GENERALIZATIONS OF DIAGONAL DOMINANCE IN MATRIX THEORY

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Abstract

A matrix $A \in \mathbf{C}^{n,n}$ is called generalized diagonally dominant or, more commonly, an $H-$matrix if there is a positive vector $x = (x_1, \cdots, x_n)^t$ such that

$$|a_{ii}|x_i > \sum_{j \neq i} |a_{ij}|x_j, \quad i = 1, 2, \cdots, n.$$

In this thesis, we first give an efficient iterative algorithm to calculate the vector x for a given H-matrix, and show that this algorithm can be used effectively as a criterion for H-matrices. When A is an H-matrix, this algorithm determines a positive diagonal matrix D such that AD is strictly (row) diagonally dominant; its failure to produce such a matrix D signifies that A is not an H-matrix. Subsequently, we consider the class of doubly diagonally dominant matrices (abbreviated d.d.d.). We give necessary and sufficient conditions for a d.d.d. matrix to be an H-matrix. We show that the Schur complements of a d.d.d matrix are also d.d.d. matrices, which can be viewed as a natural extension of the corresponding result on diagonally dominant matrices. Lastly, we obtain some results on the numerical stability of incomplete block LU-factorizations of H-matrices and answer a question posed in the literature.

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Contents

Chapter 1

INTRODUCTION

As is well-known, diagonal dominance of matrices arises in various applications (cf [29]) and plays an important role in the mathematical sciences, especially in numerical linear algebra. There are many generalizations of this concept. The most well-studied generalization of a diagonal dominant matrix is the so called H-matrix.

In the present work, we concentrate on new criteria and algorithms for H-matrices. We also consider a further generalization of diagonal dominance, called double diagonal dominance.

1.1 Basic Definitions and Notation

Throughout this thesis, we will use the notation introduced in this section. Given a positive integer n, let $\langle n \rangle = \{1, \cdots, n\}$. Let $\mathbf{C}^{n,n}$ denote the collection of all $n \times n$ complex matrices and let $\mathbf{Z}^{n,n}$ denote the collection of all $n \times n$ real matrices $A = [a_{ij}]$ with $a_{ij} \leq 0$ for all distinct $i, j \in \langle n \rangle$. Let $A = [a_{ij}] \in \mathbf{C}^{n,n}$. We denote by $\sigma(A)$ the *spectrum* of A, namely, the set of all eigenvalues of A. The *spectral radius* of A, $\rho(A)$, is defined by $\rho(A) = max\{|\lambda| : \lambda \in \sigma(A)\}$.

We write $A \geq 0$ $(A > 0)$ if $a_{ij} \geq 0$ $(a_{ij} > 0)$ for $i, j \in \langle n \rangle$. We also write $A \geq B$ if $A - B \geq 0$. We call $A \geq 0$ a *nonnegative matrix.* Similar notation will be used for vectors in $\mathbf{C}^n$.

Also, we define $R_i(A) = \sum_{k \neq i} |a_{ik}|$ $(i \in \langle n \rangle)$ and denote $|A| = [|a_{ij}|]$.

We will next introduce various types of diagonal dominance, and some related concepts and terminology.

1.2 Diagonal Dominance and Double Diagonal Dominance

A matrix P is called a *permutation matrix* if it is obtained by permuting rows and columns of the identity matrix. A matrix $A \in \mathbf{C}^{n,n}$ is called *reducible* if either

(i) $n = 1$ and $A = 0$; or
(ii) there is a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square and non-vacuous. If a matrix is not reducible, then we say that it is *irreducible*. An equivalent definition of irreducibility, using the directed graph of a matrix, will be given in Chapter 3.

We now recall that A is called (row) *diagonally dominant* if

$$|a_{ii}| \geq R_i(A) \quad (i \in \langle n \rangle). \tag{1.2.1}$$

If the inequality in (1.2.1) is strict for all $i \in \langle n \rangle$, we say that A is *strictly diagonally dominant*. We say that A is *irreducibly diagonally dominant* if A is irreducible and at least one of the inequalities in (1.2.1) holds strictly.

Now we can introduce the definitions pertaining to double diagonal dominance.

Definition 1.2.1 ([26]) The matrix $A \in \mathbf{C}^{n,n}$ is *doubly diagonally dominant* (we write $A \in \mathbf{G}^{n,n}$) if

$$|a_{ii}||a_{jj}| \geq R_i(A)R_j(A), \quad i, j \in \langle n \rangle, \; i \neq j. \tag{1.2.2}$$

If the inequality in (1.2.2) is strict for all distinct $i, j \in \langle n \rangle$, we call A *strictly doubly diagonally dominant* (we write $A \in \mathbf{G}_1^{n,n}$). If A is an irreducible matrix that satisfies (1.2.2) and if at least one of the inequalities in (1.2.2) holds strictly, we call A *irreducibly doubly diagonally dominant* (we write $A \in \mathbf{G}_2^{n,n}$).

We note that double diagonal dominance is referred to as bidiagonal dominance in [26].

1.3 Generalized Diagonal Dominance and H-matrices

We will next be concerned with the concept of an H-matrix, which originates from Ostrowski (cf [30]). We first need some more preliminary notions and notation.

The *comparison matrix* of $A = [a_{ij}]$, denoted by $\mathcal{M}(A) = [\alpha_{ij}] \in \mathbf{C}^{n,n}$, is defined by

$$\alpha_{ij} = \begin{cases} |a_{ii}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

If $A \in \mathbf{Z}^{n,n}$, then A is called an (resp., a nonsingular)*M-matrix* provided that it can be expressed in the form $A = sI - B$, where B is a nonnegative matrix and

$s \geq \rho(B)$(resp., $s > \rho(B)$). The matrix A is called an *H-matrix* if $\mathcal{M}(A)$ is a nonsingular M-matrix. We denote by $\mathcal{H}_n$ the set of all $H-$matrices of order n.

James and Riha (cf [19]) defined $A = [a_{ij}] \in \mathbf{C}^{n,n}$ to have *generalized (row) diagonal dominance* if there exists an entrywise positive vector $x = [x_k] \in \mathbf{C}^n$ such that

$$|a_{ii}|x_i > \sum_{k \neq i} |a_{ik}|x_k \quad (i \in \langle n \rangle). \tag{1.3.3}$$

This notion obviously generalizes the notion of (row) *strict diagonal dominance*, in which $x = e$ (i.e., the all ones vector). In fact, if A satisfies (1.3.3) and if $D = \text{diag}(x)$ (i.e., the diagonal matrix whose diagonal entries are the entries of x in their natural order), it follows that AD is a strictly diagonally dominant matrix or, equivalently, that $\mathcal{M}(A)x > 0$. As we will shortly claim (in Theorem 1.3.1), the latter inequality is equivalent to $\mathcal{M}(A)$ being a nonsingular M-matrix and thus equivalent to A being an H-matrix.

Since James and Riha published their paper [19] in 1974, numerous papers in numerical linear algebra, regarding iterative solutions of large linear systems, have appeared (see [2],[5], [22]-[25]). In these papers, several characterizations of H-matrices were obtained, mainly in terms of convergence of iterative schemes.

For a detailed analysis of the properties of M-matrices and H-matrices and related material one can refer to Berman and Plemmons [6], and Horn and Johnson [17]. Here we only collect some conditions that will be frequently used in later chapters.

Theorem 1.3.1 *Let $A = [a_{ij}] \in \mathbf{C}^{n,n}$. Then the following are equivalent.*

(i) *A is an H-matrix.*

(ii) *A is a generalized diagonally dominant matrix.*

(iii) $\mathcal{M}(A)^{-1} \geq 0$.

(iv) *$\mathcal{M}(A)$ is a nonsingular M-matrix.*

(v) *There is a vector $x \in \mathbf{R}^n$ with $x > 0$ such that $\mathcal{M}(A)x > 0$. Equivalently, letting $D = diag(x)$, AD is strictly diagonally dominant.*

(vi) *There exist upper and lower triangular nonsingular M-matrices L and U such that $\mathcal{M}(A) = LU$.*

(vii) *Let $D = diag(A)$. Then $\rho(|I - D^{-1}A|) < 1$.*

Proof. Details of the proof can be found in [6] and [30]. Here we shall prove that (iii) and (vii) are equivalent, for the benefit of better understanding the various notions.

Define $D = diag(A)$ and $D_1 = diag(\mathcal{M}(A))$. Assume that $\mathcal{M}(A)^{-1} \geq 0$. Let $\mathcal{M}(A)^{-1} = [r_{ij}]$. As $\mathcal{M}(A)$ has nonpositive off-diagonal entries,

$$r_{ii}|a_{ii}| - \sum_{j\neq i} r_{ij}|a_{ji}| = 1, \quad 1 \leq i \leq n.$$

Thus $M(A)^{-1} \geq 0$ implies that all diagonal entries of $\mathcal{M}(A)$ are positive, i.e., $D_1 > 0$. It follows that $B = I - D_1^{-1}\mathcal{M}(A) \geq 0$. Moreover, as $I - B = D_1^{-1}\mathcal{M}(A)$, $I - B$ is nonsingular and

$$(I - B)^{-1} = \mathcal{M}(A)^{-1}D_1 \geq 0.$$

As $B \geq 0$, there is an eigenvector $x \geq 0$ such that $Bx = \rho(B)x$. Thus

$$(I - B)^{-1}x = \frac{1}{1 - \rho(B)}x \geq 0.$$

This implies $\rho(B) < 1$. Notice that $\rho(|I - D_1^{-1}\mathcal{M}(A)|) = \rho(B) = \rho(|I - D^{-1}A|)$. Hence $\rho(|I - D^{-1}A|) < 1$.
Conversely, assume that $\rho(|I - D^{-1}A|) < 1$. Then

$$\rho(|I - D_1^{-1}\mathcal{M}(A)|) = \rho(|I - D^{-1}A|) < 1.$$

Let $B = I - D_1^{-1}\mathcal{M}(A)$. Then $B \geq 0$, $\rho(B) < 1$ and hence

$$\mathcal{M}(A)^{-1}D_1 = (I - B)^{-1} = I + B + B^2 + \cdots \geq 0.$$

This implies $\mathcal{M}(A)^{-1} \geq 0$. ■

Remark 1.3.2 If A is an H-matrix, then A is nonsingular: Since by Theorem 1.3.1(v) AD is strictly diagonally dominant, it follows from the Lévy-Desplanques theorem (see [16]) that AD is nonsingular and thus so is A. [1]

Remark 1.3.3 We emphasize that from the above theorem it follows that the notions of an H-matrix and a generalized diagonally dominant matrix are equivalent. Also, if we let $\mathcal{D}_A$ denote the set of all positive diagonal matrices D satisfying Theorem 1.3.1(v), we have that A is an H-matrix iff $\mathcal{D}_A$ is not empty.

[1] We caution the reader that some authors define an H-matrix by what amounts to requiring that the eigenvalues of $\mathcal{M}(A)$ have nonnegative real parts, thus allowing for singular "H-matrices".

1.4 Incomplete Block (Point) LU-factorizations

Consider the solution of the linear system

$$Ax = b, \tag{1.4.4}$$

where $A \in \mathbf{C}^{n,n}$ is a nonsingular large sparse matrix, by iterative schemes of the form

$$Bx^{(m+1)} = Cx^{(m)} + b, \quad m = 0, 1, 2, \ldots, \tag{1.4.5}$$

where B is nonsingular and $A = B - C$. $A = B - C$ is usually referred to as a *splitting* of A. The aim is to find B and C so that the iterates in (1.4.5) are easily computable and converge to a solution of (1.4.4). For example, if we choose $B = diag(A)$ and $C = B - A$, then the iterative method defined above becomes the classical Jacobi method, which converges to a solution if and only if B is invertible and $\rho(B^{-1}C) < 1$. We usually implement (1.4.5) via an equivalent two-step process, namely,

$$\begin{cases} Bl^{(m)} = b - Ax^{(m)} \\ x^{(m+1)} = x^{(m)} + l^{(m)}. \end{cases} \tag{1.4.6}$$

In each iteration of (1.4.6), we solve for $l^{(m)}$ from the first equation and update the iterate by the second equation. It is desirable that the matrix B, called a *preconditioning matrix* of A, is sparse, that is, it has as few nonzero entries as possible. An "incomplete block LU-factorization" of A can be used for efficiently finding such a preconditioning matrix (cf [2], [5] and [13]). This method is a generalization of the "elemental incomplete (point) LU-factorization" (which in turn is based on Gaussian elimination).

To describe the incomplete block LU-factorization of a matrix, we first need to formally define the notions of block triangular and block diagonal matrices. For that purpose, let $A \in \mathbf{C}^{n,n}$ be partitioned into block matrix form as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}. \tag{1.4.7}$$

Here $A_{ij} \in \mathbf{C}^{n_i,n_j}$, $1 \le n_i,\ n_j \le n$ and $\sum n_i = \sum n_j = n$. If $A_{ij} = 0$ for all $i < j$ (resp., $i > j$), then we say A is a *lower (resp., an upper) block triangular matrix.* Similarly, if $A_{ij} = 0$ for all $i \neq j$, we say that A is a *block diagonal matrix* and write $A = diag(A_{11}, \ldots, A_{mm})$.

Let $\alpha^{n,n}$ be the set of all (0,1) matrices with all diagonal entries equal to one (a (0,1) matrix means all of its entries are 0's and 1's). Also denote by $A * B$ the

Hadamard product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same dimensions, defined by

$$A * B = [a_{ij} b_{ij}], \quad i, j \in \langle n \rangle.$$

Now let $A = [a_{ij}]$ and $\alpha \in \alpha^{n,n}$ be partitioned conformally, as in (1.4.7). Consider the following algorithm.

Algorithm 1.1

`Set`

$$\begin{cases} A_0 = A \\ \texttt{for } r = 1, 2, \cdots, m-1 \\ \tilde{A}_r = \alpha * A_{r-1} \\ A_r = L_r \tilde{A}_r \end{cases}$$

`where, if` $A_r = [A_{ij}^{(r)}], \tilde{A}_r = [\tilde{A}_{ij}^{(r)}]$`, then` $L_r = [L_{ij}^{(r)}]$ `is defined by`

$$\begin{cases} L_{ir}^{(r)} & = & -\tilde{A}_{ir}^{(r)} (\tilde{A}_{rr}^{(r)})^{-1} & \texttt{for } i = r+1, \cdots, m \\ L_{ii}^{(r)} & = & I_{n_i} & \texttt{for } i \in \langle m \rangle \\ L_{ij}^{(r)} & = & 0 & \texttt{otherwise.} \end{cases}$$

`At step` $m-1$`, set`

$$\begin{cases} U & = & A_{m-1} \\ L & = & (\prod_{r=1}^{n-1} L_{n-r})^{-1} \\ N & = & LU - A \end{cases}$$

The matrices L and U in the above algorithm are, respectively, a lower block triangular matrix all of whose diagonal blocks equal the identity and an upper block triangular matrix. The *incomplete block LU-factorization of A* is defined to be the product LU. We have in essence computed a block LU-factorization of (the sparser matrix) $\alpha * A$ so that $A = LU - N$. If $n_i = 1$ for all $i \in \langle m \rangle$, then the corresponding factorization is called the *incomplete point LU-factorization of A*.

Given $A \in \mathbf{C}^{n,n}$, assume that $|B|$ is an approximation of $|A|$ (cf Chapter 5). Let L_1U_1 and L_2U_2 be incomplete point LU-factorizations of A and B, respectively. According to [22], if $|L_1| \leq |L_2|$, then we say L_1U_1 is "at least as stable" as L_2U_2. Messaoudi [23] showed that an incomplete point LU-factorization of an H-matrix is at least as stable as the incomplete point LU-factorization of its comparison matrix $\mathcal{M}(A)$ and raised the question whether a matrix that admits a convergent incomplete point LU-factorization for all $\alpha \in \alpha^{n,n}$ is an H-matrix. We answer this question negatively in Chapter 5.

1.5 Outline of the Thesis

In Chapter 2, we give an iterative algorithm for finding a positive diagonal matrix in $\mathcal{D}_A$ for a given H-matrix A, namely a positive diagonal matrix D such that AD is strictly diagonally dominant. We show that this algorithm converges for an H-matrix, meaning that the solution can be found in a finite number of iterations. This algorithm itself can be viewed as a new characterization of H-matrices because it fails exactly when A is not an H-matrix. Chapter 2 is based on [20].

