SPECTRAL RADII OF FIXED FROBENIUS NORM PERTURBATIONS OF NONNEGATIVE MATRICES

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Abstract

Let A be an $n \times n$ nonnegative matrix. In this paper we consider the problems of maximizing the spectral radii of (i) $A + X$ and (ii) $A + D$, where X is a real $n \times n$ matrix whose Frobenius norm is restricted to be 1 and where D is as X, but is further constrained to be a diagonal matrix. As for both problems the maximums occur at nonnegative X and D, we use tools of nonnegative matrices, most notably due to Levinger and Fiedler, as well as the Kuhn–Tucker criterion for constrained optimization, to find upper and lower bounds on the maximums, and, when A is additionally assumed to be irreducible, to characterize cases of equalities in these bounds. In the case of the first problem, when A is irreducible, we characterize a matrix which gives the global maximum. A matrix which yields a global maximum to the second problem is more complicated to characterize as, depending on A, the problem admits local maximums within the nonnegative diagonal matrices of Frobenius norm 1.

1 INTRODUCTION

Let A be an $n \times n$ nonnegative matrix. In this paper we consider optimization questions about the spectral radius of $A+X$, where X is a matrix whose **Frobenius norm** is constrained to be 1. After examining the case when X is any such matrix, we turn to the more restricted case of when X is diagonal. To be specific, we shall be concerned here with the following problems:

Maximization Problem I: Let A be an $n \times n$ nonnegative matrix. Then determine

$$\max_{\substack{X \in \mathbf{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A+X). \tag{1.1}$$

Maximization Problem II: Let A be an $n \times n$ nonnegative matrix. Then determine

$$\max_{\substack{D \in \mathbf{R}^{n \times n}, \|D\|_F = 1 \\ D \text{ diagonal}}} \rho(A+D). \tag{1.2}$$

As the sets over which the maximizations are desired in both problems are compact and the spectral radius is a continuous function in the matrix entries, we are guaranteed that both problems have a solution within the restricted sets. Moreover, since $\|X\|_F = \||X|\|_F$ for any $X \in \mathbb{R}^{n,n}$, where $|X|$ is the nonnegative matrix whose entries are the absolute values of the corresponding entries in X, we have from the Perron–Frobenius theory that

$$\rho(A+X) \ \leq \ \rho(A+|X|).$$

Thus the solutions to the the above maximization problems must occur at a nonnegative element whose Frobenius norm is 1 .

In Section 3 we analyze Maximization Problem I and prove a string of results that can be summarized as follows:

THEOREM 1.1 *Let A be an $n \times n$ nonnegative matrix and let*

$$\mu \ := \max_{\substack{X \in \mathbf{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A+X). \tag{1.3}$$

Then

$$\rho(A) + 1 \ \leq \ \mu \ \leq \ \rho\left(\frac{A+A^T}{2}\right) + 1 \tag{1.4}$$

and μ is the unique value in the interval $(\rho(A), \infty)$ for which

$$\left\|(\mu I - A)^{-1}\right\|_2 \;=\; 1. \tag{1.5}$$

Moreover, if A is also irreducible, then μ is attained at a matrix W, $\|W\|_F = 1$, of rank 1. Furthermore, for the case when A is irreducible, equality holds in either of the inequalities in (1.4) if and only if A has a common right and left Perron vector.

In Section 4 we examine the Maximization Problem II. We can summarize our observations there as follows:

THEOREM 1.2 *Let A be an $n \times n$ nonnegative matrix. Set*

$$\nu \;:=\; \max_{\substack{D \in \mathbf{R}^{\mathrm{n\times n}}, \|\mathrm{D}\|_{\mathrm{F}}=1 \\ D \text{ diagonal}}} \rho(A+D). \tag{1.6}$$

Then

$$\rho(A) + \frac{1}{\sqrt{n}} \;\le\; \nu \;\le\; \rho(A) + 1. \tag{1.7}$$

Furthermore, suppose in addition that $n \ge 2$ and that A is irreducible and let $x = (x_1 \ldots x_n)^T$ and $y = (y_1 \ldots y_n)^T$ be, respectively, right and left Perron vectors of A normalized so that $y^T x = 1$. Then

$$\rho(A) + \frac{1}{\sqrt{n}} \;\le\; \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2} \;\le\; \nu < \rho(A) + 1. \tag{1.8}$$

Moreover, equality holds in the inequality

$$\rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2} \;\le\; \nu$$

if and only if

$$\rho(A) + \frac{1}{\sqrt{n}} \;=\; \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2} \;=\; \nu$$

and, under either equivalent conditions, we have that

$$x_i y_i \;=\; \frac{1}{n}, \quad i = 1, \ldots, n. \tag{1.9}$$

Lastly, equality holds in the leftmost inequality in (1.8) if and only if condition (1.9) holds.

Our results in this paper are made possible with the help of three useful tools in analysis and in the theory of nonnegative matrices: The Kuhn–Tucker criterion for constrained optimization, an inequality for the spectral radius of an $n \times n$ nonnegative matrix due to Levinger, and a characterization for the spectral radius of an $n \times n$ nonnegative and irreducible matrix due to Fiedler. For convenience we quote these required results in Section 2, where we also present other essential preliminaries.

