AN IDENTITY FOR THE DETERMINANT

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Abstract

When the directed graph of an n–by–n matrix A does not contain a Hamilton cycle, we exhibit a formula for $detA$ in terms of sums of products of proper principal minors of A. The set of minors involved depends upon the zero/nonzero pattern of A.

1 Introduction

There are known situations in which the determinant of a matrix can be expressed entirely in terms of its proper principal minors. An obvious example is when the matrix is reducible. It is also true when the directed graph (digraph) of the matrix has a critical subdigraph, a concept which generalizes a cut–point (see [MOVW]). More generally, we exhibit such a formula whenever the digraph has no Hamilton cycle; this includes many situations in which the matrix is irreducible. Our formula is qualitative in nature, in that it depends only on the zero/nonzero pattern of the matrix (equivalently, on its digraph). For some patterns, our formula involves rather few proper principal minors. However, it is not intended as a computational tool; our interest is in the existence of such a formula and its implications.

2 The Determinantal Identity

We begin by introducing necessary notation. For a positive integer n, let $\langle n\rangle \equiv \{1,2,\ldots,n\}$. A *partition*, p, of $\langle n\rangle$ is a collection of pairwise disjoint subsets of $\langle n\rangle$, called the *components* of p, whose union is $\langle n\rangle$. The number of components of p is denoted by $|p|$. We denote the set of all partitions of $\langle n\rangle$ by $P_{\langle n\rangle}$. We define on $P_{\langle n\rangle}$ a *partial order*, $\preceq$, as follows; for $p_1,p_2 \in P_{\langle n\rangle},\ p_1 \preceq p_2$ if every component of p_1 is a subset of a component of p_2 (i.e., p_1 is at least as *fine* as p_2). We also define the *intersection* of p_1 and p_2, $p_1 \cap p_2$, as a partition of $\langle n\rangle$ whose components are the intersections of the components of p_1 and p_2. Notice that $p_1 \cap p_2 \preceq p_i,\ i = 1,2$.

Consider now an n–by–n matrix $A = (a_{ij})$. By a *cycle partition* of A we mean an ordered sequence

$$q : i_1,\ldots,i_u;\ j_1,\ldots,j_v;\ldots;\ k_1,\ldots,k_w$$

in which each element of $\langle n\rangle$ appears exactly once and for which the *cyclic product*

$$C(q) = (a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{u-1}i_u}a_{i_ui_1})\ (a_{j_1j_2}a_{j_2j_3}\cdots a_{j_{v-1}j_v}a_{j_vj_1})\ \cdots$$
$$\cdots\ (a_{k_1k_2}a_{k_2k_3}\cdots a_{k_{w-1}k_w}a_{k_wk_1})$$

is nonzero. With q (which is ordered) we identify the (unordered) partition

$$p : \gamma_1 = \{i_1,\ldots,i_u\};\ \gamma_2 = \{j_1,\ldots,j_v\};\ldots;\ \gamma_t = \{k_1,\ldots,k_w\} \qquad (2.1)$$

of $\langle n \rangle$, having components $\gamma_1, \gamma_2, \ldots, \gamma_t$. The *cyclic expansion* of $detA$ is an immediate (and classical, see [MOVW]) fact:

$$detA = \sum (-1)^{n+|p|} C(q), \tag{2.2}$$

where the sum is over all cycle partitions q of A. Now let $P_{\langle n \rangle}(A)$ be the set of all partitions p corresponding to the set of all cycle partitions q of A. If $P_{\langle n \rangle}(A) \neq \emptyset$, we denote the set of all maximal elements (with respect to the partial order defined on $P_{\langle n \rangle}$) of $P_{\langle n \rangle}(A)$ by $P_M(A)$. For $\alpha \subseteq \langle n \rangle$, $A[\alpha]$ denotes the principal submatrix of A lying in rows and columns α. For $p \in P_{\langle n \rangle}(A)$, we define

$$A(p) = \prod_{j=1}^{t} detA[\gamma_j],$$

which is the product of principal minors of A corresponding to the partition p in (2.1).

Our main observation is the following formula.

Theorem. Let A be an n–by–n matrix and let $P_M(A) = \{p_1, p_2, \ldots, p_m\}$. Then

$$detA = \sum_{s=1}^{m} (-1)^{s-1} \sum_{1 \le i_1 < \ldots < i_s \le m} A(p_{i_1} \cap \ldots \cap p_{i_s}). \tag{2.3}$$

Proof.

We prove the theorem by showing that the right hand side of (2.3) is equal to the right hand side of (2.2). We do this by considering the cyclic expansion of (each principal minor in) every summand $A(p_{i_1} \cap \ldots \cap p_{i_s})$ in (2.3).