In Chapter 3, we consider the notion of double diagonal dominance. We characterize H-matrices in $\mathbf{G}_2^{n,n}$ by using the directed graph of a matrix, and compare diagonal dominance to double diagonal dominance. Chapter 3 is based on [21].

In Chapter 4, we list some interesting subclasses of H-matrices, each of which is determined by some sufficient condition. We also investigate the relations among these classes.

In Chapter 5, we extend some definitions and theorems regarding numerical stability (found in [23]) to incomplete block LU-factorizations, and answer negatively a question posed in [23] by giving a counterexample. This is done in the context of the so called block OBV factorizations, which include most other methods of incomplete (block) factorizations as special cases.

In the last chapter, we review the main results we have obtained in this thesis, and discuss some unsolved problems.

In Appendix I, the reader can find MATLAB functions implementing the algorithms mentioned or developed in this thesis. In Appendix II we have included a test table cited in Chapter 5.

Chapter 2

AN ITERATIVE CRITERION FOR H-MATRICES

2.1 Introduction

The H-matrices, which can be defined by any one of the equivalent conditions in Theorem 1.3.1, generalize the widely studied classes of strictly diagonally dominant matrices and of nonsingular M-matrices. In this chapter, we will introduce a simple algorithmic characterization of H-matrices.

Recall that $\mathcal{D}_A$ denotes the set of all positive diagonal matrices D satisfying Theorem 1.3.1(v) and that

$$A \text{ is an } H\text{-matrix if and only if } \mathcal{D}_A \neq \emptyset.$$

Suppose for a moment that A is an H-matrix and let $B = \mathcal{M}(A)$, $x \in \mathbf{C}^n$ be an entrywise positive vector, and $y = B^{-1}x$. Then, as B^{-1} is an entrywise nonnegative matrix (see Theorem 1.3.1(iii)), y is also entrywise positive. It follows that $D_y = \text{diag}(y) \in \mathcal{D}_A$. However, the computation of such a vector y can be a relatively intense numerical exercise since we need to solve the linear system

$$By = x. \tag{2.1.1}$$

A partial analysis of this computation is included in Section 2.3.

In [10, Theorem 1], a sufficient condition is given for strict generalized diagonal dominance of $A \in \mathbf{C}^{n,n}$. The proof of that result proceeds with the construction of a matrix $D \in \mathcal{D}_A$. However, the condition in [10] is not necessary. Moreover, the construction of D depends on knowing a partition of $\langle n \rangle$ for which the sufficient condition is satisfied, making the computational complexity prohibitive. Similar remarks are valid for the sufficient conditions for H-matrices presented in [11] and [18].

In view of the preceding comments, we find ourselves in pursuit of another method for computing a matrix in $\mathcal{D}_A$. Ideally, we want this method to be computationally convenient, and we also want the possible failure of the algorithm to produce a matrix in $\mathcal{D}_A$ to signify that the input matrix A is not an H-matrix. In other words, we are in pursuit of an algorithmic characterization of an H-matrix, which can be effectively implemented on a computer. The algorithm that we will introduce in the following section has these features.

2.2 Algorithm $\mathbb{H}$

Given a matrix $X = [x_{ij}] \in \mathbf{C}^{n,n}$ we use the notation

$$\mathbf{N}_1(X) = \{i \in \langle n \rangle \ : \ |x_{ii}| > R_i(X)\}, \quad \text{and} \quad \mathbf{N}_2(X) = \langle n \rangle \setminus \mathbf{N}_1(X).$$

An algorithmic approach to computing a matrix in $\mathcal{D}_A$ was proposed in [14], where the columns of the m-th iterate, $A^{(m)}$, are scaled by post-multiplication with a suitable diagonal matrix $\mathrm{diag}(d)$. The entries of $d \in \mathbf{C}^n$ satisfy

$$d_i = \begin{cases} 1-\epsilon & \text{if } i \in \mathbf{N}_1(A^{(m)}) \\ 1 & \text{if } i \in \mathbf{N}_2(A^{(m)}). \end{cases}$$

Assuming that $\epsilon > 0$ is sufficiently small, and that A is an H-matrix, the algorithm produces a strictly diagonally dominant matrix. Thus the product of the intermediate diagonal matrices yields a matrix in $\mathcal{D}_A$. The main drawback of this method is that the choice of ϵ may lead to a large number of required iterations. Moreover, when it is not a priori known whether A is an H-matrix, a possible failure of the algorithm to produce a matrix in $\mathcal{D}_A$ after a large number of iterations cannot necessarily be attributed to the choice of ϵ

We will next introduce a different algorithmic procedure for the computation of a matrix in $\mathcal{D}_A$, in which the above drawbacks are addressed.

There are two cases where A is easily seen not to be an H-matrix. First, if A has no diagonally dominant rows, then all the entries of $\mathcal{M}(A)e$ are nonpositive, violating the monotonicity condition for nonsingular M-matrices (see [6, Theorem 6.2.3]). It follows that A is not an H-matrix. Second, if a diagonal entry of A is zero, then A is not an H-matrix since $\mathcal{D}_A = \emptyset$. Consequently, the algorithm below is designed to terminate (at step 1 - before any iterations take place) if either of these cases occurs. Otherwise, it quantifies the diagonal dominance in certain rows of the m-th iterate, $A^{(m)}$, by computing the ratios $R_i(A^{(m)})/|a_{ii}^{(m)}|$. Then the algorithm proceeds to re-distribute the (collective) diagonal dominance among all rows by rescaling the columns of $A^{(m)}$, thus producing $A^{(m+1)}$.

<u>`Algorithm` $\mathbb{H}$</u>

`INPUT: a matrix` $A = [a_{ij}] \in \mathbf{C}^{n,n}$ `and` <u>`any`</u> $\theta : 0 < \theta < 1$.

```
OUTPUT: D = D^(1) D^(2) ··· D^(m) ∈ 𝒟_A if A is an H-matrix.
1.    if N_1(A) = ∅ or a_ii = 0 for some i ∈ ⟨n⟩, 'A is not an H-matrix',
      STOP; otherwise
2.    set A^(0) = A, D^(0) = I, m = 1
3.    compute A^(m) = A^(m-1) D^(m-1) = [a_ij^(m)]
4.    if N_1(A^(m)) = ⟨n⟩, 'A is an H-matrix', STOP; otherwise
5.    set d = [d_i], where
```

$$d_i = \begin{cases} 1-\theta\left(1-\frac{R_i(A^{(m)})}{|a_{ii}|^{(m)}}\right) & \texttt{if } i \in \mathbf{N}_1(A^{(m)}) \\ 1 & \texttt{if } i \in \mathbf{N}_2(A^{(m)}) \end{cases}$$

```
6.    set D^(m) = diag(d), m = m + 1; go to step 3
```

The theoretical basis for the functionality of Algorithm $\mathbb{H}$ as a criterion for H-matrices is provided by the following theorem and the two lemmata that precede its proof.

Theorem 2.2.1 *The matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is an H-matrix if and only if Algorithm $\mathbb{H}$ terminates after a finite number of iterations by producing a strictly diagonally dominant matrix.*

Lemma 2.2.2 *The Algorithm $\mathbb{H}$ either terminates or it produces an infinite sequence of distinct matrices $\{A^{(m)} = [a_{ij}^{(m)}]\}$ such that $\lim_{m\to\infty} |a_{ij}^{(m)}|$ exists for all $i, j \in \langle n \rangle$.*

Proof. Suppose that Algorithm $\mathbb{H}$ does not terminate, that is, it produces an infinite sequence of matrices. Recall that this means $\mathbf{N}_1(A) \neq \emptyset$ and $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. For notational convenience, we can assume that $A = \mathcal{M}(A)$ and that

$$A = \begin{pmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where $a_{ii} > 0$ and $a_{ij} \geq 0$ for all $i, j \in \langle n \rangle$. By the definition of d_i in step 5, it readily follows that for all $i \in \mathbf{N}_1(A^{(m)})$, $d_i \in (0, 1)$ and also that

$$d_i = 1 - \theta\left(1 - \frac{R_i(A^{(m)})}{a_{ii}^{(m)}}\right).$$

Hence, as $\theta \in (0,1)$, for $i \in \mathbf{N}_1(A)$ we have that for any $m = 1, 2, \cdots$,

$$\begin{aligned} a_{ii}^{(m+1)} &= d_i a_{ii}^{(m)} = a_{ii}^{(m)} - \theta(a_{ii}^{(m)} - R_i(A^{(m)})) \\ &= (1-\theta)a_{ii}^{(m)} + \theta R_i(A^{(m)}) \\ &> (1-\theta)R_i(A^{(m)}) + \theta R_i(A^{(m)}) \\ &= R_i(A^{(m)}) \geq R_i(A^{(m+1)}). \end{aligned}$$

In other words, we have shown that

$$\mathbf{N}_1(A) = \mathbf{N}_1(A^{(1)}) \subseteq \mathbf{N}_1(A^{(2)}) \subseteq \cdots \subseteq \mathbf{N}_1(A^{(m)}) \subseteq \cdots .$$

Consequently, there exists a smallest integer ℓ such that $\mathbf{N}_1(A^{(\ell)}) = \mathbf{N}_1(A^{(\ell+p)})$ for all $p = 1, 2, \cdots$. Since Algorithm $\mathbb{H}$ terminates for the input matrix A if and only if it terminates for the input matrix $A^{(\ell)}$, we may without loss of generality assume that $\ell = 1$. Further, we may suppose that

$$\mathbf{N}_1(A) = \mathbf{N}_1(A^{(1)}) = \{1, 2, \cdots, k\} \text{ for some } k < n$$

(otherwise we can consider a permutation similarity of A). Under this assumption, the algorithm yields

$$A^{(m+1)} = A^{(m)} D^{(m)} \quad (m = 1, 2, \cdots),$$

where

$$D^{(m)} = \operatorname{diag}(d_m), \quad d_m = [d_1^{(m)}, d_2^{(m)}, \cdots, d_k^{(m)}, 1, 1, \ldots, 1]^t,$$

and $d_i^{(m)} \in (0,1)$ for all $i \in \langle n \rangle$. Thus,

$$a_{st}^{(m+1)} = \begin{cases} d_t^{(m)} a_{st}^{(m)} & \text{if } s \in \langle n \rangle \text{ and } t \in \mathbf{N}_1(A^{(1)}) \\ a_{st} & \text{if } s \in \langle n \rangle \text{ and } t \in \mathbf{N}_2(A^{(1)}). \end{cases}$$

It follows that for any $s, t \in \langle n \rangle$, $\{a_{st}^{(m)}\}$ is a non-increasing and bounded sequence. Thus $\lim_{m\to\infty} a_{st}^{(m)}$ exists for all $s, t \in \langle n \rangle$. ■

Lemma 2.2.3 *If Algorithm $\mathbb{H}$ produces the infinite sequence $\{A^{(m)} = [a_{ij}^{(m)}]\}$, then for all $i \in \mathbf{N}_1(A)$,*

$$\lim_{m\to\infty} \left(|a_{ii}^{(m)}| - R_i(A^{(m)}) \right) = 0.$$

Proof. Assume that A is as in the proof of Lemma 2.2.2 and suppose, by way of contradiction, that for some $i \in \mathbf{N}_1(A)$, $\lim_{m\to\infty} \left(a_{ii}^{(m)} - R_i(A^{(m)}) \right) \neq 0$. Notice that $a_{ii}^{(m)} > R_i(A^{(m)})$ and recall that, from Lemma 2.2.2, both sequences $\{a_{ii}^{(m)}\}$ and $\{R_i(A^{(m)})\}$ converge. We can therefore conclude that there exists $\epsilon_0 > 0$ such that

$$a_{ii}^{(m)} - R_i(A^{(m)}) > \epsilon_0 \quad (m = 1, 2, \cdots). \tag{2.2.2}$$

In particular, $a_{ii}^{(m)} > \epsilon_0 + R_i(A^{(m)}) \geq \epsilon_0$. ¿From Algorithm $\mathbb{H}$ we then obtain

$$\begin{aligned} 0 < a_{ii}^{(m+1)} &= d_i^{(m)} a_{ii}^{(m)} \\ &= a_{ii}^{(m)} - \theta \left(a_{ii}^{(m)} - R_i(A^{(m)})\right) \\ &\leq a_{ii}^{(m)} - \theta\epsilon_0 \quad \text{(by (2.2.2))} \\ &= a_{ii}^{(m)} - c, \end{aligned}$$

where $c = \theta\epsilon_0$. Note that c is positive and therefore, as

$$a_{11} \geq a_{11}^{(1)} + c \geq \cdots \geq a_{11}^{(m)} + mc \geq mc,$$

by letting $m \to \infty$ we obtain a contradiction. ■

We are now able to prove our main result in this chapter.

Proof of Theorem 2.2.1:

Sufficiency: Suppose that Algorithm $\mathbb{H}$ terminates after k iterations. That is, we have obtained a strictly diagonally dominant matrix $A^{(k)} = AD$, where $D = D^{(1)}D^{(2)} \cdots D^{(k-1)}$ is by construction a positive diagonal matrix. By our introductory remarks, it follows that A is an H-matrix.

Necessity: Let A be an H-matrix and assume that A is as in the proof of Lemma 2.2.2. Furthermore, by way of contradiction, assume that Algorithm $\mathbb{H}$ yields the infinite sequences

$$\{A^{(m)}\},\ \{a_{ii}^{(m)}\},\ \{R_i(A^{(m)})\},\ \{\mathbf{N}_1(A^{(m)})\}.$$

As in the proof of Lemma 2.2.2, we can without loss of generality assume that $\mathbf{N}_1(A^{(m)}) = \mathbf{N}_1(A) = \{1, 2, \cdots, k\}$ for some $k < n$ and all $m = 1, 2, \cdots$. Notice that

$$A^{(m+1)} = A^{(m)}D^{(m)} = AD^{(1)}D^{(2)} \cdots D^{(m)} = AF^{(m)},$$

where $F^{(m)}$ is a positive diagonal matrix $\text{diag}(d_m)$ with $d_m = [f_1^{(m)}, \cdots, f_k^{(m)}, 1, \cdots, 1]^T$. ¿From Lemma 2.2.2, it follows that $\lim_{m\to\infty} A^{(m)}$ exists and so $\lim_{m\to\infty} F^{(m)}$ also exists. Say these limits are B and $F = \text{diag}(d)$, where $d = [f_1, \cdots, f_k, 1, \cdots, 1]^t$. We thus have $AF = B$. Now notice that B is of the form

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1k} & -a_{1,k+1} & \cdots & -a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{k1} & -b_{k2} & \cdots & b_{kk} & -a_{k,k+1} & \cdots & -a_{kn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{n1} & -b_{n2} & \cdots & -b_{nk} & -a_{n,k+1} & \cdots & a_{nn} \end{pmatrix},$$

where, by Lemma 2.2.3, $b_{ii} = R_i(B)$ for all $i \in \mathbf{N}_1(A)$, and $b_{ii} = a_{ii} \leq R_i(B)$ for all $i \in \mathbf{N}_2(A)$. Hence $\mathbf{N}_1(B) = \emptyset$, implying that B is not an H-matrix.

Claim: $f_1 = f_2 = \cdots = f_k = 0$.

Proof of claim: First, note that if all $f_i > 0$, then $B = AF$ would be an H-matrix, a contradiction. So at least one of the f_i's equals zero. Without loss of generality, assume that $f_1 = f_2 = \cdots = f_p = 0$ for some $p < k$ and that $f_q > 0$ for all $q = p+1, p+2, \cdots, k$ (otherwise we can consider a permutation similarity of A that symmetrically permutes the first p rows and columns of A, leaving $\mathbf{N}_1(A)$ invariant). Then $B = AF$ has the block form

$$AF = \begin{pmatrix} 0 & * \\ 0 & \tilde{A}_{n-p} \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & B_{n-p} \end{pmatrix} = B,$$

where $\tilde{A}_{n-p}$ and B_{n-p} are $(n-p) \times (n-p)$. As $\tilde{A}_{n-p}$ is an H-matrix, so is B_{n-p}. This is a contradiction, because $b_{ii} \leq R_i(B_{n-p})$ for all $i \in \langle n \rangle \setminus \langle p \rangle$. This completes the proof of the claim.