We may view Theorems 1.1 and 1.2 as non–local in the sense that they tell us about the maxima of the spectral radii "far away" from A. In Section 2 we also make the simple observation that the steepest ascent of the spectral radius at A occurs in the direction of the transpose of the eigenprojection onto the Perron eigenspace of A.

We comment that in Section 3 it will be argued, using (1.5), that when A is irreducible, the Maximization Problem I has no local maxima in the nonnegative orthant other than, of course, the global maximum.[1] However, Maximization Problem II can have local maxima. We further mention that the results of Theorems 1.1 and 1.2 can be easily modified to solve the respective maximization problems subject to the perturbation matrices X or D having Frobenius norm $t > 0$ rather than 1. All that needs to be done is to replace that matrix A in (1.1) and (1.2) by the matrix A/t.

It should finally be remarked that in two recent papers, Johnson, Stanford, Olesky, and van den Driessche [7] and Johnson, Loewy, Olesky, and van den Driessche [8] consider optimization problems for the spectral radius of $A + D$, where A is an $n \times n$ nonnegative and irreducible matrix, but now D is a diagonal matrix whose **trace** is constrained in some fashion rather than its Frobenius norm.

[1] Note the different sense in which the notion "local" is used now to the sense in which it was used in the previous paragraph.

2 PRELIMINARIES

In this section we shall present some preliminary notations, some known results from the literature, as well as some observations which should aid in the understanding of the paper.

The following basic tenants of the Perron–Frobenius theory are taken from the book of Berman and Plemmons [1]. Let B be an $n \times n$ nonnegative irreducible matrix. The latter condition is equivalent to the existence of **no** permutation matrix P for which

$$PBP^T \;=\; \begin{pmatrix} \tilde{B}_{1,1} & \tilde{B}_{1,2} \\ 0 & \tilde{B}_{2,2} \end{pmatrix},$$

where both diagonal blocks are square and non–vacuous. The spectral radius of B, denoted by $\rho(B)$, is a simple eigenvalue, frequently called the *Perron root of* B, and it possesses right and left Perron eigenvectors having only positive entries. We shall use "$\gg 0$" to denote an array, square or rectangular, whose entries are all positive and we shall use "$\circ$" to denote the entrywise product of two arrays whose dimensions coincide.

As a **function** of any of the matrix entries, the Perron root is strictly increasing at B. Moreover, as it is a **simple** eigenvalue we know, for instance from Wilkinson [11], that $\rho(\cdot)$ is an analytic function in each of the entries in an $\mathbb{R}^{n,n}$–neighborhood of B and hence the partial derivative of $\rho(\cdot)$ with respect to each of the entries exists at B.

Let x and y be positive right and left Perron vectors of B, respectively. Then according to, e.g., Stewart [10, Ex.1, p.305], we have that

$$\frac{\partial \rho(B)}{\partial_{i,j}} \;=\; \frac{y_i x_j}{x^T y}, \quad i, j = 1, \ldots, n. \tag{2.1}$$

Thus, since the eigenprojection onto the Perron eigenspace is given by

$$E \;:=\; \frac{xy^T}{y^T x},$$

it follows at once, relying on the said differentiability of the Perron root at B, that

$$\rho'(B) \;=\; \nabla \rho(B) \;=\; E^T \;=\; \frac{yx^T}{y^T x}. \tag{2.2}$$

With an expression for the gradient of the Perron root at our disposal, we can compute its directional derivatives at B. For that purpose let us recall that the inner product on $\mathbb{R}^{n,n}$ is given by

$$P \cdot Q \;=\; \mathrm{trace}(P^T Q)$$

which induces the matrix norm:

$$\|P\|_F \;=\; \sqrt{\mathrm{trace}(P^T P)}.$$

Now let $X \in \mathbb{R}^{n,n}$ with $\|X\|_F = 1$. Then the directional derivative in the direction of X at B is given by:

$$\rho'_X(B) \;=\; \lim_{t\to 0} \frac{\rho(B+tX)-\rho(B)}{t} \;=\; \nabla\rho(B)\cdot X \;=\; E^T \cdot X = \mathrm{trace}(EX). \tag{2.3}$$

Multivariate calculus tells us now that the steepest rate of increase at B is in the direction of the gradient and hence it is given by:

$$\rho'_{E^T/\|E\|_F}(B) \;=\; \lim_{t\to 0} \frac{\rho\left(B + t\dfrac{E^T}{\|E\|_F}\right) - \rho(B)}{t} \;=\; E^T \cdot \frac{E^T}{\|E\|_F} \;=\; \|E\|_F. \tag{2.4}$$

By the Cauchy–Schwarz inequality we know that

$$\rho'_X(B) \;<\; \rho'_{E^T/\|E\|_F}(B) \tag{2.5}$$

unless for $X \in R^{n,n}$ with $\|X\|_F = 1$, $X = kE$ for some constant k. Thus we can conclude the following about the local behavior of the Perron root in the vicinity of B:

THEOREM 2.1 *Let B be an $n \times n$ nonnegative irreducible matrix and let E be the eigenprojection onto the eigenspace of $\rho(B)$. Then for any $X \in \mathbb{R}^{n\times n}$ with $\|X\|_F = 1$ and $X \neq kE^T$, k a constant, there exists $t_X > 0$ such that for all $0 < t \le t_X$,*

$$\rho\left(B + t\frac{E^T}{\|E\|_F}\right) \;>\; \rho(B+tX). \tag{2.6}$$

Proof: Let X be as prescribed. The conclusion follows at once from (2.3), (2.5), (2.4). □

If we regard for a moment the Perron root as a function of the diagonal entries only, then from (2.1) we see that

$$\rho'(B) \;=\; \nabla\rho(B) \;=\; \frac{1}{y^T x}\begin{pmatrix} x_1y_1 \\ \vdots \\ x_ny_n \end{pmatrix} \;:=\; E_D. \tag{2.7}$$

Moreover, we see that the rate of steepest ascent of the spectral radius as a function of the diagonal entries at B is given by

$$\rho'_{\mathrm{diag}(E_D)/\|\mathrm{diag}(E_D)\|_F}(B) \;=\; \frac{\|x \circ y\|_2}{y^T x}.$$

Now using similar argumentation which led us to Theorem 2.1, we have the following local behavior of the Perron root with regard to diagonal perturbations:

THEOREM 2.2 *Let B be an $n \times n$ nonnegative irreducible matrix. Then for any diagonal matrix $\Delta \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ with $\|\Delta\|_F = 1$ and $\Delta \neq kE_D^T = kE_D$, k a constant, there exists $t_\Delta > 0$ such that for all $0 < t \leq t_\Delta$,*

$$\rho\left(B + t\frac{\mathrm{diag}\,(E_D)}{\|\mathrm{diag}\,(E_D)\,\|_F}\right) \;>\; \rho(B + t\Delta). \tag{2.8}$$

To obtain our goal of proving Theorems 1.1 and 1.2, we shall rely on the following results on the spectral radius of an $n \times n$ nonnegative matrices:

Theorem Levinger ([9]) *Let B be an $n \times n$ nonnegative matrix. Then*

$$\rho(B) \;\leq\; \rho\left(\frac{B + B^T}{2}\right) \tag{2.9}$$

with equality holding in the inequality if and only if B has common right and left Perron vectors.

Theorem Fiedler ([4, Theorem]) *Let B be an irreducible $n \times n$ nonnegative matrix with x and y positive right and left Perron vectors, respectively, viz. $Bx = \rho(B)x$ and $B^T y = \rho(B)y$. Then:*

(a)

$$\rho(B) \;=\; \max_{z \gg 0} \min_{\substack{u,v \gg 0 \\ u \circ v = z}} \frac{v^T B u}{v^T u}. \tag{2.10}$$

(b) *$v^T Bu \geq y^T Bx = \rho(B) y^T x$ whenever $u, v \gg 0$ and $u \circ v = x \circ y$, and equality holds if and only if $u = kx$ for some positive scalar k.*

(c) *In particular, $x^T By \geq y^T Bx = \rho(B) y^T x$, with equality if and only if B has same right and left Perron vectors.*

We mention that while no proof of Theorem Levinger is given in [9], a variety of proofs of his result can be found in the literature. Actually, Theorem Fiedler, whose statement and proof are given in [4, Theorem] comes from Fiedler [2]. For a more recent treatment of both the Levinger and Fiedler theorem see [3].

Finally, as both Problems I and II are ones of **constrained optimization**, we shall make use of the Kuhn–Tucker constrained optimization criteria subject to one equality constrain:

Theorem Kuhn–Tucker ([6, p.81]) Consider the equality constrained maximization problem

$$\begin{cases} \max_{t \in \mathbb{R}^k} F(t) \\ \\ \text{subject to } c(t) = 0. \end{cases} \tag{2.11}$$

Suppose the functions F and c are differentiable in some $\mathbb{R}^k$–neighborhood of the optimal solution t_*. Then

$$\nabla F(t_*) \;=\; \lambda \nabla c(t_*), \tag{2.12}$$

for some real scalar λ.

3 MAXIMIZATION PROBLEM I

The goal of this section is to provide all claims and proofs that are necessary to establish Theorem 1.1.

To get an initial lower estimate on the value of μ in (1.3) is trivial. All we need to do is observe that:

OBSERVATION 3.1 *Let A be an $n \times n$ nonnegative matrix and let x be a right Perron vector of A with $\|x\|_2 = 1$. Then*

$$\rho(A + xx^T) \;=\; \rho(A) + 1.$$

Proof: Observe that $\|xx^T\|_F = 1$ and that

$$(A + xx^T)x \;=\; [\rho(A) + 1]x.$$

$\square$

The above observation implies at once that

$$\max_{\substack{X \in \mathbf{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A + X) \;\geq\; \rho(A) + 1. \tag{3.2}$$

For the symmetric case we can obtain with the aid of (3.2) that:

COROLLARY 3.2 *Let A be an $n \times n$ symmetric nonnegative matrix. Then*

$$\max_{\substack{X \in \mathbf{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A + X) \;=\; \rho(A) + 1. \tag{3.3}$$