First take a summand from (2.2) involving some cycle partition q from an (unordered) partition $p \in P_{\langle n \rangle}(A)$:

$$\alpha_q \equiv (-1)^{n+|p|} C(q), \quad p \in P_{\langle n \rangle}(A).$$

By definition of $P_M(A)$, for any such $p \in P_{\langle n \rangle}(A)$ there exists $i \in \{1, 2, \ldots, m\}$ such that $p \preceq p_i$. Let p_i have components γ_j, $j = 1, 2, \ldots, t$, and consider the cyclic expansion of each $detA[\gamma_j]$. Then, on letting k_j denote the number of components of p that are contained in γ_j, we have that

$$\sum_{j=1}^{t} |\gamma_j| = n \quad \text{and} \quad \sum_{j=1}^{t} k_j = |p|,$$

and thus the cyclic expansion of $A(p_i) = \prod_{j=1}^{t} detA[\gamma_j]$ contains the term

$$\left(\prod_{j=1}^{t} (-1)^{|\gamma_j| + k_j} \right) C(q) = (-1)^{n+|p|} C(q) = \alpha_q.$$

This in particular shows that every summand α_q on the right hand side of (2.2) is a term in the cyclic expansion of a summand on the right hand side of (2.3) with $s = 1$.

Conversely, we claim that every term in the cyclic expansion of $A(p_{i_1} \cap \ldots \cap p_{i_s})$ is equal to a term in the cyclic expansion (2.2). For that purpose, let $\delta_1, \delta_2, \ldots, \delta_r$ be the components of the partition $p_{i_1} \cap \ldots \cap p_{i_s}$, recall that

$$A(p_{i_1} \cap \ldots \cap p_{i_s}) = \prod_{j=1}^{r} detA[\delta_j],$$

and consider the cyclic expansion of every $detA[\delta_j]$. The claim then follows from an argument similar to the first part of the proof and the fact that every component δ_j is contained in a component of a partition belonging to $P_M(A)$.

To complete the proof we must show that, after we expand and collect terms on the right hand side of (2.3), the coefficient of each α_q is equal to one. Since $\{p_1, p_2, \ldots, p_m\} = P_M(A)$, there exist $p_{i_1}, p_{i_2}, \ldots, p_{i_k}, \quad k \leq m$, such that

$$p \begin{cases} \preceq p_{i_j} & \text{if } \ j \in \{1, 2, \ldots, k\} \\ \\ \not\preceq p_{i_j} & \text{otherwise.} \end{cases}$$

Observe then that $(-1)^{l-1}\alpha_q$ is a term in the cyclic expansion of $\binom{k}{l}$ summands of (2.3), where $l = 1, 2, \ldots, k$. Therefore, the coefficient of α_q on the right hand side of (2.3) is given by

$$\tbinom{k}{1} - \tbinom{k}{2} + \ldots (-1)^{k-1}\tbinom{k}{k} \ = \ 1,$$

completing the proof of the theorem. $\square$

We remark that if the digraph of A contains a Hamilton cycle, then $P_M(A) = \{\langle n \rangle\}$, and (2.3) reduces to the trivial identity $detA = detA[\langle n \rangle]$. When $P_{\langle n \rangle}(A)$ is nonempty and the digraph of A does not contain a Hamilton cycle, then $P_M(A)$ has cardinality at least one. When $P_{\langle n \rangle}(A)$ is empty, then $detA = 0$.

Many terms in (2.3) may be identically zero. In fact, $A(p_{i_1} \cap \ldots \cap p_{i_s}) = 0$ *unless* $p_{i_1} \cap \ldots \cap p_{i_s} \in P_{\langle n \rangle}(A)$ *or* there is an element of $P_{\langle n \rangle}(A)$ below $p_{i_1} \cap \ldots \cap p_{i_s}$ in the partial order on $P_{\langle n \rangle}$. In particular, if all the diagonal entries of A are nonzero, then $\{\{1\}, \{2\}, \ldots, \{n\}\} \in P_{\langle n \rangle}(A)$ and no term in (2.3) is identically zero. In contrast, if $P_M(A) = P_{\langle n \rangle}(A)$, all terms in (2.3) involving $s > 1$ are zero, so that

$$detA = \sum_{i=1}^{m} A(p_i).$$

Corollary. Let A be an n–by–n complex matrix. If λ is an eigenvalue of at least one principal submatrix in each of the nonzero summands of (2.3), then λ is an

eigenvalue of A.
Proof.
Apply (2.3) to $det(A - \lambda I)$. $\square$
Note that this eigenvalue result is true independently of the values of the nonzero entries of A not contained in the particular set of principal submatrices with common eigenvalue λ. This result generalizes the statement that if A is reducible and in Frobenius normal form, and if λ is an eigenvalue of an irreducible diagonal block, then λ is an eigenvalue of A.

3 Examples

In the following examples we apply the determinantal identity of our theorem to matrices with a specified zero/nonzero pattern.