We now have that

$$AF = A \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & \tilde{A}_{n-k} \end{pmatrix} = B = \begin{pmatrix} 0 & * \\ 0 & B_{n-k} \end{pmatrix}.$$

Once again, we have a contradiction because $\tilde{A}_{n-k}$ is an H-matrix but B_{n-k} is not. This shows that Algorithm $\mathbb{H}$ must terminate after a finite number of iterations, completing the proof of the theorem. ■

2.3 Some Numerical Examples

We illustrate Algorithm $\mathbb{H}$ and its performance when applied to an H-matrix A. Let k denote the number of iterations required by the MATLAB function in Appendix I, namely the number of iterations required by the algorithm to produce a matrix in $\mathcal{D}_A$.

Example 2.3.1 (this example appeared in [15]) Let

$$A = \begin{bmatrix} 1 & 0.1 & 0.1 & 0.1 & 0.8 \\ 0.35 & 1 & 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 1 & 0.1 & 0.02 \\ 0.1 & 0.06 & 0.03 & 1 & 0.02 \\ 0.1 & 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}.$$

We have $k = 1$ ($\theta = 0.9$) for Algorithm $\mathbb{H}$; 18 iterations with $\epsilon = 0.02$ are required by the algorithm presented in [14].

Example 2.3.2 (this example was given in [14]) Let

$$A = \begin{bmatrix} 0.9 & 0.1 & 0.05 & 0.05 & 0.1 & 0.1 \\ 0.1 & 1.05 & 0.05 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.9 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.7 & 0 & 0 \\ 0.5 & 0.4 & 0.02 & 0.3 & 0.98 & 0.01 \\ 0.5 & 0.5 & 0.01 & 0.3 & 0 & 0.92 \end{bmatrix}.$$

Here we have $k = 1$ ($\theta = 0.9$) for Algorithm $\mathbb{H}$; $19(\epsilon = 0.02)$ iterations are needed if we use the algorithm proposed in [14].

Example 2.3.3 (this example was given [14]) Let

$$A = \begin{bmatrix} 1 & -0.2 & -0.1 & -0.2 & -0.1 \\ -0.4 & 1 & -0.2 & -0.1 & -0.1 \\ -0.9 & -0.2 & 1 & -0.1 & -0.1 \\ -0.3 & -0.7 & -0.3 & 1 & -0.1 \\ -1 & -0.3 & -0.2 & -0.4 & 1 \end{bmatrix}.$$

In this example, $k = 13$ ($\theta = 0.9$) for Algorithm $\mathbb{H}$; 46 iterations are needed by the algorithm in [14] with $\epsilon = 0.02$.

Remark 2.3.4 We can calculate and compare the numbers of operations required in the solution of (2.1.1) by using Gaussian elimination and Algorithm $\mathbb{H}$, as methods to identify an H-matrix. It is well-known that the solution of (2.1.1) by Gaussian elimination (LU-factorization) with partial pivoting requires $\frac{2}{3}n^3 + O(n^2)$ operations (cf [12]). Algorithm $\mathbb{H}$ requires at most $k(2n^2 + O(n))$ operations, where k is the number of iterations required. Thus for large n when $k \leq \frac{1}{3}n$, Algorithm $\mathbb{H}$ requires less operations than the direct solution of the equation (2.1.1). In Example 2.3.1 and Example 2.3.2 above, this is indeed the case. More remarks and advice on the implementation of Algorithm $\mathbb{H}$ can be found in the next section.

2.4 Further Comments and a MATLAB Function

It is clear from the definition of Algorithm $\mathbb{H}$ and Theorem 2.2.1 that the termination or not of Algorithm $\mathbb{H}$ is irrespective of the choice of the positive parameter $\theta \in (0, 1)$. However, the column scalings and the re-distribution of the diagonal dominance at each iteration are done according to the ratios

$$1 - \theta \left(1 - \frac{R_i(A^{(m)})}{|a_{ii}|^{(m)}}\right).$$

Also, for $0 < b < a$, $1 - \theta(1 - b/a)$ is a decreasing function of $\theta \in (0, 1)$. Hence, larger choices of the parameter $\theta \in (0, 1)$ result in at least as large a set $\mathbf{N}_1(A^{(m+1)})$. Nevertheless, it is <u>not</u> generally true that by choosing θ close enough to 1 the number of further iterations required for the termination of the algorithm is 1, even if A is an H-matrix. To see this formally, let $A \in \mathbf{C}^{n,n}$ be an H-matrix and suppose that $\ell \in \mathbf{N}_2(A^{(m)})$ for some positive integer m. Observe then that

$$R_\ell(A^{(m+1)}) \;=\; \sum_{k \in \mathbf{N}_1(A^{(m)})} \frac{(1-\theta)|a_{kk}^{(m)}| + \theta R_k(A^{(m)})}{|a_{kk}^{(m)}|}\, |a_{\ell k}^{(m)}| \;+\; \sum_{k \in \mathbf{N}_2(A^{(m)}),\ k \neq \ell} |a_{\ell k}^{(m)}|.$$

So, if the entries of $A^{(m)}$ satisfy

$$\sum_{k \in \mathbf{N}_1(A^{(m)})} \frac{R_k(A^{(m)})}{|a_{kk}^{(m)}|}\, |a_{\ell k}^{(m)}| \;+\; \sum_{k \in \mathbf{N}_2(A^{(m)}),\ k \neq \ell} |a_{\ell k}^{(m)}| \;>\; |a_{\ell\ell}^{(m)}|,$$

then at least 2 more iterations of Algorithm $\mathbb{H}$ are required, regardless of the choice of $\theta \in (0, 1)$. We illustrate this situation with the following example.

Example 2.4.1 Consider the H-matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ -1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

and notice that $\mathbf{N}_1(A) = \{1, 2\}$, $\mathbf{N}_2(A) = \{3\}$. As

$$\begin{aligned} \lim_{\theta \to 1-} \sum_{k \in \mathbf{N}_1(A)} \frac{(1-\theta)|a_{kk}| + \theta R_k(A)}{|a_{kk}|}\, |a_{3k}| \;&=\; \lim_{\theta \to 1-} \left(\frac{4(1-\theta) + 2\theta}{4} + \frac{3(1-\theta) + 2\theta}{3} \right) \\ &=\; \frac{7}{6} \;>\; 1 \;=\; |a_{33}|, \end{aligned}$$

it follows that a first pass of the Algorithm $\mathbb{H}$ will not result in a strictly diagonally dominant third row. That is, at least 2 iterations are needed for the algorithm to terminate by producing $D \in \mathcal{D}_A$, regardless of the choice of $\theta \in (0, 1)$. In fact, for $\theta = 0.9$ exactly 2 iterations are needed.

The next practical aspect of Algorithm $\mathbb{H}$ we want to discuss is the situation when the input matrix $A \in \mathbf{C}^{n,n}$ is not (known to be) an H-matrix. When the computed diagonal matrix $D^{(m)}$ is approximately equal to the identity (and the algorithm has not terminated), it means that the present iterate is not diagonally dominant and there is little numerical hope that it will become one. Based on Theorem 2.2.1, we can then stop and declare A not an H-matrix.

We also comment that Algorithm $\mathbb{H}$ can be modified so that step 6 takes place every time an $i \in \mathbf{N}_1(A^{(m)})$ is encountered; then it proceeds by searching for the first index in $\mathbf{N}_1(A^{(m+1)})$. This usually results in fewer iterations until a matrix $D \in \mathcal{D}_A$ is found.

We provide a MATLAB function (in Appendix I) implementing Algorithm $\mathbb{H}$ with a fixed parameter θ. The termination criteria regarding the computation of a $D \in \mathcal{D}_A$ or the decision that A is not an H-matrix are handled by the default relative accuracy of MATLAB.

Chapter 3

DOUBLY DIAGONALLY DOMINANT MATRICES

3.1 Preliminaries

The theorem of Geršgorin and the theorem of Brauer are two classical results about regions in the complex plane that include the spectrum of a matrix (see e.g., Horn and Johnson [16]). To summarize, they, respectively, locate the eigenvalues of an $n \times n$ complex matrix $A = [a_{ij}]$ in the union of n closed discs (known as the Geršgorin discs),

$$\{z \in \mathbf{C} : \ |z - a_{ii}| \ \leq \ R_i(A)\} \ \ (i = 1, 2, \ldots, n),$$

or in the union of $n(n-1)/2$ ovals (known as the ovals of Cassini),

$$\{z \in \mathbf{C} : \ |z - a_{ii}| \, |z - a_{jj}| \ \leq \ R_i(A) R_j(A)\} \ \ (i, j = 1, 2, \ldots, n; \ i \neq j).$$

As a consequence of either of these theorems, but more precisely as a consequence of Geršgorin's theorem, every strictly diagonally dominant matrix is invertible. In geometric terms, strict diagonal dominance means that the origin does not belong to the union of the Geršgorin discs and hence it cannot be an eigenvalue. In this chapter we will consider a condition weaker than diagonal dominance, whose geometric interpretation regards the location of the origin relative to the ovals of Cassini. This condition gives rise to the class of doubly diagonally dominant matrices and its subclasses, whose precise definitions were given in Chapter 1(cf Definition 1.2.1).

We begin by introducing some further definitions and notations.

With A we associate its (loopless) *directed graph*, $D(A)$, defined as follows. The vertices of $D(A)$ are $1, 2, \ldots, n$. There is an arc (i, j) from i to j when $a_{ij} \neq 0$ and $i \neq j$. A *path (of length p) from i to j* is a sequence of distinct vertices $i = i_0, i_1, \ldots, i_p = j$ such that $(i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p)$ are arcs of $D(A)$. We denote such a path by $P_{ij} = (i_0, i_1, \ldots, i_p)$. A *circuit* γ of $D(A)$ consists of the distinct vertices $i_0, i_1, \cdots, i_p$, $p \geq 1$, provided that $(i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p)$, and (i_p, i_0) are arcs of $D(A)$. We write $\gamma = (i_0, i_1, \ldots, i_p, i_0)$ and denote the set of all circuits of $D(A)$ by $\mathcal{E}(A)$.

For $n \geq 2$, the matrix $A \in \mathbf{C}^{n,n}$ is called *irreducible* if its directed graph is strongly connected, i.e., for every pair of distinct vertices i, j, there is a path P_{ij} in $D(A)$. this definition is equivalent to the one given in Chapter 1 (see e.g. [31]).

A particular directed graph which will arise in our subsequent discussion is the directed graph of a matrix $A \in \mathbf{C}^{n,n}$ whose diagonal entries are nonzero, the entries of the i_0–th row and column (for some $i_0 \in \langle n \rangle$) are nonzero, and all other entries are zero. Prompted by its shape, we refer to $D(A)$ as a *star centered at* i_0.

Recall $\mathbf{G}^{n,n}, \mathbf{G}_1^{n,n}$ and $\mathbf{G}_2^{n,n}$ respectively denote the classes of doubly diagonally dominant (abbrev. d.d.d.), strictly doubly diagonally dominant (abbrev. s.d.d.d.) and irreducibly doubly diagonally dominant matrices (abbrev. i.d.d.d.)(cf Definition 1.2.2). Notice that the diagonal entries of every matrix in $\mathbf{G}_1^{n,n}$ or $\mathbf{G}_2^{n,n}$ are nonzero.

Let us now review some classical results and note some similarities and differences between diagonal dominance and double diagonal dominance:

(1) If A is strictly diagonally dominant then det$A \neq 0$ (Lévy–Desplanques theorem). If $A \in \mathbf{G}_1^{n,n}$ then det$A \neq 0$ (by Brauer's theorem).

(2) If A is irreducibly diagonally dominant then det$A \neq 0$ (see Taussky [28] and [29]). However, a matrix in $\mathbf{G}_2^{n,n}$ is not necessarily nonsingular as the following example shows:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

If $A \in \mathbf{G}_2^{n,n}$ and if (1.2.2) holds strictly for at least one pair of the vertices of some circuit $\gamma \in \mathcal{E}(A)$, we can conclude that det$A \neq 0$ (see Zhang and Gu [33, Theorem 1]).

(3) If A is strictly diagonally dominant or irreducibly diagonally dominant then A is an H–matrix (see e.g., Varga [30]). More precisely, by Theorem 1.3.1 A is an H–matrix if and only if there exists a positive diagonal matrix D such that AD is strictly diagonally dominant. In the literature the latter property is referred to as 'generalized diagonal dominance' (see e.g., [6]), because it reduces to diagonal dominance when D is the identity. The example in (2) above also shows that not every matrix in $\mathbf{G}_2^{n,n}$ is an H–matrix.

(4) When A is irreducible, a form of diagonal dominance based on the circuits of $D(A)$, introduced by Brualdi in [7], implies the invertibility of A:

Theorem 3.1.1 ([7, Theorem 2.9]) *Let $A = [a_{ij}] \in \mathbf{C}^{n,n}$ be irreducible. Suppose*

$$\prod_{i \in \gamma} |a_{ii}| \;\geq\; \prod_{i \in \gamma} R_i(A) \quad (\gamma \in \mathcal{E}(A)),$$

with strict inequality holding for at least one circuit γ. Then $detA \neq 0$.

In what follows we will characterize H–matrices in $\mathbf{G}^{n,n}$ and $\mathbf{G}_2^{n,n}$, and will describe the singular matrices in $\mathbf{G}_2^{n,n}$ (section 2). In section 3 we will prove several results regarding the Schur complements of doubly diagonally dominant matrices, leading up to the fundamental result that the Schur complements of matrices in $\mathbf{G}^{n,n}$ are also doubly diagonally dominant.

3.2 Double Diagonal Dominance, Singularity and H–Matrices

We begin with some basic observations regarding matrices in $\mathbf{G}^{n,n}$.

Theorem 3.2.1 *Let $A \in \mathbf{G}^{n,n}$. Then the following hold.*

(i) *$\mathcal{M}(A)$ is an M–matrix.*

(ii) *A is an H–matrix if and only if $\mathcal{M}(A)$ is nonsingular.*

(iii) *If $A \in \mathbf{G}_1^{n,n}$, then A is an H–matrix.*

(iv) *If $A \in \mathbf{G}_2^{n,n}$ is such that (1.2.2) holds strictly for at least one pair of vertices i, j that lie on a common circuit of $D(A)$, then A is an H–matrix.*

Proof. To show (i), for $\epsilon > 0$, let $B_\epsilon = \mathcal{M}(A) + \epsilon I = [b_{ij}]$. Since $|b_{ii}||b_{jj}| > R_i(B_\epsilon)R_j(B_\epsilon)$ for all i, j, $i \neq j$, it follows from Brauer's theorem that $B_\epsilon \in \mathbf{Z}^{n,n}$ is nonsingular for every $\epsilon > 0$, which implies that $\mathcal{M}(A)$ is an M–matrix (see e.g., condition (C_9) of Theorem 4.6 in [6, Chapter 6]). Parts (ii) and (iii) are immediate consequences of part (i) and Brauer's theorem. Part (iv) follows from part (ii) and Theorem 3.1.1 applied to $\mathcal{M}(A)$. ■

Some results related to Theorem 3.2.1 appear in [26]. There it is claimed that matrices in $\mathbf{G}_2^{n,n}$ are H–matrices, which is false as we have seen by an example in section 2. In Chapter 4, we shall derive two other characterizations of H-matrices in $\mathbf{G}^{n,n}$.

Next we will characterize the singular matrices in $\mathbf{G}_2^{n,n}$. First we need the following lemma.

Lemma 3.2.2 *Consider $A \in \mathbf{C}^{n,n}$ such that $D(A)$ is a star centered at $i_0 \in \langle n \rangle$. Then*

$$\det A = \prod_{j \neq i_0} a_{jj} \left[a_{i_0 i_0} - \sum_{k \neq i_0} \frac{a_{k i_0} a_{i_0 k}}{a_{kk}} \right].$$

Proof. The terms in the expansion of the determinant of a matrix A as prescribed are

$$\prod_{j=1}^{n} a_{jj} \quad \text{and} \quad -(a_{ki_0}a_{i_0k}\prod_{m\neq k,i_0} a_{mm}) \quad (k \in \langle n\rangle \setminus \{i_0\})$$

and the formula for the determinant follows readily. ■

Theorem 3.2.3 *Let $A = [a_{ij}] \in \mathbf{G}_2^{n,n}$. Then A is singular if and only if $D(A)$ is a star centered at some $i_0 \in \langle n\rangle$ and the following hold:*

$$|a_{i_0i_0}||a_{jj}| = R_{i_0}(A)R_j(A) \quad (j \in \langle n\rangle \setminus \{i_0\}) \tag{3.2.1}$$

and

$$a_{i_0i_0} - \sum_{k\neq i_0} \frac{a_{ki_0}a_{i_0k}}{a_{kk}} = 0. \tag{3.2.2}$$

Proof.