Proof: On the one hand we know by (3.2) that

$$\max_{\substack{X \in \mathbf{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A + X) \;\geq\; \rho(A) + 1.$$

On the other hand, we can write that for any nonnegative $X \in \mathbb{R}^{n,n}$ with Frobenius norm 1,

$$\begin{aligned}
\rho(A + X) \;&\leq\; \|A + X\|_2 \;\leq\; \|A\|_2 + \|X\|_2 \\
&\leq\; \rho\left(\frac{A + A^T}{2}\right) + \|X\|_F \;=\; \rho\left(\frac{A + A^T}{2}\right) + 1 \qquad (3.4) \\
&=\; \rho(A) + 1.
\end{aligned}$$

This completes the proof. $\square$

Now if A is **not necessarily symmetric**, then we can use Theorem Levinger (see Section 2) to obtain the following upper estimate:

$$\rho(A+X) \;\leq\; \rho\left(\frac{A+A^T}{2}+\frac{X+X^T}{2}\right).$$

However,

$$\left\|\frac{X+X^T}{2}\right\|_2 \;\leq\; 1$$

so that we can certainly increase elements of $(X+X^T)/2$ if necessary to arrive at a matrix $\tilde{X}$ with $\|\tilde{X}\|_F = 1$, in which case we see from (3.4) that

$$\begin{aligned}\rho(A+X) \quad &\leq \quad \rho\left(\frac{A+A^T}{2}+\frac{X+X^T}{2}\right)\\ &\leq \quad \rho\left(\frac{A+A^T}{2}+\tilde{X}\right) \;\leq\; \rho\left(\frac{A+A^T}{2}\right)+1.\end{aligned}$$

We have thus shown that:

PROPOSITION 3.3 *Let A be an $n\times n$ nonnegative matrix. Then*

$$\max_{X\in\mathbb{R}^{n\times n},\|X\|_F=1} \rho(A+X) \;\leq\; \rho\left(\frac{A+A^T}{2}\right)+1. \tag{3.5}$$

We turn now to the problem of showing that the value of μ given in (1.3) is given by the unique value of μ which satisfies (1.5). To this end, we begin by assuming that A is also irreducible and we characterize a nonnegative matrix W with $\|W\|_F = 1$ which solves (1.1). We can use the remarks preceding Theorem 1.1 to reason as follows. As there exists a matrix W that furnishes a global solution and which is nonnegative and as A is irreducible and nonnegative, at such a W, $A+W$ is nonnegative and irreducible thus securing the differentiability of the Perron root at $A+W$ in each of the matrix entries. This means that we can use the Kuhn–Tucker necessary conditions for the solution of the constrained optimization problem (see Theorem Kuhn–Tucker in Section 2) to make an observation essential to this section:

LEMMA 3.4 *Let A be an $n \times n$ nonnegative and irreducible matrix and let W be a nonnegative global maximizer for* (1.1). *Then $W \gg 0$ and* $\mathrm{rank}(W) = 1$.

Proof: On the one hand we know, by (2.2), that the gradient of $\rho(\cdot)$ at $A + W$ is the eigenprojection onto the Perron eigenspace of $(A + W)^T$ and hence a positive matrix of rank 1. On the other hand, the gradient of $c(X) = \|X\|_F^2 - 1$ at W is $2W$ from which our conclusion simply follows. □

Lemma 3.4 leads us to the following characterization for the maximal value attained in (1.1):

THEOREM 3.5 *Let A be an $n \times n$ nonnegative matrix. If*

$$\mu \;=\; \max_{\substack{X \in \mathbf{R}^{n,n} \\ \|X\|_F = 1}} \rho(A + X), \tag{3.6}$$

then

$$\left\| (\mu I - A)^{-1} \right\|_2 \;=\; 1. \tag{3.7}$$

Proof: Let us begin by assuming that A is (also) irreducible and suppose that W is a maximizing nonnegative solution to (1.1). Then, by Lemma 3.4, W is positive, of rank 1, and Frobenius norm 1 and therefore can be taken without loss of generality to be $W = qp^T$, where $\|q\|_2 = \|p\|_2 = 1$. Moreover, it follows from the arguments in the proof of Lemma 3.4 that W is a multiple of the eigenprojection of $(A + W)^T$ onto its Perron eigenspace and so

$$(A + W)^T W \;=\; \mu W \tag{3.8}$$

so that

$$(A^T + pq^T)q \;=\; \mu q. \tag{3.9}$$

It is easily realized that W^T is the global optimal solution to (1.1) when $A \to A^T$. Thus we also have that

$$(A + qp^T)p \;=\; \mu p. \tag{3.10}$$

But then (3.9) and (3.10) together give that

$$\left(\mu I - A^T\right)^{-1} (\mu I - A)^{-1} q \;=\; q.$$

As both matrices in the product above are positive, 1 is the spectral radius of this product and hence $\left\|(\mu I - A)^{-1}\right\|_2 = 1$.

