since column $k$ is generic, we know the structure of the signings for which column $k$ is not balanced. From this structure it is easy to verify that there exist two signings $E$ and $F$ such that $E$ and $F$ differ in just one entry, and both $EA$ and $FA$ have column $k$ as their only column that is not balanced. Let $\ell$ be the row that $E$ and $F$ differ in. It now follows that there is a strict $(1,-1)$-signing $F$ (namely the one obtained from $E$ by deleting row and column $\ell$) such that columns 2,3,4,5 of $FA(\{\ell\},\{k\})$ are strictly balanced, and column 1 of $FA(\{\ell\},\{k\})$ is weakly balanced. By perturbing $F$ slightly we can obtain a strict complex signing $B'$ such that each column of $B'A(\{\ell\},\{k\})$ is strongly balanced. The result now follows from Lemma 3.7. ■

Proposition 4.4 implies that if a full 5 by 5 ray-nonsingular matrix $A$ exists, then each row and each column of $A$ intersects a 2 by 2 matrix of the form

$$\begin{bmatrix} x & y \\ z & \pm zy/x \end{bmatrix}.$$

This indicates that if a full 5 by 5 ray-nonsingular matrix exists, it has a very specialized structure.

At this stage it is natural to consider the best that we can do so far. Using the following method, we can construct 5 by 5 ray–nonsingular matrices with only three zeros. Let $A$ be a full 4 by 4 ray–nonsingular matrix whose first column is $A_1$. Consider the ray–pattern matrix

$$B = \left[ \begin{array}{ccccc} 1 & -1 & 0 & 0 & 0 \\ & & & & \\ A_1 & & A & & \\ & & & & \end{array} \right]$$

Let $D = \mathrm{diag}(d_1, d_2, d_3, d_4, d_5)$ be a complex signing of order 5. Let $E = \mathrm{diag}(d_2, d_3, d_4, d_5)$. If one of columns 2, 3, 4 of $EA$ is not balanced then the corresponding column of $DB$ is not balanced. If column 1 of $EA$ is not balanced, then either the column 1 or 2 of $DA$ is not balanced. Hence, by Lemma 3.2, $B$ is a ray-nonsingular matrix.

# 5 Regular ray-nonsingular matrices

In this section we consider ray-nonsingular matrices which are *regular* in the sense that each row and column contains the same number of nonzero entries.

We will use the Lovász Local Lemma in the form stated below (see [AES]).

**Lemma 5.1** *Let $E_1, E_2, \ldots, E_n$ be events in an arbitrary probability space. Assume that $p$ and $t$ are numbers with $t \leq n-1$ such that for each $i$ the probability of event $E_i$ is at most $p$ and that the event $E_i$ is mutually independent of a set of at least $n-1-t$ other events $E_j$. If*

$$e\,p\,(t+1) \;<\; 1,$$

*then the probability that none of the events occur is positive.*

**Theorem 5.2** *Let $A$ be an $n$ by $n$ ray–nonsingular matrix with exactly $k$ nonzero entries in each row and column. Then $k \leq 13$.*

**Proof.** By Lemma 3.3 we may assume that $A$ is a generic, ray–pattern matrix. Consider the probability space, $X$, consisting of all strict $(1,-1)$-signings, $D$, each with probability $1/2^n$. Let $E_j$ be the event consisting of all strict $(1,-1)$-signings $D$ such that column $j$ of $DA$ is not balanced. It follows from Lemma 3.4 that

$$\mathrm{Prob}(E_j) = \frac{k2^{n-k+1}}{2^n} = \frac{k}{2^{k-1}}.$$

By Lemma 3.2, $\text{Prob}(\cap_{j=1}^{n}\overline{E_j}) = 0$. Note that the event $E_j$ is mutually independent of the set of events $E_\ell$ such that column $\ell$ and column $j$ of $A$ have no nonzero rows in common, and that there are at least $\max\{0, n-1-(k^2-k)\}$ such events. First suppose that $n-1 > k^2-k$. By Lemma 5.1 (with $p = k/2^{k-1}$ and with $t = k^2-k$),

$$\frac{ek(k^2-k+1)}{2^{k-1}} = ep(t+1) \geq 1$$

implying that $k \leq 13$. Next suppose that $n-1 \leq k^2-k$. By Lemma 5.1 (with $p = k/2^{k-1}$ and with $t = n-1$) ),

$$\frac{ek(k^2-k+1)}{2^{k-1}} \geq \frac{ekn}{2^{k-1}} = ep(t+1) \geq 1,$$

implying that $k \leq 13$. Therefore, $k \leq 13$. ■

Results analogous to Theorem 5.2 for sign-nonsingular matrices are proven in [AL, F].

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