Sufficiency: If $D(A)$ is a star centered at $i_0 \in \langle n\rangle$ and (3.2.2) holds, then by Lemma 3.2.2, A is singular.

Necessity: Assume that $A = [a_{ij}] \in \mathbf{G}_2^{n,n}$ is singular. Since $A \in \mathbf{G}_2^{n,n}$, one of the following two cases must occur. Either $|a_{ii}| \geq R_i(A)$ for all $i \in \langle n\rangle$ with at least one strict inequality holding, or there exists one and only one $i_0 \in \langle n\rangle$ such that

$$|a_{i_0i_0}| < R_{i_0} \quad \text{and} \quad |a_{jj}| > R_j(A) \quad (j \in \langle n\rangle \setminus \{i_0\}). \tag{3.2.3}$$

In the former case A is an irreducibly diagonally dominant matrix and hence nonsingular, contradicting our assumption. Therefore (3.2.3) holds. It also follows from the definition of $\mathbf{G}_2^{n,n}$ that

$$\prod_{i\in\gamma} |a_{ii}| \;\geq\; \prod_{i\in\gamma} R_i(A) \quad (\gamma \in \mathcal{E}(A)). \tag{3.2.4}$$

If $\gamma \in \mathcal{E}(A)$ and $i_0 \notin \gamma$, it follows by (3.2.3) that

$$\prod_{i\in\gamma} |a_{ii}| > \prod_{i\in\gamma} R_i(A). \tag{3.2.5}$$

Then Theorem 3.1.1, (3.2.4) and (3.2.5) imply that $\det A \neq 0$, contradicting our assumption. Hence for every $\gamma \in \mathcal{E}(A)$, $i_0 \in \gamma$.

We now claim that every $\gamma \in \mathcal{E}(A)$ is of the form $\gamma = (i_0, j, i_0)$ for some $j \in \langle n\rangle \setminus \{i_0\}$. Indeed if $\gamma = (i_0, i_1, \cdots, i_p, i_0)$ with $p \geq 2$, then

$$\begin{aligned}
\prod_{i\in\gamma} |a_{ii}| &= |a_{i_0i_0}||a_{i_1i_1}| \prod_{i\in\gamma\setminus\{i_0,i_1\}} |a_{ii}| \\
&> |a_{i_0i_0}||a_{i_1i_1}| \prod_{i\in\gamma\setminus\{i_0,i_1\}} R_i(A) \\
&\geq \prod_{i\in\gamma} R_i(A),
\end{aligned}$$

so, by Theorem 3.1.1, $\det A \neq 0$, contradicting again our assumption that A is singular.

As is well known, since $D(A)$ is by assumption strongly connected, every vertex i lies on some circuit $\gamma \in \mathcal{E}(A)$. Therefore we deduce that

$$\mathcal{E}(A) = \{\gamma_j : \gamma_j = (i_0, j, i_0),\ j \in \langle n \rangle \setminus \{i_0\}\}. \tag{3.2.6}$$

In particular, it follows that there are no arcs (i_1, i_2) in $D(A)$ with $i_1 \neq i_0$ and $i_2 \neq i_0$, otherwise $\gamma = (i_0, i_1, i_2, i_0) \in \mathcal{E}(A)$, contradicting (3.2.6). Thus $D(A)$ is a star centered at i_0.

If for some j, $|a_{i_0 i_0}||a_{jj}| > R_{i_0}(A)R_j(A)$, then, by Theorem 3.1.1, we are led to the contradiction that $\det A \neq 0$. Thus for each $j \in \langle n \rangle \setminus \{i_0\}$, we have

$$|a_{i_0 i_0}||a_{jj}| = R_{i_0}(A)R_j(A).$$

Finally, by Lemma 3.2.2, we can now assert that A satisfies (3.2.2). ■

We note that the necessity part of Theorem 3.2.3 also follows from the results in Tam, Yang, and Zhang [27]. The next theorem offers a characterization of the H–matrices in $\mathbf{G}_2^{n,n}$.

Theorem 3.2.4 *Let $A = [a_{ij}] \in \mathbf{G}_2^{n,n}$. Then A is not an H–matrix if and only if $D(A)$ is a star centered at some $i_0 \in \langle n \rangle$ and*

$$|a_{i_0 i_0}||a_{jj}| = R_{i_0}(A)R_j(A) \quad (j \in \langle n \rangle \setminus \{i_0\}). \tag{3.2.7}$$

Proof.

Necessity: Suppose A is not an H–matrix. Note that if $A \in \mathbf{G}_2^{n,n}$, then $\mathcal{M}(A) \in \mathbf{G}_2^{n,n}$. The result follows by Theorem 3.2.1 part (ii) and Theorem 3.2.3 applied to $\mathcal{M}(A)$.

Sufficiency: By assumption, $D(\mathcal{M}(A))$ is a star centered at some $i_0 \in \langle n \rangle$ and (3.2.7) holds. Consider the vector $x = [x_1, x_2, \ldots, x_n]^T$, where $x_{i_0} = R_{i_0}(A)$ and $x_i = |a_{i_0 i_0}|$ for all $i \neq i_0$. Then $\mathcal{M}(A)x = 0$, $x \neq 0$, and thus, by Theorem 3.2.1 part (ii), A is not an H–matrix. ■

If $A \in \mathbf{G}^{n,n}$ is singular, by Theorem 3.2.1 part (ii), $\mathcal{M}(A)$ is singular. The converse of this statement is not necessarily true. More specifically, $A \in \mathbf{G}_2^{n,n}$ being nonsingular does not in general imply that A is an H–matrix (i.e., that $\mathcal{M}(A)$ is nonsingular). This situation occurs in the next example.

Example 3.2.5 The following matrices illustrate the use of Theorems 3.2.3 and 3.2.4 in checking whether an irreducibly doubly diagonally dominant matrix is an H–matrix or not. Consider the following matrices in $\mathbf{G}_2^{3,3}$:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 0 \\ -1 & 0 & 3 \end{bmatrix},\ E = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 0 \\ 1 & 1 & 4 \end{bmatrix}.$$

The directed graph of A is a star centered at $i_0 = 1$ and A satisfies (3.2.2). ¿From Lemma 3.2.2, A is singular. Since $\mathcal{M}(B) = A$, B is not an H–matrix (even though B is nonsingular). Note that $D(C)$ is a star centered at $i_0 = 1$ but $|c_{11}||c_{33}| = 3 > 2 = R_1(C)R_3(C)$. Hence, by Theorem 3.2.4, C is an H–matrix. Finally, $D(E)$ is not a star centered at any $i_0 \in \{1, 2, 3\}$ and so E must be an H–matrix.

3.3 Schur Complements

Let $A = [a_{ij}] \in \mathbf{C}^{n,n}$ be partitioned as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{3.3.8}$$

where A_{11} is the leading $k \times k$ principal submatrix of A, for some $k \in \langle n \rangle$. Assuming that A_{11} is invertible we can reduce A (using elementary row operations) to the matrix

$$\begin{bmatrix} U_k & * \\ \mathbf{0} & A/A_{11} \end{bmatrix}, \tag{3.3.9}$$

where $U_k \in \mathbf{C}^{k,k}$ is upper triangular and A/A_{11}, known as the *Schur complement of A relative to A_{11}*, is given by $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. In particular, if $a_{11} \neq 0$, we can reduce A to the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}, \tag{3.3.10}$$

where $b_{ij} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}$, $2 \leq i, j \leq n$. The trailing $(n-1) \times (n-1)$ submatrix of the matrix above is the Schur complement of A relative to $A_{11} = [a_{11}]$, which we will subsequently denote by $B = [b_{ij}]$, and index its entries by $2 \leq i, j \leq n$.

In this section, we shall prove that if A belongs to $\mathbf{G}^{n,n}$ and $\det A_{11} \neq 0$, then A/A_{11} belongs to $\mathbf{G}^{n-k,n-k}$. We will first consider the Schur complements of matrices in $\mathbf{G}_1^{n,n}$. We note that our proofs rely on the fact that if $A \in \mathbf{G}_1^{n,n}$, then all principal submatrices of A are invertible and so the associated Schur complements are well defined. The following is a well known fact in numerical linear algebra.

Lemma 3.3.1 *If $A \in \mathbf{C}^{n,n}$ is strictly diagonally dominant and partitioned as in (3.3.8), then $\det A_{11} \neq 0$ and A/A_{11} is also strictly diagonally dominant.*

Lemma 3.3.2 *Let $A = [a_{ij}] \in \mathbf{G}_1^{3,3}$. Then*

$$\left|a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right|\left|a_{33} - \frac{a_{31}a_{13}}{a_{11}}\right| > \left|a_{23} - \frac{a_{21}a_{13}}{a_{11}}\right|\left|a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right|. \tag{3.3.11}$$

Proof. Since $A = [a_{ij}] \in \mathbf{G}_1^{3,3}$, from Theorem 3.2.1 part (iii), A is an H–matrix. Hence there is a positive diagonal matrix $D = \mathrm{diag}(d_1, d_2, d_3)$ such that AD is a strictly diagonally dominant matrix. Since $d_1 a_{11} \neq 0$, we can reduce AD to the matrix

$$\begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ 0 & d_2 a_{22} - \frac{d_2 a_{21} a_{12}}{a_{11}} & d_3 a_{23} - \frac{d_3 a_{21} a_{13}}{a_{11}} \\ 0 & d_2 a_{32} - \frac{d_2 a_{31} a_{12}}{a_{11}} & d_3 a_{33} - \frac{d_3 a_{31} a_{13}}{a_{11}} \end{bmatrix},$$

which, by Lemma 3.3.1, is also strictly diagonally dominant and (3.3.11) follows. ■

Theorem 3.3.3 *Let* $A \in \mathbf{G}_1^{n,n}$ *and let* $B \in \mathbf{C}^{n-1,n-1}$ *as in* (3.3.10). *Then* $B \in \mathbf{G}_1^{n-1,n-1}$.

Proof. Since $A = [a_{ij}] \in \mathbf{G}_1^{n,n}$, one of the following two cases must occur. Either there exists $i \in \langle n \rangle$ such that $|a_{ii}| \leq R_i(A)$ or $|a_{ii}| > R_i(A) \quad (i \in \langle n \rangle)$. In the latter case, by Lemma 3.3.1, B is strictly diagonally dominant and hence $B \in \mathbf{G}_1^{n-1,n-1}$. We now consider the former case in two subcases:

(i) $i = 1$.

In this case, we shall also prove that B is strictly diagonally dominant (and hence in $\mathbf{G}_1^{n-1,n-1}$). It suffices to prove that

$$|b_{22}| > \sum_{j=3}^{n} |b_{2j}|, \tag{3.3.12}$$

where $b_{22} = a_{22} - \frac{a_{21} a_{12}}{a_{11}}$, and $b_{2j} = a_{2j} - \frac{a_{21} a_{1j}}{a_{11}}$, with $j \geq 3$.

Since

$$|a_{11}||a_{22}| > \sum_{j=2}^{n} |a_{1j}| \sum_{j \neq 2} |a_{2j}| = \sum_{j=2}^{n} |a_{1j}| \sum_{j=3}^{n} |a_{2j}| + |a_{21}| \sum_{j=2}^{n} |a_{1j}|$$

$$\geq |a_{11}| \sum_{j=3}^{n} |a_{2j}| + |a_{21}||a_{12}| + |a_{21}| \sum_{j=3}^{n} |a_{1j}|,$$

(where we used the assumption $|a_{11}| \leq R_1(A)$ for the last inequality), we have

$$|a_{11} a_{22}| - |a_{12} a_{21}| > |a_{11}| \sum_{j=3}^{n} \left(|a_{2j}| + \frac{|a_{21} a_{1j}|}{|a_{11}|} \right) \geq |a_{11}| \sum_{j=3}^{n} \left| a_{2j} - \frac{a_{21} a_{1j}}{a_{11}} \right|.$$

That is,

$$\left| a_{22} - \frac{a_{21} a_{12}}{a_{11}} \right| \geq |a_{22}| - \frac{|a_{21} a_{12}|}{|a_{11}|} > \sum_{j=3}^{n} \left| a_{2j} - \frac{a_{21} a_{1j}}{a_{11}} \right|,$$

which is equivalent to (3.3.12).

(ii) $i \geq 2$.

In this case we will see that B belongs to $\mathbf{G}_1^{n-1,n-1}$. Without loss of generality, we can assume that $i = 2$. Set

$$A_1 = \begin{bmatrix} |a_{11}| & -\sum_{j\neq 1,3} |a_{1j}| & -|a_{13}| \\ -|a_{21}| & |a_{22}| & -\sum_{j=3}^{n} |a_{2j}| \\ -|a_{31}| & -\sum_{j\neq 1,3} |a_{3j}| & |a_{33}| \end{bmatrix}.$$

Since $A \in \mathbf{G}_1^{n,n}$ it follows that $A_1 \in \mathbf{G}_1^{3,3} \cap \mathbf{Z}^{3,3}$ and that A_1 has positive diagonal entries. Applying Lemma 3.3.2 to A_1 we obtain

$$\left[|a_{22}| - \frac{|a_{21}|}{|a_{11}|} \sum_{j\neq 1,3} |a_{1j}|\right] \left[|a_{33}| - \frac{|a_{13}a_{31}|}{|a_{11}|}\right] >$$

$$\left[\sum_{j=3}^{n} |a_{2j}| + \frac{|a_{21}a_{13}|}{|a_{11}|}\right] \left[\sum_{j\neq 1,3} |a_{3j}| + \frac{|a_{31}|}{|a_{11}|} \sum_{j\neq 1,3} |a_{1j}|\right]. \tag{3.3.13}$$

Setting

$$\gamma_1 = \frac{|a_{21}|}{|a_{11}|} \sum_{j=4}^{n} |a_{1j}| \left[|a_{33}| - \frac{|a_{31}a_{13}|}{|a_{11}|}\right], \ \gamma_2 = \sum_{j=3}^{n} |a_{2j}| + \frac{|a_{21}a_{13}|}{|a_{11}|},$$

$$\gamma_3 = \sum_{j\neq 1,3} |a_{3j}| + \frac{|a_{31}|}{|a_{11}|} \sum_{j\neq 1,3} |a_{1j}|,$$

we see that (3.3.13) is equivalent to

$$\left[|a_{22}| - \frac{|a_{21}a_{12}|}{|a_{11}|}\right] \left[|a_{33}| - \frac{|a_{31}a_{13}|}{|a_{11}|}\right] > \gamma_1 + \gamma_2\gamma_3. \tag{3.3.14}$$

For γ_1 we have

$$\gamma_1 = \frac{|a_{21}|}{|a_{11}|} \sum_{j=4}^{n} |a_{1j}| \left||a_{33}| - \frac{|a_{31}a_{13}|}{|a_{11}|}\right|$$

$$\overset{|a_{33}|>R_3(A)}{\geq} \frac{|a_{21}|}{|a_{11}|} \sum_{j=4}^{n} |a_{1j}| \left[\sum_{j\neq 1,3} |a_{3j}| + \frac{|a_{31}|}{|a_{11}|}(|a_{11}| - |a_{13}|)\right]$$

$$\overset{|a_{11}|>R_1(A)}{\geq} \frac{|a_{21}|}{|a_{11}|} \sum_{j=4}^{n} |a_{1j}| \left[\sum_{j\neq 1,3} |a_{3j}| + \frac{|a_{31}|}{|a_{11}|} \sum_{j\neq 1,3} |a_{1j}|\right] = \frac{|a_{21}|}{|a_{11}|} \sum_{j=4}^{n} |a_{1j}|\gamma_3. \tag{3.3.15}$$

¿From (3.3.14) and (3.3.15), it follows that

$$\left|a_{22}-\frac{a_{21}a_{12}}{a_{11}}\right|\left|a_{33}-\frac{a_{31}a_{13}}{a_{11}}\right| \geq \left[|a_{22}|-\frac{|a_{21}a_{12}|}{|a_{11}|}\right]\left[|a_{33}|-\frac{|a_{31}a_{13}|}{|a_{11}|}\right] > \gamma_1+\gamma_2\gamma_3$$

$$\geq \left[\frac{|a_{21}|}{|a_{11}|}\sum_{j=4}^{n}|a_{1j}|+\sum_{j=3}^{n}|a_{2j}|+\frac{|a_{21}a_{13}|}{|a_{11}|}\right]\gamma_3$$

$$= \left[\frac{|a_{21}|}{|a_{11}|}\sum_{j=3}^{n}|a_{1j}|+\sum_{j=3}^{n}|a_{2j}|\right]\left[\sum_{j\neq 1,3}|a_{3j}|+\frac{|a_{31}|}{|a_{11}|}\sum_{j\neq 1,3}|a_{1j}|\right]$$

$$\geq \sum_{j=3}^{n}\left|a_{2j}-\frac{a_{21}a_{1j}}{a_{11}}\right|\sum_{j\neq 1,3}\left|a_{3j}-\frac{a_{31}a_{1j}}{a_{11}}\right|,$$

or equivalently $|b_{22}||b_{33}| > R_2(B)R_3(B)$. Similarly, $|b_{22}||b_{jj}| > R_2(B)R_j(B)$ for $j = 4, 5, \ldots, n$. In general, since row reduction with respect to a strictly diagonally dominant row preserves strict diagonal dominance, we have that $|b_{ii}||b_{jj}| > R_i(B)R_j(B)$ for $i, j = 3, 4, \ldots, n$ and $i \neq j$. Hence, $B \in \mathbf{G}_1^{n-1,n-1}$. ■

Corollary 3.3.4 *If $A = [a_{ij}] \in \mathbf{G}_1^{n,n}$ and $|a_{11}| \leq R_1(A)$, then B, as in (3.3.10), is strictly diagonally dominant.*

Proof. This is subcase (i) in the proof of the previous theorem. ■

We continue now with general Schur complements of matrices in $\mathbf{G}_1^{n,n}$.