Suppose now that A is reducible and let J be the $n \times n$ matrix of all 1's. For $\epsilon > 0$, set $A_\epsilon = A + \epsilon J$, and let

$$\mu_\epsilon \;:=\; \max_{\substack{X \in \mathbb{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A_\epsilon + X) \;=\; \rho\left(A_\epsilon + X_\epsilon\right).$$

Then by intial part of the proof,

$$\left\|(\mu_\epsilon I - A)^{-1}\right\|_2 \;=\; 1.$$

Since the set

$$\{X \in \mathbb{R}^{n,n} \mid X \geq 0,\; \|X\|_F = 1\}$$

is compact, there exists a monotonically decreasing sequence $\{\epsilon_i\}_{i=1}^{\infty}$ such that $X_{\epsilon_i} \to \tilde{X}$ with $\|\tilde{X}\|_F = 1$ as $\epsilon_i \to 0$. Note that $A_{\epsilon_i} \to A$ and so, by the continuity of the spectral radius,

$$\mu_{\epsilon_i} \;=\; \rho\left(A_{\epsilon_i} + X_{\epsilon_i}\right) \;\to \rho\left(A + \tilde{X}\right) \;=\; \tilde{\mu}$$

with

$$\left\|(\tilde{\mu} I - A)^{-1}\right\|_2 \;=\; 1.$$

Next, clearly for some $\hat{X} \in \mathbb{R}^{n,n}$ with $\hat{X} \geq 0$ and with $\|\hat{X}\|_F = 1$,

$$\tilde{\mu} \;\leq\; \max_{\substack{X \in \mathbb{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A + X) \;=\; \rho\left(A + \hat{X}\right) \;=\; \mu.$$

On the other hand,

$$\mu_\epsilon \;\geq\; \rho(A_\epsilon + \hat{X})$$

and so

$$\tilde{\mu} \;\geq\; \lim_{i \to \infty} \rho\left(A_{\epsilon_i} + \hat{X}\right) \;=\; \mu.$$

Thus $\mu = \tilde{\mu}$ and $\left\|(\mu I - A)^{-1}\right\|_2 = 1$ and our proof is complete. □

COROLLARY 3.6 *Let A be an $n \times n$ nonnegative matrix. Then*

$$\mu \;:=\; \max_{\substack{X \in \mathbb{R}^{n \times n} \\ \|X\|_F = 1}} \rho(A + X)$$

if and only if

$$\left\|(\mu I - A)^{-1}\right\|_2 \;=\; 1.$$

Since for $\nu > \rho(A)$,

$$\left\|(\nu I - A)^{-1}\right\|_2^2 \;=\; \rho\left\{(\nu I - A)^{-1}\left(\nu I - A^T\right)^{-1}\right\},$$

and due to Neumann's expansion, as a function of ν in the interval $(\rho(A), \infty)$, $\|(\nu I - A)^{-1})\|_2$ is strictly decreasing and hence for only one value of ν, namely $\nu = \mu$, the function assumes the value 1. □

REMARK 3.7 If A is an $n \times n$ irreducible matrix, then on the set of nonnegative X's in $\mathrm{IR}^{\mathrm{n,n}}$ with Frobenius norm 1, $\rho(A + X)$ has no local maxima in addition to its global maximum. This is because, once again, the use of the Kuhn–Tucker conditions as in Lemma 3.4 would give that a local maximum has to occur at a rank 1 element. An analysis such as in the proof of Theorem 3.5 would then imply that the value of the local maximum has to be equal to the only value of μ that can render $\left\|(\mu I - A)^{-1}\right\|_2 = 1$.

As an example for the results of Lemma 3.4 and Theorem 3.5, consider that matrix

$$A \;=\; \begin{bmatrix} 12 & 1 \\ 99 & 2 \end{bmatrix}.$$

Our computations yield that the maximizing matrix is given by the rank 1 matrix

$$W \;\approx\; \begin{bmatrix} 0.1963 & 0.9756 \\ 0.01946 & 0.09675 \end{bmatrix}$$

and

$$\rho(A + W) - \rho(A) \approx 22.0166 - 18.1355 \;=\; 3.8811.$$

We remark that the upper bound $\rho((A + A)^T/2) + 1$ on $\mu = \rho(A + W)$ given on the right–hand side of (1.4) comes to approximately 53.2494. Finally, if we attempt to verify (3.8) for this example, then we get that

$$(A + W)^T W - \rho(A + W)W \;\approx\; 10^{-3}\begin{bmatrix} -0.09520 & -0.4730 \\ 0.01910 & 0.09520 \end{bmatrix}.$$

We computed W above using a function routine in conjunction with the MATLAB Optimization Toolbox. In our function, the search was done only over the rank 1 nonnegative matrices of Frobenius norm 1. However, to confirm the results, as well as the results for other examples, we used a version of the SQP method, see Fletcher [5], devised by M. J. D. Powell, FRS. We implemented the algorithm on an IBM mainframe with single precision arithmetic. This usually gave nonsingular solutions W to our various example. A quick check showed that the computed W's had a high condition number.