Example 3.1 Let

$$A = \begin{bmatrix} \times & \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 & 0 \\ \times & 0 & \times & 0 & 0 & \times \\ 0 & 0 & 0 & \times & \times & 0 \\ \times & 0 & 0 & \times & \times & \times \\ \times & 0 & 0 & 0 & 0 & \times \end{bmatrix}.$$

The set $P_M(A)$ consists of the partitions

$$p_1 \ : \ \{1,2,4,5,6\},\ \{3\} \ \text{ and } \ p_2 \ : \ \{1,2,3,6\},\ \{4,5\}.$$

According to the theorem we compute the intersection $p_1 \cap p_2$,

$$p_1 \cap p_2 \ : \ \{1,2,6\},\ \{4,5\},\ \{3\}$$

and the determinant of A can be expressed in terms of its principal minors as follows:

$$\begin{aligned} detA \;=\; & detA[1,2,4,5,6]\ detA[3] + detA[1,2,3,6]\ detA[4,5] \\ & -detA[1,2,6]\ detA[4,5]\ detA[3]. \ \square \end{aligned}$$

Example 3.2 Let A have the same zero/nonzero pattern as in example 3.1 except that the (6,6) entry is set equal to zero. Now $P_M(A) = P_{\langle n \rangle}(A)$, and thus as noted after the theorem,

$$detA = detA[1,2,4,5,6]detA[3] + detA[1,2,3,6]detA[4,5]. \quad \square$$

Example 3.3 Consider a matrix A with the following zero/nonzero pattern:

$$A = \begin{bmatrix} \times & \times & \times & \times & 0 \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & \times & \times \\ \times & 0 & 0 & 0 & \times \end{bmatrix}.$$

Here $P_M(A)$ consists of

$$p_1 \;:\; \{1,2,3\},\; \{4\},\; \{5\} \quad \text{and} \quad p_2 \;:\; \{1,4,5\},\; \{2,3\}.$$

From the theorem we obtain

$$detA = detA[1,2,3]\; detA[4]\; detA[5] + detA[1,4,5]\; detA[2,3]$$
$$-detA[1]\; detA[2,3]\; detA[4]\; detA[5]. \tag{3.1}$$

Applying the corollary, if $\lambda = A[4]$ (or $A[5]$) is an eigenvalue of $A[2,3]$, then λ is also an eigenvalue of A, independently of the values of the other nonzero entries in A (even though A is irreducible).

We remark that the digraph of A contains a cut–point (vertex 1). In this case, a formula for $detA$ in terms of principal minors is known (see [MOVW]), which yields

$$detA = detA[1,2,3]\; detA[4,5] + detA[1,4,5]\; detA[2,3]$$
$$-detA[1]\; detA[2,3]\; detA[4,5].$$

By comparison with (3.1), this cut–point formula does not recognize the fact that for this pattern $detA[4,5] = detA[4]detA[5]$. $\square$

As in the previous example, whenever $|P_M(A)| = 2$, the identity (2.3) expresses $detA$ as a sum of at most three products of principal minors of A. Our next example by contrast illustrates that (2.3) may give a very complicated expression for $detA$, and it also illustrates the possibility of multiple occurrences of a summand in (2.3).

Example 3.4 Let

$$A = \begin{bmatrix} \times & \times & \times & 0 & 0 \\ 0 & \times & \times & 0 & \times \\ \times & 0 & \times & \times & 0 \\ \times & \times & 0 & \times & \times \\ 0 & \times & 0 & \times & \times \end{bmatrix}.$$

$P_M(A)$ has cardinality 6 and is given by

$$p_1 \ : \ \{1,2,3,4\}, \ \{5\}, \quad p_2 \ : \ \{1\}, \ \{2,3,4,5\},$$

$$p_3 \ : \ \{1,2,4,5\}, \ \{3\}, \quad p_4 \ : \ \{1,2,3\}, \ \{4,5\},$$

$$p_5 \ : \ \{2,4,5\}, \ \{1,3\} \ \text{ and } \ p_6 \ : \ \{1,3,4\}, \ \{2,5\}.$$

Applying our theorem, there are $2^6 - 1$ possible summands on the right hand side of (2.3). Many of the intersections of these partitions are repeated (e.g., $p_2 \cap p_3 = p_2 \cap p_5 = p_3 \cap p_5 = p_2 \cap p_3 \cap p_5 : \{1\}, \ \{3\}, \ \{2,4,5\}$). As a consequence, (2.3) reduces to 27 distinct summands. Not all of these summands occur with a coefficient of ± 1. For example, the reduced identity contains a summand of the form

$$-2detA[1] \ detA[3] \ detA[2,4,5]. \qquad \square$$

In conclusion, we have demonstrated via formula (2.3) that when the digraph of A does not contain a Hamilton cycle, $detA$ can always be expressed as a sum of products of principal minors of A. As our examples illustrate, the simplicity of this expression for $detA$ depends on the digraph, and in particular on $P_M(A)$.

References

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