Theorem 3.3.5 *Let $J = \{i \in \langle n\rangle : |a_{ii}| \leq R_i(A)\}$, where $A = [a_{ij}] \in \mathbf{G}_1^{n,n}$ is partitioned as in* (3.3.8)*. Then*

(i) *A/A_{11} is strictly diagonally dominant if $J \subset \{1, 2, \cdots, k\}$.*

(ii) *$A/A_{11} \in \mathbf{G}_1^{n-k,n-k}$ if $\emptyset \neq J \subset \{k+1, \ldots, n\}$.*

Proof.

(i) If $J = \emptyset$, then A is strictly diagonally dominant and hence the result follows by Lemma 3.3.1. If $J \neq \emptyset$, then J can only contain one element. Without loss of generality, assume that $i = 1 \in J$ (otherwise we can symmetrically permute the first k rows and columns of A, an operation that leaves the Schur complement in question unaffected.) ¿From Corollary 3.3.4, B (as defined in (3.3.10)) is strictly diagonally dominant. The result follows by noting that A/A_{11} is equal to a Schur complement of B (see e.g., Fiedler [8, Theorem 1.25]) and by applying Lemma 3.3.1 to B.

(ii) ¿From Theorem 3.3.3 we have that $B \in \mathbf{G}_1^{n-1,n-1}$. Inductively, since A/A_{11} is equal to a Schur complement of B, it follows that if $\emptyset \neq J \subset \{k+1, \ldots, n\}$ then $A/A_{11} \in \mathbf{G}_1^{n-k,n-k}$.

■

Remark 3.3.6 If $\emptyset \neq J \subset \{k+1, \ldots, n\}$, A/A_{11} is not necessarily strictly diagonally dominant. For example, consider

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1.1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \in \mathbf{G}_1^{3,3}$$

Taking $A_{11} = [2]$ with $J = \{2\}$ we have that $A/A_{11} = \begin{bmatrix} 0.6 & -1 \\ 0 & 2 \end{bmatrix}$ which is not strictly diagonally dominant.

We can now turn our attention to Schur complements of matrices in $\mathbf{G}^{n,n}$.

Theorem 3.3.7 *If* $A \in \mathbf{G}^{n,n}$ *is partitioned as in* (3.3.8) *with* $\det A_{11} \neq 0$*, then* $A/A_{11} \in \mathbf{G}^{n-k,n-k}$.

Proof. Let $A = [a_{ij}]$ be as prescribed above. We first observe that $a_{ii} \neq 0$ for $i \in \{1, 2, \ldots, k\}$. Indeed, if $a_{ii} = 0$ for some $i \in \{1, 2, \ldots, k\}$, then $0 \geq R_i(A)R_j(A)$ for all $j \in \langle n \rangle \setminus \{i\}$. Also $R_i(A) \neq 0$ since $\det A_{11} \neq 0$ and hence $R_j(A) = 0$ for all $j \in \langle n \rangle \setminus \{i\}$. Thus the i–th column of A_{11} is zero, a contradiction.

Now set $D = \mathrm{diag}(e^{i\arg a_{11}}, \ldots, e^{i\arg a_{kk}}, \delta_{k+1}, \ldots, \delta_n)$, where, for $j \in \{k+1, k+2, \ldots, n\}$,

$$\delta_j = \begin{cases} e^{i\arg a_{jj}} & \text{if } a_{jj} \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Note that $A + \epsilon D \in G_1^{n,n}$, for every $\epsilon > 0$. Suppose that we row reduce $A + \epsilon D$ and obtain the matrix

$$\begin{bmatrix} (|a_{11}| + \epsilon)e^{i\arg a_{11}} & a_{12} & \cdots & a_{1n} \\ 0 & b_{22}(\epsilon) & \cdots & b_{2n}(\epsilon) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b_{n2}(\epsilon) & \cdots & b_{nn}(\epsilon) \end{bmatrix}.$$

Set $B(\epsilon) = [b_{ij}(\epsilon)]$. For $2 \leq i \leq k$ we have

$$b_{ii}(\epsilon) = (|a_{ii}| + \epsilon)e^{i\arg a_{ii}} - \frac{a_{i1}a_{1i}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}}, \tag{3.3.16}$$

$$b_{ij}(\epsilon) = a_{ij} - \frac{a_{i1}a_{1j}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}} \quad (j \neq i;\ j \geq 2). \tag{3.3.17}$$

For $k + 1 \leq i \leq n$,

$$b_{ii}(\epsilon) = a_{ii} + \epsilon\delta_i - \frac{a_{i1}a_{1i}}{(|a_{11}| + \epsilon)e^{i\arg a_{11}}}, \tag{3.3.18}$$

$$b_{ij}(\epsilon) = a_{ij} - \frac{a_{i1}a_{1j}}{(|a_{11}| + \epsilon)e^{i \arg a_{11}}} \quad (j \neq i;\ j \geq 2). \tag{3.3.19}$$

¿From Theorem 3.3.5 we obtain

$$|b_{ii}(\epsilon)||b_{jj}(\epsilon)| > R_i(B(\epsilon))R_j(B(\epsilon)) \quad (i \neq j;\ i,j \geq 2). \tag{3.3.20}$$

The combination of (3.3.16)–(3.3.19) gives

$$\lim_{\epsilon \to 0} |b_{ii}(\epsilon)| = \left| a_{ii} - \frac{a_{i1}a_{1i}}{a_{11}} \right| = |b_{ii}|,\ i \geq 2$$

and

$$\lim_{\epsilon \to 0} |b_{ij}(\epsilon)| = \left| a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}} \right| = |b_{ij}| \quad (i \neq j;\ i,j \geq 2),$$

(recalling B from (3.3.10)). Hence, by taking the limit in (3.3.20) as $\epsilon \to 0$, we have

$$|b_{ii}||b_{jj}| \geq R_i(B)R_j(B) \quad (i \neq j).$$

Thus $B \in \mathbf{G}^{n-1,n-1}$. The theorem follows by noting that A/A_{11} is equal to a Schur complement of B, and by applying the above argument inductively. ■

3.4 A Property of Inverse H-matrices

We will close this chapter by proving a theorem (Theorem 3.4.2) that generalizes a classical result (Theorem 3.4.1) in a way that parallels our generalization of diagonal dominance to double diagonal dominance. We must comment however that the result of Theorem 3.4.1 is implicit in the proof of a result by Fiedler and Pták in [9].

A matrix $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is said to be *strictly diagonally dominant of its row (*resp., *of its column) entries* if

$$|a_{ii}| > |a_{ij}| \quad (\text{resp.}, |a_{ii}| > |a_{ji}|),$$

for all $i \in \langle n \rangle$ and all $j \in \langle n \rangle \setminus \{i\}$ (see [17]). We will call $A = [a_{ij}] \in \mathbf{C}^{n,n}$ *strictly doubly diagonally dominant of its entries* if

$$|a_{ii}||a_{jj}| > |a_{ij}||a_{ji}| \quad (i,j \in \langle n \rangle;\ i \neq j).$$

We will show that the inverse of an H–matrix (and hence of every matrix in $\mathbf{G}_1^{n,n}$) is strictly doubly diagonally dominant of its entries. First we extend Theorem 2.5.12 in [17] from matrices with real entries to matrices with entries from the complex field. As was mentioned above, this result has been proven in [9] implicitly.

Theorem 3.4.1 *If $A \in \mathbf{C}^{n,n}$ is strictly diagonally dominant, then A^{-1} is strictly diagonally dominant of its column entries.*

Proof. By the assumption of strict diagonal dominance, $A = [a_{ij}]$ is invertible. Let $A^{-1} = [\alpha_{ij}]$. Since $\alpha_{ij} = (-1)^{i+j}\text{det}A_{ji}/\text{det}A$, where A_{ji} denotes the submatrix of A obtained by deleting row j and column i, it suffices to prove that $|\text{det}A_{ii}| > |\text{det}A_{ij}|$ for all $j \neq i$. Without loss of generality, we only consider the case where $i = 1$ and $j = 2$. Since A is strictly diagonally dominant, so is A_{ii}, $i \in \langle n\rangle$. Hence $\text{det}A_{ii} \neq 0$. Suppose that $0 < |\text{det}A_{11}| \leq |\text{det}A_{12}|$. Then there is a positive number $\epsilon_0 :\ 0 < \epsilon_0 \leq 1$ such that

$$|\text{det}A_{11}| - \epsilon_0|\text{detA}_{12}| = 0.$$

Hence $\text{det}A_{11} + \epsilon_0 e^{i\varphi_0}\text{det}A_{12} = 0$, where $\varphi_0 = \arg\frac{\text{detA}_{11}}{\text{detA}_{12}} + \pi$. Note that for every $\epsilon \in [0,1]$ and every $\varphi \in \mathbf{R}$,

$$\begin{aligned}
&\text{det}A_{11} + \epsilon e^{i\varphi}\text{det}A_{12} = \\
= \ &\text{det}\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} + \text{det}\begin{bmatrix} a_{21}\epsilon e^{i\varphi} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}\epsilon e^{i\varphi} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \\
= \ &\text{det}\begin{bmatrix} a_{22} + \epsilon e^{i\varphi}a_{21} & a_{23} & \cdots & a_{2n} \\ a_{32} + \epsilon e^{i\varphi}a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n2} + \epsilon e^{i\varphi}a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \triangleq \text{det}C.
\end{aligned}$$

The matrix C is strictly diagonally dominant for every $\epsilon \in [0,1]$ and every $\varphi \in \mathbf{R}$ because

$$|a_{22} + \epsilon e^{i\varphi}a_{21}| \geq |a_{22}| - |a_{21}| > \sum_{k=3}^{n} |a_{2k}|,$$

and because for $i = 3, 4, \ldots, n$, $|a_{ii}|$ dominates the sum of the off-diagonal moduli of C by the triangle inequality. Hence $\text{det}C \neq 0$ or $\text{det}A_{11} + \epsilon e^{i\varphi} detA_{12} \neq 0$. In particular, $\text{det}A_{11} + \epsilon_0 e^{i\varphi_0}\text{det}A_{12} \neq 0$, which is a contradiction. Thus $|\text{det}A_{11}| > |\text{det}A_{12}|$, completing the proof of the theorem. ■

Theorem 3.4.2 *If $A = [a_{ij}] \in \mathbf{C}^{n,n}$ is an H–matrix, then A^{-1} is strictly doubly diagonally dominant of its entries.*

Proof. Since A is an H–matrix, by Theorem 1.3.1(v) there is a positive diagonal matrix $D = \text{diag}(d_1, \cdots, d_n)$ such that AD is strictly diagonally dominant. Note that $(AD)^{-1} = D^{-1}A^{-1} = [d_i^{-1}\alpha_{ij}]$ so from Theorem 3.4.1

$$\left|\frac{1}{d_i}\alpha_{ii}\right|\left|\frac{1}{d_j}\alpha_{jj}\right| > \left|\frac{1}{d_j}\alpha_{ji}\right|\left|\frac{1}{d_i}\alpha_{ij}\right|,$$

which is equivalent to $|\alpha_{ii}||\alpha_{jj}| > |\alpha_{ij}||\alpha_{ji}|,\ i, j \in \langle n\rangle,\ i \neq j.$ ■

Remark 3.4.3 The converse of Theorem 3.4.2 is not necessarily true. For example consider

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Notice that A^{-1} is strictly doubly diagonally dominant of its entries. However A is not an H–matrix since

$$(\mathcal{M}(A))^{-1} = \begin{bmatrix} -1 & -5 & -2 \\ -1 & -2 & -1 \\ -1 & -3 & -1 \end{bmatrix}$$

is not a nonnegative matrix (which is necessary and sufficient for $\mathcal{M}(A)$ to be a nonsingular M–matrix, see Theorem 1.3.1(iii)).

Chapter 4

SUBCLASSES OF H-MATRICES

4.1 Introduction

As we mentioned in Chapter 1, there are many equivalent characterizations of H-matrices. However, except for special cases, none of them can be easily applied in practice. This reason motivates the quest for sufficient conditions, which on some occasions can be more useful in identifying H-matrices (cf [10],[18] and [11]). In this chapter, we collect and compare some subclasses of H-matrices, each of which contains $\mathbf{G}_1^{n,n}$ as a subclass and is determined by some sufficient condition. We also derive two new characterizations for H-matrices in $\mathbf{G}^{n,n}$.

We first need some further notation. We say that N_1 and N_2 constitute *a partition* of $\langle n \rangle$ if $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = \langle n \rangle$. Let then

$$\alpha_i = \sum_{k \in N_1 \setminus \{i\}} |a_{ik}|, \text{ and } \beta_i = \sum_{k \in N_2 \setminus \{i\}} |a_{ik}|. \tag{4.1.1}$$

Notice that $R_i(A) = \alpha_i + \beta_i, \quad i \in \langle n \rangle$.

Finally, let $N \subseteq \langle n \rangle$. We denote by $A(N)$ the principal submatrix of A whose rows and columns are indexed by N.

4.2 M-matrices and their Schur Complements

For future reference, we state a proposition on nonsingular M-matrices and its Schur complements.

Proposition 4.2.1 *Let $A = [a_{ij}] \in \mathbf{Z}^{n,n}$. Then the following are equivalent.*
(1) *A is a nonsingular M-matrix.*
(2) *There exists a partition $\langle n \rangle$ into N_1 and N_2 such that both $A(N_1)$ and $A/A(N_1)$ are nonsingular M-matrices.*
(3) *For any partition of $\langle n \rangle$ into N_1 and N_2 , both $A(N_1)$ and $A/A(N_1)$ are nonsingular M-matrices.*

Proof. The equivalence of (1) and (2) is well-known (e.g. see [1]). Obviously, (3) implies (2), and hence (1). If (1) holds, then for an arbitrary partition N_1 and N_2 of $\langle n\rangle$, it follows from Theorem 3.1 in [1] that both $A(N_1)$ and $A/A(N_1)$ are nonsingular M-matrices. ■

4.3 Some Subclasses of H-matrices

We will now consider some subclasses $\mathbf{C_i}$ of the H-matrices. For notational simplicity we let $\mathbf{C_i}$ denote both the ith class and its defining condition. As usual, we refer to a matrix $A = [a_{ij}]$.