In our next results we shall characterize precisely when equalities hold in (3.2) and (3.5) for the case when A is irreducible:

THEOREM 3.8 *Let A be an $n \times n$ nonnegative and irreducible matrix. Then equality holds in* (3.2) *if and only if A has a common left and right eigenvector.*

Proof: Since A is irreducible, there exist unique n–vectors $x \gg 0$ and $y \gg 0$ with $\|x\|_2 = \|y\|_2 = 1$ such that $Ax = \rho(A)x$ and $A^T y = \rho(A)y$. Suppose that $\mu = \rho(A) + 1$. Then

$$(\mu I - A)^{-1} x \; = \; x \tag{3.11}$$

and

$$\left(\mu I - A^T\right)^{-1} y \; = \; y. \tag{3.12}$$

Now from the proof of Theorem 3.5, there exists a unique n–vector $z \gg 0$ with $\|z\|_2 = 1$ such that

$$\left(\mu I - A^T\right)^{-1} (\mu I - A)^{-1} z \; = \; z. \tag{3.13}$$

But then from (3.12) and (3.13) and the irreducibility of $\left(\mu I - A^T\right)^{-1}$ and $\left(\mu I - A^T\right)^{-1} (\mu I - A)^{-1}$ we get that $y = z$ and $y = (\mu I - A)^{-1} z$. Whence $y = (\mu I - A)^{-1} y$ so that, by (3.11), $x = y$ from which we conclude that A has a common left and right Perron vector.

Conversely, if $x = y$, then

$$\left((1 + \rho(A))I - A^T\right)^{-1} ((1 + \rho(A)I - A)^{-1} x \; = \; x.$$

Hence

$$\|(1 + \rho(A)I - A)^{-1}\|_2 \; = \; 1$$

which implies that $\mu = \rho(A) + 1$. $\square$

THEOREM 3.9 *Let A be an $n \times n$ nonnegative and irreducible matrix. Then equality holds in* (3.5) *if and only if A has a common right and left Perron eigenvector.*

Proof: Let W be a nonnegative global maximizer for the maximization problem (1.1). Then by Theorem Levinger inequality and by inequality (3.5) as applied to the matrix $(A + A^T)/2$ and because $\|(W + W^T)/2\|_F \leq 1$, we can write that

$$\rho(A + W) \ \leq \ \rho\left(\frac{A + A^T}{2} + \frac{W + W^T}{2}\right) \ \leq \ \rho\left(\frac{A + A^T}{2}\right) + 1. \qquad (3.14)$$

But then, by the hypothesis of the theorem,

$$\rho(A + W) \ = \ \rho\left(\frac{A + A^T}{2} + \frac{W + W^T}{2}\right). \qquad (3.15)$$

It now follows by the equality case in Levinger's inequality that $A + W$ has a common right and left Perron vector. Now according to our characterization of the maximizing element using the Kuhn–Tucker condition, $W = qp^T$, with $(A + W)p = \rho(A + W)p$ and $(A^T + W^T)q = \rho(A + W)q$, so that we must have that $p = q$. But then, using again the fact that $W = qp^T$, it easily follows that p (or $q = p$) has to be concurrently a right and left Perron vector of A.

Conversely, if A has a common right and left Perron vector, then by Levinger's inequality, $\rho(A) = \rho((A + A^T)/2)$. That equality holds in (3.5) now follows from Theorem 3.8 and (3.5). $\square$

4 DIAGONAL PERTURBATIONS

In this section we consider the solution to Maximization Problem II. We begin by establishing the results which give the lower bounds in (1.8) in the irreducible case.

LEMMA 4.1 *Let A be an $n \times n$ nonnegative and irreducible matrix and suppose that $x = (x_1 \ldots x_n)^T$ and $y = (y_1 \ldots y_n)^T$ are, respectively, right and left Perron vectors of A normalized so that $y^T x = 1$. Then*

$$\rho(A) + \frac{1}{\sqrt{n}} \ \leq \ \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2} \ \leq \ \nu. \tag{4.1}$$

Moreover, for equality to hold in the rightmost inequality it is necessary and sufficient that equalities hold throughout (4.1)*, that is, $\nu = \rho(A) + 1/\sqrt{n}$, and, under either equivalent conditions,*

$$x_i y_i \ = \ \frac{1}{n}, \quad i = 1, \ldots, n. \tag{4.2}$$

Lastly, equality holds in the leftmost inequality in (4.1) *if and only if* (4.2) *holds.*

Proof: We begin by showing the validity of the rightmost inequality in (4.1). For that purpose let $z = x \circ y$ and let $D = \mathrm{diag}\left(\sqrt{z_1}, \ldots, \sqrt{z_n}\right)$ so that D is a diagonal matrix whose Frobenius norm is 1. Then for any two positive vectors $\xi = (\xi_1 \ldots \xi_n)^T$ and $\eta = (\eta_1 \ldots \eta_n)^T$ with $\xi_i \eta_i = z_i$, $i = 1, \ldots, n$, we have, by Theorem Fiedler part (b) of Section 2 as applied to the matrix A, that

$$\xi^T (A + D) \eta \ \geq \ \rho(A) + \sum_{i=1}^{n} \xi_i \left(\sqrt{z_i}\right) \eta_i \ = \ \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2}. \tag{4.3}$$

Next, the proof of Theorem Fiedler given in [4], as applied now to the matrix $A + D$, assures us that for the vector $z \gg 0$, there exist positive vectors $\tilde{\xi}$ and $\tilde{\eta}$ such that $\tilde{\xi} \circ \tilde{\eta} = z$ and such that

$$\tilde{\xi}^T (A + D) \tilde{\eta} \ = \ \min_{\substack{\xi \circ \eta = z \\ \xi \gg 0, \eta \gg 0}} \xi^T (A + D) \eta. \tag{4.4}$$

Thus, from the maximization that occurs in Theorem Fiedler part (a) when applied to the matrix $A + D$ and by (4.3) we can write that:

$$\nu \ \geq \ \rho(A + D) \ \geq \ \tilde{\xi}^T (A + D) \tilde{\eta} \geq \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2}. \tag{4.5}$$

We have thus proved that the rightmost inequality in (4.1) holds.

Suppose now that equality holds in the rightmost inequality in (4.1) so that

$$\nu \;=\; \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2}.$$

Then, by (4.4) and (4.5), we have that

$$\nu \;=\; \tilde{\xi}^T (A+D)\tilde{\eta} \;=\; \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2}. \tag{4.6}$$

As $\tilde{\xi}^T A\tilde{\eta} \geq \rho(A)$, we must actually have that $\tilde{\xi}^T A\tilde{\eta} = \rho(A)$. But then by part (b) of Theorem Fiedler, now as applied to the matrix A, it follows that $\tilde{\eta} = kx$ for some positive scalar k and therefore also $\tilde{\xi} = (1/k)y$. Without loss of generality we can assume that $k = 1$. Hence, by (4.6) and part (b) of Theorem Fiedler as applied again to the matrix $A + D$, it follows that

$$(A+D)\tilde{\eta} \;=\; \nu\tilde{\eta} \;\text{ and }\; (A^T + D)\tilde{\xi} \;=\; \nu\tilde{\xi}.$$

This means that x and y are also right and left Perron eigenvectors of $A+D$, respectively, and so it holds that

$$Dx \;=\; (\nu - \rho(A))x \;\text{ and }\; Dy \;=\; (\nu - \rho(A))y.$$

As the diagonal entries of D are $\sqrt{x_i y_i}$, $i = 1, \ldots, n$, the above relations easily yield that

$$x_i y_i \;=\; (\nu - \rho(A))^2, \quad i = 1, \ldots, n. \tag{4.7}$$

Whence

$$\sum_{i=1}^{n} (x_i y_i)^{3/2} \;=\; n(\nu - \rho(A))^3 \tag{4.8}$$

which easily yields, using (4.6), that $\nu - \rho(A) = 1/\sqrt{n}$. This shows that equality must hold also in the leftmost inequality in (4.1). Observe too that (4.7) validates (4.2) under the present assumptions.

Let us now show the leftmost inequality in (4.1). Consider the problem of minimizing the function $\sum_{i=1}^{n} u_i^{3/2}$ subject to the conditions the $u_i > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} u_i = 1$. The method of the Lagrange multipliers easily gives that we must have the vector equality $(u_1^{1/2} \ldots u_n^{1/2})^T = \lambda(1 \ldots 1)^T$ for some nonzero scalar λ so that $u_1 = \ldots = u_n$. As $\sum_{i=1}^{n} u_i = 1$, it is necessary that $u_i = 1/n$, $i = 1, \ldots, n$ and so the leftmost inequality in (4.1) holds.

That (4.2) is a necessary and sufficient condition for equality to hold in the leftmost equality of (4.1) is an immediate consequence of the arguments in the preceding paragraph.

Finally, if $\nu = \rho(A) + 1/\sqrt{n}$, then by (4.3) and the last paragraph but one, we have that

$$\nu \;=\; \rho(A) + \sum_{i=1}^{n} (x_i y_i)^{3/2}$$

which implies, again by using the preceding paragraph, that $x_i y_i = 1/n$, $i = 1, \ldots, n$. □

We remark that the lower bound

$$\nu \;\geq\; \rho(A) + \frac{1}{\sqrt{n}}$$

holds also when A is reducible by obvious continuity arguments.

We next provide two examples to illustrate the behavior of some matrices in the presence of equalities in some or all of the inequalities of (4.1). We begin with the case when $\nu = \rho(A) + (1/\sqrt{n})$, i.e., equalities hold throughout (4.1). This can occur without A being either symmetric or doubly stochastic. To see this let

$$A_1 \;=\; \begin{bmatrix} 2.0 & 1.0 \\ 2.0 & 2.0 \end{bmatrix}.$$

On computing right and left eigenvectors of A_1, say $x = (0.6124\ 0.8660)^T$ and $y = (0.8165\ 0.5774)^T$, we see $x \circ y = (1/2\ 1/2)^T$. In our second example we show that equality in the leftmost inequality of (4.1) **does not** imply equality also in the rightmost inequality of (4.1). Take

$$A_2 \;=\; \begin{bmatrix} 0 & 0 & 0 & 0.6124 & 0.2494 & 0.1381 \\ 0 & 0 & 0 & 0.2494 & 0.3685 & 0.3821 \\ 0 & 0 & 0 & 0.1381 & 0.3821 & 0.4797 \\ 0.6124 & 0.2494 & 0.1381 & 0 & 0 & 0 \\ 0.2494 & 0.3685 & 0.3821 & 0 & 0 & 0 \\ 0.1381 & 0.3821 & 0.4797 & 0 & 0 & 0 \end{bmatrix}. \tag{4.9}$$

Then A_2 is symmetric, doubly stochastic, and even has a constant diagonal, but $\nu > \rho(A_2) + 1/\sqrt{6}$ as can be seen by selecting

$$D \;=\; \mathrm{diag}\,(0.07869, 0.1312, 0.8979, 0.08612, 0.2206, 0.3382)$$

for which $\|D\|_F = 1$ and $\rho(A + D) - \rho(A) = 0.4245 > 1/\sqrt{6}$.