$\mathbf{C_1}$: $|a_{ii}| > R_i(A) \;\; \forall i \in \langle n\rangle$, (s.d.d.).

$\mathbf{C_2}$: $|a_{ii}||a_{jj}| > R_i(A)R_j(A) \;\; \forall i, j \in \langle n\rangle,\; i \neq j$, (s.d.d.d.).

$\mathbf{C_3}$: There is an $i \in \langle n\rangle$ such that $|a_{ii}|(|a_{jj}| - \beta_j) > R_i(A)|a_{ji}|, \;\; \forall j \in \langle n\rangle\backslash\{i\}$.

$\mathbf{C_4}$: ([10]) There exists a partition of $\langle n\rangle$ into N_1 and N_2, such that

$$(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) > \beta_i\alpha_j, \;\; \forall i \in N_1, \; \forall j \in N_2.$$

$\mathbf{C_5}$: ([18])

(a) There exists a partition of $\langle n\rangle$ into $N_1 = \{i_1, i_2, \cdots, i_k\}$ and $N_2 = \langle n\rangle\backslash N_1$, such that $A_1 \triangleq \mathcal{M}(A)(N_1)$ is a nonsingular M-matrix,

(b) For all $i \in N_1$ and $j \in N_2$, $(A_1^{-1}\mathbf{u})_i < \gamma_j$, where $(A_1^{-1}\mathbf{u})_i$ denotes the ith component of $A_1^{-1}\mathbf{u}$ and

$$\begin{aligned} \gamma_j &= \frac{|a_{jj}| - \beta_j}{\alpha_j}, \quad j \in N_2, \\ \mathbf{u} &= (\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_k})^t, \end{aligned}$$

and where $\alpha_j, \beta_j, j \in \langle n\rangle$ are defined in (4.1.1). Also $|a_{jj}| - \beta_j > 0$ and $\gamma_j = \infty$ when $\alpha_j = 0$.

(Note that if N_2 has only one element, $\beta_j = 0$.)

$\mathbf{C_6}$:

(a) There exists a partition of $\langle n\rangle$ into $N_1 = \{i_1, i_2, \cdots, i_k\}$ and $N_2 = \langle n\rangle\backslash N_1$, such that $A_1 \triangleq \mathcal{M}(A)(N_1)$ is a nonsingular M-matrix.

(b) $\mathcal{M}(A)/A_1$ is strictly diagonally dominant.

We will now examine the relation between the above mentioned subclasses of H-matrices.

Theorem 4.3.1 $\mathbf{C_i} \subseteq \mathbf{C_{i+1}}, \quad i \in \langle 5 \rangle$.

Proof. It is obvious that $\mathbf{C_1} \subseteq \mathbf{C_2}$. The proof of $\mathbf{C_2} \subseteq \mathbf{C_3} \subseteq \mathbf{C_4}$ was given in [10]. The relation $\mathbf{C_4} \subseteq \mathbf{C_5}$ has been proved in [11]. We will now prove $\mathbf{C_5} \subseteq \mathbf{C_6}$. For simplicity, we can assume that $\mathcal{M}(A)$ is of the form

$$\begin{bmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & a_{nn} \end{bmatrix}, \tag{4.3.2}$$

where $a_{ij} \geq 0 \; \forall i, j \in \langle n \rangle$. Recall the fact that A is a nonsingular M-matrix iff for any permutation matrix P, $P^t AP$ is a nonsingular M-matrix. Therefore, we can assume that $N_1 = \{1, 2, \cdots, k\}$, $N_2 = \langle n \rangle \backslash N_1$. Then

$$\begin{aligned} \mathcal{M}(A)/A_1 &= \begin{bmatrix} a_{k+1,k+1} & -a_{k+1,k+2} & \cdots & -a_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n,k+1} & -a_{n,k+2} & \cdots & a_{n,n} \end{bmatrix} \\ &- \begin{bmatrix} -a_{k+1,1} & -a_{k+1,2} & \cdots & -a_{k+1,k} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n,1} & -a_{n,2} & \cdots & -a_{n,k} \end{bmatrix} A_1^{-1} \begin{bmatrix} -a_{1,k+1} & -a_{1,k+2} & \cdots & -a_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k,k+1} & -a_{k,k+2} & \cdots & -a_{k,n} \end{bmatrix} \\ &\triangleq [b_{ij}]_{i,j \geq k+1}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} b_{ii} &= a_{ii} - (a_{i1}, \cdots, a_{ik}) A_1^{-1} \mathbf{v}_i \\ b_{ij} &= -a_{ij} - (a_{i1}, \cdots, a_{ik}) A_1^{-1} \mathbf{v}_j \quad i \neq j, \end{aligned}$$

where $\mathbf{v}_j = (a_{1j}, \cdots, a_{kj})^t \geq 0, j \in N_2$. Since $A_1^{-1} \geq 0$, by Theorem 1.3.1(iii) and $\mathbf{v}_j \leq \mathbf{u}$, we have $0 \leq A_1^{-1}\mathbf{v}_j \leq A_1^{-1}\mathbf{u}$, and hence it follows that

$$\begin{aligned} b_{ii} &= a_{ii} - (a_{i1}, \cdots, a_{ik}) A_1^{-1} \mathbf{v}_i \\ &\geq a_{ii} - (a_{i1}, \cdots, a_{ik}) A_1^{-1} \mathbf{u} \\ &= a_{ii} - \sum_{t \in N_1} a_{it} (A_1^{-1}\mathbf{u})_t. \end{aligned}$$

Thus, using the assumptions of $\mathbf{C_5}$, we have that b_{ii}

$$\begin{cases} > a_{ii} - \left(\sum_{t \in N_1} a_{it}\right) \gamma_i = a_{ii} - \alpha_i \gamma_i & \text{if } \alpha_i \neq 0 \\ = a_{ii} & \text{if } \alpha_i = 0 \end{cases}$$
$$\begin{cases} = a_{ii} - (a_{ii} - \beta_i) = \beta_i \geq 0 & \text{if } \alpha_i \neq 0 \\ = a_{ii} > 0 & \text{if } \alpha_i = 0 \end{cases},$$

where we have used some assumptions about $\mathbf{C_5}$. Note that for all $i,\ j \in N_2,\ i \neq j$,

$$b_{ij} = -a_{ij} - (a_{i1}, \cdots, a_{ik})A_1^{-1}\mathbf{v}_j \leq 0.$$

Hence $\mathcal{M}(A)/A_1 \in \mathbf{Z}^{n-k,n-k}$ with all diagonal entries positive. For $i \in N_2$, we have

$$\begin{aligned}
b_{ii} - \sum_{t \in N_2 \backslash \{i\}} |b_{it}| &= a_{ii} - (a_{i1}, \cdots, a_{ik})A_1^{-1}\mathbf{v}_i \\
&\quad - \sum_{t \in N_2 \backslash \{i\}} a_{it} - (a_{i1}, \cdots, a_{ik}) \sum_{t \in N_2 \backslash \{i\}} \left(A_1^{-1}\mathbf{v}_t\right) \\
&= a_{ii} - \sum_{t \in N_2 \backslash \{i\}} a_{it} - (a_{i1}, \cdots, a_{ik})A_1^{-1} \left(\sum_{t \in N_2} \mathbf{v}_t \right) \\
&= a_{ii} - \sum_{t \in N_2 \backslash \{i\}} a_{it} - (a_{i1}, \cdots, a_{ik})A_1^{-1}\mathbf{u} \\
&= (a_{ii} - \beta_i) - (a_{i1}, \cdots, a_{ik})A_1^{-1}\mathbf{u} \\
&\begin{cases} = \gamma_i \alpha_i - \sum_{t \in N_1} a_{it}(A_1^{-1}u)_t & \text{if } \alpha_i \neq 0 \\ = a_{ii} - \beta_i & \text{if } \alpha_i = 0 \end{cases} \\
&\begin{cases} > \gamma_i \alpha_i - \gamma_i \alpha_i = 0 & \text{if } \alpha_i \neq 0 \\ > 0 & \text{if } \alpha_i = 0 \end{cases}.
\end{aligned}$$

Hence $b_{ii} > \sum_{t \in N_2 \backslash \{i\}} |b_{it}|,\ \forall i \in N_2$ and thus $\mathcal{M}(A)/A_1$ is a strictly diagonally dominant M-matrix (see [31, Theorem 5.14]). This completes the proof. ■

Remark 4.3.2 By Proposition 4.2.1, we know that $\mathbf{C_6} \subseteq \mathcal{H}_n$, and hence the proof of the above theorem implies Theorem 1 in [18]. Also Theorem 4.3.1, to some extent, reveals how strong the assumptions in Theorem 1 are when compared to Proposition 4.2.1.

We continue with some illustrative examples.

Example 4.3.3

$$A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 3 & -\frac{4}{3} \\ -\frac{1}{2} & -\frac{1}{2} & -1 & 3 \end{bmatrix}.$$

Take $N_1 = \{1, 2\}, N_2 = \{3, 4\}$ so that

$$A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}.$$

Then

$$\begin{aligned} A/A_1 &= \begin{bmatrix} 3 & -\frac{4}{3} \\ -1 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -\frac{11}{6} \\ -2 & \frac{5}{2} \end{bmatrix} \end{aligned}$$

is strictly diagonally dominant. Notice that for this partition, (b) in $\mathbf{C_5}$ is not satisfied since

$$\begin{aligned} A_1^{-1}\mathbf{u} &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \gamma_3 &= \frac{3 - \frac{4}{3}}{1} = \frac{5}{3}. \end{aligned}$$

Obviously $(A_1^{-1}\mathbf{u})_1 = 2 > \gamma_3$. However, if we take $N_1 = \{1, 3\}$ and $N_2 = \{2, 4\}$, both (a) and (b) in $\mathbf{C_5}$ are satisfied.

Example 4.3.4

$$A = \begin{bmatrix} \frac{3}{2} & -1 & -1 \\ -1 & \frac{4}{3} & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

It is easy to check that for any partition of $\langle 3 \rangle$, (b) in $\mathbf{C_5}$ can not be satisfied, but A is an H-matrix. This example shows that Theorem 1 in [18] is only a sufficient condition for H-matrices. If we take $N_1 = \{1, 2\}$ and $N_2 = \{3\}$, then $A \in \mathbf{C_6}$, and hence $\mathbf{C_5}$ is a proper subset of $\mathbf{C_6}$.

4.4 Two criteria for H-matrices in $\mathbf{G}^{n,n}$

Huang [18] has proved that $\mathbf{C_5}$ is also a necessary condition for diagonal dominance. This result can be generalized to matrices in $\mathbf{G}^{n,n}$. We now proceed to prove this fact. We first need four technical results.

Lemma 4.4.1 *Let*

$$A = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & a_{22} & 0 & \cdots & 0 \\ -a_{31} & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{4.4.3}$$

be doubly diagonally dominant, where $a_{kk} > 0, a_{1k} > 0$ *and* $a_{k1} \geq 0,\ k \in \langle n \rangle$. *Let* $a_{11} < R_1(A)$. *Then* A *is a nonsingular* M*-matrix iff* $J_0(A) \triangleq \{t \in \langle n \rangle \backslash \{1\} : a_{11}a_{tt} > R_1(A)R_t(A)\} \neq \phi$.

Proof. Define $J_1(A) = \{t \in \langle n \rangle \backslash \{1\} : a_{t1} = 0\}$. One of two cases will occur:
Case 1: $J_1(A) = \phi$. In this case, the result follows from Theorem 3.2.4.
Case 2: $J_1(A) \neq \phi$. In this case, $J_0(A) \neq \phi$ because $a_{11}a_{tt} > 0 = R_1(A)R_t(A)$ for $t \in J_1(A)$ and hence necessity becomes trivial. Sufficiency is true without any assumption on $J_0(A)$: Notice that there exists a permutation matrix P which keeps the order of the first row and column such that

$$P^tAP = \begin{bmatrix} A_{11} & * \\ \mathbf{0} & D \end{bmatrix},$$

where D is a positive diagonal matrix and where $A_{11} \triangleq [b_{ij}]_{1 \leq i,j \leq n-|J_1(A)|}$[1] has the form (4.4.3) and satisfies that A_{11} is doubly diagonally dominant and $J_1(A_{11}) = \phi$. Since A is a nonsingular M-matrix iff A_{11} is a nonsingular M-matrix, it follows from case 1 that A_{11} is a nonsingular M-matrix iff $J_0(A_{11}) \neq \phi$. Note that $b_{11}b_{tt} = a_{11}a_{tt} \geq R_1(A)R_t(A) > R_1(A_{11})R_t(A_{11}),\ t \in \langle n \rangle \backslash J_1(A),\ t \neq 1,$ and hence $J_0(A_{11}) \neq \phi$, completing the proof. ■

Lemma 4.4.2 *Let* $A \in \mathbf{Z}^{n,n}$ *be doubly diagonally dominant of the form (4.3.2). Suppose that* $a_{11} < R_1(A)$ *and that for some* $k > 1$, $a_{1k} \neq 0$ *and* $\sum_{t \in \langle n \rangle \backslash \{1,k\}} a_{kt} \neq 0$. *Also suppose that* $A \in \mathcal{H}_n$. *Then* $\mathbf{C_5}$ *holds.*

Proof. Suppose that A is as prescribed in the statement of the lemma. Without loss of generality, assume that $k = 2$, i.e., $a_{12} \neq 0$ and $\beta_2 = \sum_{j=3}^{n} a_{2j} \neq 0$. Take $N_1 = \{1, 2\}$ and $N_2 = \langle n \rangle \backslash N_1$. Then

$$A_1 = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$$

is a nonsingular M-matrix[2] and $\mathbf{u} = (\beta_1, \beta_2)^t$. Hence it suffices to prove that

$$(A_1^{-1}\mathbf{u})_i < \gamma_j = \frac{a_{jj} - \beta_j}{\alpha_j} \quad i \in N_1,\ j \in N_2.$$

The latter are equivalent to

$$(a_{jj} - \beta_j)(a_{11}a_{22} - a_{12}a_{21}) > \alpha_j(a_{22}\beta_1 + a_{12}\beta_2),\ j \in N_2 \qquad (4.4.4)$$

[1] $|J_1(A)|$ denotes the number of elements in $J_1(A)$.
[2] This follows from a well-known property of M-matrices, i.e., that all principal submatrices of a nonsingular M-matrix are nonsingular M-matrices.

and

$$(a_{jj}-\beta_j)(a_{11}a_{12}-a_{12}a_{21}) > \alpha_j(a_{21}\beta_1+a_{11}\beta_2),\ j\in N_2 \tag{4.4.5}$$

where $\alpha_i, \beta_i, i \in \langle n\rangle$, are defined by (4.1.1) (e.g., $\beta_1 = \sum_{t=3}^{n} a_{1t}, \beta_2 = \sum_{t=3}^{n} a_{2t}, \alpha_j = \sum_{t=1}^{2} a_{jt},\ j \in N_2$, etc.). Since A is doubly diagonally dominant and $a_{11} < R_1(A)$, we have $a_{jj} > R_j(A),\ j \in \langle n\rangle\backslash\{1\}$, that is, $a_{jj}-\beta_j > \alpha_j$. Thus

$$\begin{aligned}(a_{jj}-\beta_j)(a_{11}a_{22}-a_{12}a_{21}) &> \alpha_j(a_{11}a_{22}-a_{12}a_{21})\\ &\geq \alpha_j(R_1(A)R_2(A)-a_{12}a_{21})\\ &= \alpha_j(R_1(A)(a_{21}+\beta_2)-a_{12}a_{21})\\ &= \alpha_j((a_{12}+\beta_1)a_{21}-a_{12}a_{21}+\beta_2R_1(A))\\ &= \alpha_j(a_{21}\beta_1+R_1(A)\beta_2)\\ &\geq \alpha_j(a_{21}\beta_1+a_{11}\beta_2),\end{aligned}$$

where in the last step we have applied $R_1(A) > a_{11}$. Hence (4.4.5) follows. Equations (4.4.4) are trivial if $\alpha_j = 0$. Now let $\alpha_j \neq 0$. Then

$$\begin{aligned}(a_{jj}-\beta_j)(a_{11}a_{22}-a_{12}a_{21}) &= (a_{11}a_{jj}-a_{11}\beta_j)\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)\\ &\geq (R_1(A)R_j(A)-a_{11}\beta_j)\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)\\ &\geq (R_j(A)-\beta_j)\left(a_{22}R_1(A)-a_{12}a_{21}\frac{R_1(A)}{a_{11}}\right)\\ &= \alpha_j\left(a_{22}\beta_1+a_{12}\frac{a_{11}a_{22}-a_{21}R_1(A)}{a_{11}}\right)\\ &\geq \alpha_j\left(a_{22}\beta_1+a_{12}(R_2(A)-a_{21})\frac{R_1(A)}{a_{11}}\right)\\ &= \alpha_j\left(a_{22}\beta_1+a_{12}\beta_2\frac{R_1(A)}{a_{11}}\right)\\ &> \alpha_j(a_{22}\beta_1+a_{12}\beta_2),\end{aligned}$$

where we have used the assumptions that $R_1(A) > a_{11}$ in the third step and that $a_{12} > 0$, $\beta_2 \neq 0$ and $R_1(A) > a_{11}$ for the last inequality. This shows (4.4.4). ■