Obtaining the upper bound in Theorem 1.2 is simple.

LEMMA 4.2 *Let A be an $n \times n$, $n \geq 2$, nonnegative matrix. Then, in the notation of Theorem 1.2, $\nu \leq \rho(A) + 1$. If, in addition, A is irreducible, then strict inequality holds.*

Proof: Recall that

$$\nu \;:=\; \max_{\substack{D \in \mathbf{R}^{n \times n}, \|D\|_F = 1 \\ D \text{ diagonal}}} \rho(A + D). \tag{4.10}$$

Without loss of generality, we can assume that $D \geq 0$. As $D \leq I$, we have from the Perron-Frobenius theory that

$$\rho(A + D) \;\leq\; \rho(A + I) \;=\; 1 + \rho(A).$$

As $n \geq 2$, we must have that $D \neq I$ and hence strict inequality when A is irreducible.

□

Two comments are in order concerning our findings in this section:

(i) Consider for a moment the doubly stochastic and symmetric matrix A in (4.9) for which we showed that $\nu \geq 1.4245$. By Theorem Kuhn–Tucker of Section 2 and by the preceding lemma, we know that there exists a positive diagonal matrix W, $\|W\|_F = 1$, for which

$$(A + W)\, p \;=\; \nu p \;\text{ and }\; \left(A^T + W\right) q \;=\; \nu q.$$

for some positive vectors n–vectors p and q. On the other hand we know that the vector $e = (1, \ldots, 1)^T$ is both a right and left Perron eigenvector of A. Moreover, e is, in fact, both a left and right Perron vector of the matrix

$A + \text{diag}(e \circ e) := A + E$. Thus, for the Perron root viewed as a function of the diagonal entries, we observe via (2.7), that

$$\rho'(A+E) \;=\; \nabla\rho(A+E) \;=\; k_1(e \circ e) \;=\; k_1 e$$

for some positive constant k. Now as $2e$ is the gradient of the function $\|x\|_2^2 - 1$ at $x = e$, we see that not only W satisfies the necessary condition (2.11) of Theorem Kuhn–Tucker, but also the matrix E. This suggests that unlike Maximization Problem I for which we showed that in the nonnegative orthant it has no local maxima other than the global maximum, Maximization Problem II can have local maxima.

(ii) In Lemma 4.2 we show that for $n \geq 2$, the solution to Maximization Problem II always satisfies that $\nu < \rho(A)+1$ when A is irreducible, but that when A is reducible, equality can hold. As can be expected, we can show that there are irreducible A's for which ν is arbitrarily close to $\rho(A)+1$. This is born out by analyzing nonnegative diagonal perturbations of Frobenius norm 1 in the example

$$A(\epsilon) \;=\; \begin{pmatrix} 1 & \epsilon \\ \epsilon & \epsilon \end{pmatrix}, \quad \epsilon > 0,$$

and on letting $\epsilon \to 0+$.

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References

[1] A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. *SIAM Publications*, Philadelphia, 1994.

[2] M. Fiedler. An alternative proof of Levinger's theorem. *Unpublished notes*, 1970.

[3] M. Fiedler. Numerical range of matrices and Levinger's theorem. *Lin. Alg. Appl.*, 220:171–180, 1995.

[4] M. Fiedler, C. R. Johnson, T. L. Markham, and M. Neumann. A trace inequality for M–matrices and symmetrizability of a real matrix by a positive diagonal matrix. *Lin. Alg. Appl.*, 71:81–94, 1985.

[5] R. Fletcher. Practical Methods of Optimization, 2nd Edition. *Wiley Publication*, New York, 1987.

[6] P. E. Gill, W. Murray, and M. H. Wright. Practical Optimization. *Academic Press*, New York, 1981.

[7] C. R. Johnson, D. Stanford, D. D. Olesky, and P. van den Driessche. Dominant eigenvalues under trace–preserving diagonal perturbations. *Lin. Alg. Appl.*, 212/213:415–435, 1994.

[8] C. R. Johnson, R. Loewy, D. D. Olesky, and P. van den Driessche. Maximizing the spectral radius of fixed trace diagonal perturbations of nonnegative matrices. *Lin. Alg. Appl.*, 241–243;635–654, 1996.

[9] B. W. Levinger. An inequality for nonnegative matrices. *Notices Amer. Math. Soc.*, 17:260, 1970.

[10] G. W. Stewart. Introduction to Matrix Computations. *Academic Press*, New York, 1973.

[11] J. H. Wilkinson. The Algebraic Eigenvalue Problem. *Oxford Univ. Press*, London, 1965.