Lemma 4.4.3 *Let $A \in \mathcal{H}_n$ satisfy the hypotheses of Lemma 4.4.1, then* $\mathbf{C_5}$ *holds.*

Proof. Let A be as prescribed. We can assume that $2 \in J_0(A)$, i.e., $a_{11}a_{22} > R_1(A)R_2(A) = a_{21}R_1(A)$. Take $N_1 = \{1,2\}$ and $N_2 = \langle n\rangle\backslash N_1$ so that $A_1 = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22}\end{bmatrix}$

is nonsingular M-matrix and $\mathbf{u} = (\beta_1, 0)^t$. Hence

$$(A_1^{-1}\mathbf{u})_i < \gamma_j = \frac{a_{jj} - \beta_j}{\alpha_j} = \frac{a_{jj}}{a_{j1}} \quad i \in N_1, j \in N_2$$

are equivalent to

$$a_{jj}(a_{11}a_{22} - a_{21}a_{12}) > a_{j1}a_{22}\beta_1, \; j \in N_2 \tag{4.4.6}$$

and

$$a_{jj}(a_{11}a_{22} - a_{12}a_{21}) > a_{j1}a_{21}\beta_1, \; j \in N_2, \tag{4.4.7}$$

where $\alpha_i, \beta_i, \; i \in \langle n \rangle$ are defined by (4.1.1). The proof of (4.4.7) is the same as that of (4.4.5) by noting $\alpha_j = a_{j1}, \beta_j = 0, a_{jj} > \alpha_j, j \geq 2$. For (4.4.6), as in the proof of (4.4.4), we get

$$\begin{aligned} a_{jj}(a_{11}a_{22} - a_{12}a_{21}) &\geq a_{j1}\left(a_{22}\beta_1 + a_{12}\frac{a_{11}a_{22} - R_1(A)R_2(A)}{a_{11}}\right) \\ &> a_{j1}a_{22}\beta_1, \end{aligned}$$

where the last inequality holds since $a_{12} > 0$ and $a_{11}a_{22} > R_1(A)R_2(A) = R_1(A)a_{21}$. Hence (4.4.6) holds. ∎

Lemma 4.4.4 *Let $A \in \mathbf{Z}^{n,n}$ be doubly diagonally dominant of the form (4.3.2). Let $a_{11} < R_1(A)$ and $J_2(A) = \{k \in \langle n \rangle \backslash \{1\} : \; a_{1k} = 0\} \neq \phi$. Also suppose that $A \in \mathcal{H}_n$ and that $\forall l \notin J_2(A), \sum_{t \in \langle n \rangle \backslash \{1,l\}} a_{lt} = 0$. Then $\mathbf{C_5}$ holds.*

Proof. Let A be prescribed. By the hypotheses, there exists a permutation matrix P such that

$$P^tAP = \begin{bmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{bmatrix}, \tag{4.4.8}$$

where A_{11} satisfies the assumptions of Lemma 4.4.1. Since $A \in \mathcal{H}_n$ iff A_{11}, A_{22} are H-matrices, it follows from Lemma 4.4.1 that $J_0(A_{11}) \neq \phi$. For simplicity, we can assume that A has the form (4.4.8) and that A_{11} has the form (4.4.3) with $k \geq 2$ in place of n, where $a_{11}a_{22} > R_1(A)R_2(A) = R_1(A)a_{21}$. Take $N_1 = \{1, 2\}$ and $N_2 = \langle n \rangle \backslash N_1$ and then

$$A_1 = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$$

satisfies (a) of $\mathbf{C_5}$ and $\mathbf{u} = (\beta_1, 0)^t$. Thus

$$(A_1^{-1}\mathbf{u})_i < \gamma_j = \frac{a_{jj} - \beta_j}{\alpha_j} \quad i \in N_1, \; j \in N_2,$$

are equivalent to

$$(a_{jj} - \beta_j)(a_{11}a_{22} - a_{12}a_{21}) > \alpha_j a_{22}\beta_1, \ j \in N_2 \tag{4.4.9}$$

and

$$(a_{jj} - \beta_j)(a_{11}a_{22} - a_{12}a_{21}) > \alpha_j a_{21}\beta_1, \ j \in N_2, \tag{4.4.10}$$

where α_i and β_i, $i \in \langle n\rangle$ are defined by (4.1.1). Here the proof of (4.4.10) is the same as that of (4.4.5). If $\alpha_j = 0$, (4.4.9) is trivial. Assume that $\alpha_j \neq 0$. Then (4.4.9) follows from (4.4.6) for $j \in \{3, \cdots, k\}$. For $j \in \{k+1, \cdots, n\}$, in the proof of (4.4.4) we have

$$\begin{aligned}(a_{jj} - \beta_j)(a_{11}a_{22} - a_{12}a_{21}) &\geq \alpha_j \left(a_{22}\beta_1 + a_{12}\frac{a_{11}a_{22} - a_{21}R_1(A)}{a_{11}} \right) \\ &> \alpha_j a_{22}\beta_1.\end{aligned}$$

The last inequality holds since $a_{12} > 0$ and $a_{11}a_{22} - a_{21}R_1(A) > 0$. ■

We can now prove the promised result.

Theorem 4.4.5 *Let A be in $\mathbf{G}^{n,n}$. Then the following are equivalent.*
(1) $\mathbf{C_5}$ *holds for A.*
(2) $\mathbf{C_6}$ *holds for A.*
(3) $A \in \mathcal{H}_n$.

Proof. Define $J_3(A) = \{i \in \langle n\rangle : |a_{ii}| > R_i(A)\}$. As before, we can assume that $\mathcal{M}(A)$ is of the form (4.3.2).
(1) $\Longrightarrow$ (2) : This follows from Theorem 4.3.1
(2) $\Longrightarrow$ (3) : This follows from Proposition 4.2.1.
(3) $\Longrightarrow$ (1) : Since A is doubly diagonally dominant, one of the following cases must occur:
Case 1: $J_3(A) = \langle n\rangle$, *i.e.*, A is strictly diagonally dominant.
In this case take $N_1 = \{1\}, N_2 = \langle n\rangle\backslash N_1$. Then $A_1 = [a_{11}]$ is a nonsingular M-matrix and

$$A_1^{-1}\mathbf{u} = \frac{R_1(A)}{a_{11}} < 1 < \gamma_j = \frac{a_{jj} - \sum_{t\in N_2\backslash\{j\}} a_{jt}}{a_{j1}}, \ j \in N_2.$$

Hence (a) and (b) of $\mathbf{C_5}$ hold.
Case 2: $a_{ii} \geq R_i(A)$ and $J_3(A) \neq \langle n\rangle$. This is Theorem 2 in [18].
Case 3: There exists a unique $i_0 \in \langle n\rangle$ such that $a_{i_0i_0} < R_{i_0}(A), \ a_{jj} > R_j(A)$. Without loss of generality, assume that $i_0 = 1$. Recall that

$$J_2(A) = \{k \in \langle n\rangle\backslash\{1\} : a_{1k} = 0\}.$$

In this case, we can conclude that one of the following subcases must happen:
(1) there exists some $k \in \langle n \rangle \backslash \{1\}$ such that $a_{1k} \neq 0$ and $\sum_{t \in \langle n \rangle \backslash \{1,k\}} a_{kt} \neq 0$.
(2) $J_2(A) = \phi$ (i.e., $a_{1k} > 0,\ k \in \langle n \rangle$) and $\sum_{t \in \langle n \rangle \backslash \{1,k\}} a_{kt} = 0,\ k \in \langle n \rangle \backslash \{1\}$.
(3) $J_2(A) \neq \phi$ and $\sum_{t \in \langle n \rangle \backslash \{1,k\}} a_{kt} = 0,\ k \in \langle n \rangle \backslash J_2(A),\ k \neq 1$.
These three subcases correspond to Lemmas 4.4.2 ,4.4.3 and 4.4.4, and hence the result follows. ■

Chapter 5

STABILITY OF INCOMPLETE BLOCK LU-FACTORIZATIONS OF H-MATRICES

5.1 Introduction

In Chapter 1 we introduced the notion and scope of incomplete (block) LU-factorizations, as given by Meijerink and van der Vost [22]. In this chapter we will consider a more general method, called the Oliphant-Buleev-Varga or OBV method, which was introduced by Beauwens (cf [4], [5]). Many other methods of incomplete factorizations, such as the method of Axelsson [2] and that of Meijerink and van der Vost [22], can be considered as special cases of the OBV method.

Meijerink and van der Vost [22] primarily studied incomplete point LU-factorizations of M-matrices and obtained some results on numerical stability. Messaoudi [23] studied incomplete *point* LU-factorizations of H-matrices and extended the results in [22] relating to numerical stability. Messaoudi also obtained some new characterizations of H-matrices. In this chapter, we will study OBV factorizations of H-matrices and extend some results given by Messaoudi to OBV factorizations.

Recall that $\alpha^{n,n}$ denotes the set of all $(0,1)$ matrices with all diagonal entries equal to one and that given $A,\ B \in \mathbf{C}^{n,n}$, $A * B$ denotes their Hadamard product. Let $\beta^{n,n}$ be the set of all $(0,1)$ matrices and let E denote the matrix all of whose entries equal one. In the sequel, all matrices involved are partitioned into the form (1.4.7) unless otherwise specified.

Now we describe the OBV method. Consider the following algorithm applied to $A \in \mathbf{C}^{n,n}$ with $\alpha \in \alpha^{n,n}$ and $\beta \in \beta^{n,n}$.

Algorithm 5.1

Set

$$P_{11} = L_{11} = U_{11} = \alpha_{11} * A_{11},$$

$$\begin{aligned} U_{1j} &= \alpha_{1j} * A_{1j}, \quad 1 < j \le m, \\ L_{j1} &= \alpha_{j1} * A_{j1}, \quad 1 < j \le m. \end{aligned}$$

`For` $i = 2, \cdots, m$, `set`

$$P_{ii} = L_{ii} = U_{ii} = \alpha_{ii} * A_{ii} - \beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{si} \right)$$

`For` $j = i+1, \cdots, m$, `set`

$$\begin{aligned} U_{ij} &= \alpha_{ij} * A_{ij} - \beta_{ij} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{sj} \right), \\ L_{ji} &= \alpha_{ji} * A_{ji} - \beta_{ji} * \left(\sum_{s=1}^{i-1} L_{js} K_{ss} U_{si} \right), \end{aligned}$$

`where` K_{ss} `is an approximation to` P_{ss}^{-1}.

Notice that in Algorithm 5.1, P_{ii} is determined from P_{jj} $(j < i)$. Therefore, it makes sense to define K as an approximation of the inverse of a given block diagonal matrix P. Here we briefly mention three major techniques that have been proposed for determining K. For more details, one can refer to [13] and [5].
(1) **Hadamard approximation**. First compute P^{-1} and then take $K = \gamma * P^{-1}$ for some $\gamma \in \beta^{n,n}$.
(2) **von Neumann approximate inverse**. Suppose that P has a sparse (point) factorization $P = (I-L)S(I-U)$, where L, U are strictly lower and upper triangular matrices, respectively, and S is diagonal. Then take $K = (I+U+U^2+\ldots+U^s)S^{-1}(I+L+L^2+\ldots+L^t)$ as an approximation to P^{-1}, where $s,\ t$ are appropriate nonnegative integers.
(3) **Polynomial approximation**. Suppose that the matrix P admits a convergent splitting $P = B - C$ (i.e., $\rho(B^{-1}C) < 1$). Then take $K = (\sum_{i=0}^{s}(B^{-1}C)^i)B^{-1}$ as an approximation to P^{-1}, where s is an appropriate nonnegative integer.

Definition 5.1.1 Let $A \in \mathbf{C}^{n,n}$. The matrix $LP^{-1}U$, where L, U and P are, respectively, the lower block triangular, upper block triangular, and block diagonal matrices computed by Algorithm 5.1, is called *an incomplete block LU-factorization, or a block OBV factorization* of A.

Remark 5.1.2 **(i)** $LP^{-1}U$ can be written as $\tilde{L}U$, where $\tilde{L}$ is a lower block triangular matrix whose ith block diagonal entry equals the identity matrix of order n_i.

(ii) The formulae in Algorithm 5.1 can be written in a matrix form as

$$L + U - P = \alpha * A - \beta * ((P - L)P^{-1}(P - U)). \qquad (5.1.1)$$

(iii) It can be shown that when $\beta = \alpha \in \alpha^{n,n}$ and $K_{ss} = P_{ss}^{-1}$, $s \in \langle m \rangle$, Algorithm 5.1 reduces to Algorithm 1.1. In particular, if $\beta = \alpha = E$ and $K_{ss} = P_{ss}^{-1}$, then both Algorithm 1.1 and Algorithm 5.1 yield the *complete* block LU-factorization, i.e., the usual LU-factorization.

(iv) Since a triple (α, β, K) uniquely determines Algorithm 5.1, we can simply refer to (α, β, K) as an incomplete block LU-factorization or a block OBV factorization of A. Notice that in the triple (α, β, K), $K = diag(K_{11}, \cdots, K_{mm})$ is a block diagonal matrix, where $K_{ii} \in \mathbf{C}^{n_i,n_i}$.

(v) In Algorithm 5.1, α, β could be, in fact, taken to be arbitrary matrices in $\mathbf{C}^{n,n}$. However, since the main function of α and β is to control the sparsity of the factorization, we only consider $\alpha \in \alpha^{n,n}$ and $\beta \in \beta^{n,n}$.

(vi) Let $LP^{-1}U$ be an OBV factorization of A. Define $N = LP^{-1}U - A$. Then $A = LP^{-1}U - N$ is a splitting of A.

Now we can extend some definitions given by Messaoudi [23]. In Algorithm 5.1, if

1. U_{ii} is nonsingular for all $i \in \langle m \rangle$, we say that A admits *a regular block OBV factorization.*

2. U_{ii} is a nonsingular M-matrix for all $i \in \langle m \rangle$, we say that A admits *a positive block OBV factorization.*

3. U_{ii} is nonsingular for all $i \in \langle m \rangle$ and if $\rho((LP^{-1}U)^{-1}N) < 1$, we say that A admits *a convergent block OBV factorization.*

4. U_{ii} is nonsingular for all $i \in \langle m \rangle$ and if $\rho(|(LP^{-1}U)^{-1}N|) < 1$, we say that A admits *an absolutely convergent block OBV factorization.*

Analogously to the definitions in [23], we set

$$\begin{aligned}
\mathcal{F}_n &= \{A \in \mathbf{C}^{n,n} : A \text{ admits a regular block } OBV \text{ factorization for any} \\
&\qquad \alpha \in \alpha^{n,n},\ \beta = \alpha\}. \\
\mathcal{T}_n &= \{A \in \mathbf{C}^{n,n} : \mathcal{M}(A) \text{ admits a positive block } OBV \text{ factorization} \\
&\qquad \text{for any } \alpha \in \alpha^{n,n},\ \beta = \alpha\}. \\
\mathcal{J}_n &= \{A \in \mathbf{C}^{n,n} : A \text{ admits a convergent block } OBV \text{ factorization}
\end{aligned}$$

$$\text{for any } \alpha \in \alpha^{n,n},\ \beta = \alpha\}.$$

$$\mathcal{K}_n = \{A \in \mathbf{C}^{n,n} : A \text{ admits an absolutely convergent block } OBV \text{ factorization for any } \alpha \in \alpha^{n,n},\ \beta = \alpha\}.$$

$\Omega^d(A) = \{B = [B_{ij}] \in \mathbf{C}^{n,n} : diag(|B_{ii}|) = diag(|A_{ii}|) \text{ and } |B_{ij}| \le |A_{ij}| \quad i,j \in \langle m \rangle\}.$

Subsequently, we will show that

$$\mathcal{H}_n = \mathcal{T}_n = \mathcal{K}_n = \mathcal{J}_n^d,$$

where $\mathcal{J}_n^d = \{A \in \mathcal{J}_n :\ \Omega^d(A) \subseteq \mathcal{J}_n\}$.

5.2 Stability

Given $A,\ B \in \Omega^d(A)$, let $LP^{-1}U$ and $L_1P_1^{-1}U_1$ be the block OBV factorizations of A and B, respectively. If $|LP^{-1}| \le |L_1P_1^{-1}|$, then we say that the block OBV factorization of A is at least as stable as that of B. This definition of stability was given in [22] implicitly. We will consider the cases where either $B = A$ or $B = \mathcal{M}(A)$. In particular, we will focus on the stability of the OBV factorizations of an H-matrix and its comparison matrix.

Meijerink and van der Vost [22] studied the stability of incomplete *point LU*-factorizations of M-matrices. Messaoudi [23] generalized the corresponding results to H-matrices. In this section, we will further generalize some results in the above papers to block OBV factorizations of an H-matrix. Let's first recall some of their results.

Theorem 5.2.1 ([22]) *Let $A \in \mathbf{C}^{n,n}$ be a nonsingular M-matrix. Then the incomplete point LU-factorization $LP^{-1}U$ of A is "at least as stable" as the complete factorization $A = L_1U_1$ without pivoting, i.e., $|LP^{-1}| \le |L_1|$.*

Theorem 5.2.2 ([23]) *Let $A \in \mathbf{C}^{n,n}$ be an H-matrix and let $\alpha \in \alpha^{n,n}$ be given. The incomplete point LU-factorization using α and without pivoting of A is at least as stable as the corresponding factorization of $\mathcal{M}(A)$.*

For the existence of block OBV factorizations of nonsingular M-matrices and H-matrices, one can refer to [5] and [13].

Now we turn our attention to the block OBV factorization. The following theorem is our main result in this section, extending Theorem 5.2.2 above. For notational simplicity, we will momentarily use A^0 (instead of $\mathcal{M}(A)$) to denote the comparison matrix of A.

Theorem 5.2.3 *Let $A \in \mathbf{C}^{n,n}$ be an H-matrix and A^0 its comparison matrix. Let (α, β, K) and (α^0, β^0, K^0) be respectively the block OBV factorizations of A and A^0, where $\alpha,\ \alpha^0 \in \alpha^{n,n},\ \ \beta,\ \beta^0 \in \beta^{n,n},\ K$ and K^0 satisfy $I \le \alpha \le \alpha^0,\ \beta \le \beta^0,\ |K_{ss}| \le K^0_{ss} \le (P^0_{ss})^{-1}$, for all $s \in \langle m\rangle$. Then the factorization (α, β, K) of A is at least as stable as the factorization (α^0, β^0, K^0) of A^0 without block entry pivoting.*

Note: The following proof is parallel to that of Theorem 3.2 in [13].

Proof. Given two triples (α, β, K) and (α^0, β^0, K^0), by Algorithm 5.1 we can obtain two splittings $A = LP^{-1}U - N$ and $A^0 = L^0(P^0)^{-1}U^0 - N^0$. It follows from [13, Theorem 3.1] that L^0, P^0 and U^0 are all nonsingular M-matrices. Let $\tilde{L} = LP^{-1} = [\tilde{L}_{ij}]$ and $\tilde{L}^0 = L^0(P^0)^{-1} = [\tilde{L}^0_{ij}]$. Then $\tilde{L}_{ii} = \tilde{L}^0_{ii} = I_{n_i},\ i \in \langle m\rangle$. Note that $\tilde{L}_{ij} = L_{ij}P^{-1}_{jj}$ and $\tilde{L}^0_{ij} = L^0_{ij}(P^0_{ij})^{-1}$. To prove the theorem, according to the definition of stability, we only need to show that $|\tilde{L}_{ij}| \le |\tilde{L}^0_{ij}|,\ i,\ j \in \langle m\rangle,\ i > j$. Thus it is sufficient to prove that

$$|L| \le |L^0| \text{ and } |P^{-1}| \le (P^0)^{-1}.$$

Set $offdiag(A) = A - diag(A)$. From Algorithm 5.1, we have

$$\begin{aligned} |L_{11}| &= |P_{11}| = |U_{11}| = \alpha_{11} * |A_{11}| \\ &= |L^0_{11}| = |P^0_{11}| = |U^0_{11}|, \\ |L_{j1}| &= \alpha_{j1} * |A_{j1}| = |L^0_{j1}|, \\ |U_{1j}| &= \alpha_{1j} * |A_{1j}| = |U^0_{1j}|, \quad i < j \le m. \end{aligned} \tag{5.2.2}$$

Thus the inequalities

$$\begin{aligned} |diag(P_{tt})| &\ge |diag(P^0_{tt})|, \\ |offdiag(P_{tt})| &\le |offdiag(P^0_{tt})|, \\ |L_{jt}| \le |L^0_{jt}|\ , &\qquad |U_{tj}| \le |U^0_{tj}|, \quad t < j \le m. \end{aligned} \tag{5.2.3}$$

hold for $t = 1$. Now assume that (5.2.3) is true for $1 \le t \le i-1$. Then for $t = i$, using Algorithm 5.1 we have

$$\begin{aligned} |diag(P_{ii})| &\ge |diag(\alpha_{ii}) * A_{ii})| - \left|diag\left(\beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is}K_{ss}U_{si}\right)\right)\right| \\ &\ge |diag(\alpha_{ii} * A_{ii})| - \left|diag\left(\beta_{ii} * \left(\sum_{s=1}^{i-1} |L_{is}||K_{ss}||U_{si}|\right)\right)\right| \\ &\ge |diag(\alpha^0_{ii} * A^0_{ii})| - diag\left(\beta^0_{ii} * \left(\sum_{s=1}^{i-1} |L^0_{is}||K^0_{ss}||U^0_{si}|\right)\right) \\ &\ge diag(\alpha^0_{ii} * A^0_{ii}) - diag\left(\beta^0_{ii} * \left(\sum_{s=1}^{i-1} L^0_{is}K^0_{ss}U^0_{si}\right)\right) \\ &= diag(P^0_{ii}), \end{aligned}$$

where we have applied the fact that L^0, U^0 are nonsingular M-matrices, $K_{ss}^0 \geq 0,\ s \in \langle m \rangle$ and $diag(\alpha) = diag(\alpha^0) = I$. Also,

$$
\begin{aligned}
|offdiag(P_{ii})| &\leq |offdiag(\alpha_{ii} * A_{ii})| + \left| offdiag\left(\beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is}K_{ss}U_{si}\right)\right)\right| \\
&\leq offdiag\left(-\alpha_{ii} * A_{ii}^0 + \beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is}^0 K_{ss}^0 U_{si}^0\right)\right) \\
&\leq offdiag\left(-\alpha_{ii}^0 * A_{ii}^0 + \beta_{ii}^0 * \left(\sum_{s=1}^{i-1} L_{is}^0 K_{ss}^0 U_{si}^0\right)\right) \\
&= -offdiag(P_{ii}^0) = |offdiag(P_{ii}^0)|,
\end{aligned}
$$

and

$$
\begin{aligned}
|L_{ji}| &= \left| \alpha_{ji} * A_{ji} - \beta_{ji} * \left(\sum_{s=1}^{i-1} L_{is}K_{ss}U_{sj}\right)\right| \\
&\leq \alpha_{ji} * |A_{ji}| + \beta_{ji} * \left(\sum_{s=1}^{i-1} |L_{is}||K_{ss}||U_{sj}|\right) \\
&\leq -\alpha_{ji}^0 * A_{ji}^0 + \beta_{ji}^0 * \left(\sum_{s=1}^{i-1} L_{is}^0 K_{ss}^0 U_{sj}^0\right) \qquad (5.2.4) \\
&= -L_{ji}^0 = |L_{ji}^0|, \quad i < j \leq m.
\end{aligned}
$$

Similarly,

$$|U_{ij}| \leq |U_{ij}^0|, \quad i < j \leq m.$$

Therefore we have proved by induction that (5.2.3) is true for all $t \in \langle m \rangle$. ¿From above, we know that $\mathcal{M}(P) = diag(\mathcal{M}(P_{11}), \cdots, \mathcal{M}(P_{mm})) \geq P^0 = diag(P_{11}^0, \cdots, P_{mm}^0)$. Since P^0 is a nonsingular M-matrix, so is $\mathcal{M}(P)$ (cf [17, Theorem 2.5.4(a)]) and hence P is an H-matrix. Moreover, from a well-known result of Ostrowski (e.g. cf [13]) we have

$$|P^{-1}| \leq \mathcal{M}(P)^{-1} \leq (P^0)^{-1}. \qquad (5.2.5)$$

The combination of (5.2.4) and (5.2.5) implies that $|\tilde{L}| \leq |(\tilde{L}^0)^{-1}|$. This completes the proof. ■

In Theorem 5.2.3, take $\alpha^0 = \beta^0 = E$ and $K^0 = (P^0)^{-1}$. Then (α^0, β^0, K^0) is the complete block OBV factorization of A^0, i.e., $A^0 = L^0(P^0)^{-1}U^0$ and hence we have the following result.

Corollary 5.2.4 *The block OBV factorization (α, β, K) of an H-matrix satisfying $\alpha \in \alpha^{n,n}$, $\beta \in \beta^{n,n}$ and $|K| \leq (P^0)^{-1}$ is at least as stable as the complete block factorization of its comparison matrix without block entry pivoting.*

5.3 Some Characterizations of H-matrices

Messaoudi [23] gave some new characterizations of H-matrices in terms of the sets $\mathcal{T}, \mathcal{J}, \mathcal{K}$ and incomplete point LU-factorizations. We now show that those theorems also hold for block OBV factorizations. We begin with a lemma.

Lemma 5.3.1 Let $A \in \mathbf{Z}^{n,n}$ be partitioned as in (1.4.7). Then the following are equivalent.

(i) A is a nonsingular M-matrix.

(ii) There exist lower and upper block triangular matrices L and U respectively, such that $A = LP^{-1}U$, where $P = diag(P_{11}, \cdots, P_{mm})$ and where $L_{ii} = P_{ii} = U_{ii},\ i \in \langle m \rangle$, are nonsingular M-matrices.

Proof. ¿From Theorem 4.2 in [5], it follows that (i) implies (ii). Conversely, let $\tilde{L} = LP^{-1} = [R_{ij}]$ and $\tilde{U} = U = [U_{ij}]$. We will show that both $\tilde{L}$ and $\tilde{U}$ are nonsingular M-matrices. We first prove that $\tilde{L},\ \tilde{U} \in \mathbf{Z}^{n,n}$ by induction on $i + j,\ 1 \leq i, j \leq m$ (the proof is similar to that of Theorem 6.2.3 in [6]). Notice that $diag(\tilde{L}) = diag(I_{n_1}, \cdots, I_{n_m}),\ diag(U) = diag(P_{11}, \cdots, P_{mm}) \in \mathbf{Z}^{n,n}$. Hence we only need to show that $offdiag(\tilde{L}) \leq 0$, and $offdiag(U) \leq 0$. If $i + j = 3$, the equalities $R_{21} \leq 0,\ U_{12} \leq 0$ follow from $A_{12} = R_{11}U_{12}$ and $A_{21} = R_{21}U_{11}$, since $U_{12} = R_{11}^{-1}A_{12} = A_{12} \leq 0$, and $R_{21} = A_{21}U_{11}^{-1} \leq 0$ $(U_{11}^{-1} \geq 0)$. Let $i + j > 3,\ i \neq j$, and suppose the inequalities $R_{kl} \leq 0$ and $U_{kl} \leq 0,\ k \neq l$, are valid if $k + l < i + j$. Then if $j < i$, we have the relation

$$A_{ij} = R_{ij}U_{jj} + \sum_{l<j} R_{il}U_{lj}. \tag{5.3.6}$$

Since $i + l < i + j, l + j < i + j$, by the induction we have $R_{il} \leq 0$, and $U_{lj} \leq 0$. Hence it follows from (5.3.6) that $\sum_{l<j} R_{il}U_{lj} \geq 0$. Thus since $A_{ij} \leq 0$ and $U_{jj}^{-1} \geq 0,\ R_{ij} = \left(A_{ij} - \sum_{l<j} R_{il}U_{lj}\right) U_{jj}^{-1} \leq 0$. Similarly if $i < j,\ U_{ij} \leq 0$. Hence $\tilde{L} \in \mathbf{Z}^{n,n}$ and $\tilde{U} = U \in \mathbf{Z}^{n,n}$. Notice that since $\tilde{L}$ is a lower (point) triangular matrix in $\mathbf{Z}^{n,n}$ with all diagonal entries equal to one, it is a nonsingular M-matrix, i.e., $\tilde{L}^{-1} \geq 0$. Since, by Algorithm 5.1, $P = diag(U) = diag(U_{11}, \cdots, U_{mm})$, we can write $U = \tilde{U}_1 P$, where

$$\tilde{U}_1 = \begin{bmatrix} I_{n_1} & U_{12}P_{22}^{-1} & \cdots & U_{1m}P_{mm}^{-1} \\ 0 & I_{n_2} & \cdots & U_{2m}P_{mm}^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{n_m} \end{bmatrix} \in \mathbf{Z}^{n,n}$$

is also an upper triangular matrix with all diagonal entries equal to one. Hence $\tilde{U}_1$ is also a nonsingular M-matrix. Since P is a nonsingular M-matrix, $P^{-1} \geq 0$, and hence

$A^{-1} = P^{-1}\tilde{U}_1^{-1}\tilde{L}^{-1} \geq 0$, i.e., A is a nonsingular M-matrix (cf Theorem 1.3.1(iii)). ■

Theorem 5.3.2 $\mathcal{H}_n = \mathcal{T}_n$.

Proof. The relation $\mathcal{H}_n \subseteq \mathcal{T}_n$ follows from Theorem 4.3(3) in [5]. To prove $\mathcal{T}_n \subseteq \mathcal{H}_n$, let $A \in \mathcal{T}_n$. Then $\mathcal{M}(A)$ admits a *positive* block OBV factorization for any $\alpha \in \alpha^{n,n}$, $\beta = \alpha$. If we choose $\alpha = \beta = E$, then $\mathcal{M}(A)$ admits a complete *positive* block factorization, i.e., $\mathcal{M}(A) = LP^{-1}U$, where $P_{ii} = L_{ii} = U_{ii}$ are nonsingular M-matrices $i \in \langle m \rangle$. It then follows from Lemma 5.3.1 that $\mathcal{M}(A)$ is a nonsingular M-matrix, i.e., $A \in \mathcal{H}_n$. ■

Let $A \in \mathbf{C}^{n,n}$. If we take $\alpha = I$, then from Algorithm 5.1 we obtain the *block* Jacobi splitting $A = D_A - B$, where $D_A = diag(a_{11}, \cdots, a_{nn})$. This splitting is the same as the *point* Jacobi splitting of A. The combination of this fact and the proofs of Theorem 4.2 in [13] and Theorem 3.4 and Theorem 3.6 in [23] gives the following result.

Theorem 5.3.3 $\mathcal{H}_n = \mathcal{K}_n = \mathcal{J}_n^d$.

5.4 Answer to an Open Question

Recall that if $n_i = 1$, $i \in \langle m \rangle = \langle n \rangle$, and if $\beta = \alpha$ and $K = P^{-1}$, then the OBV method Algorithm 5.1 reduces to the incomplete *point* LU-factorization Algorithm 1.1. In this section, let us consider the special case in which

$$\tilde{\mathcal{J}}_n = \{A \in \mathbf{C}^{n,n} : A \text{ admits a convergent incomplete } \textit{point} \\ LU\text{-factorization (Algorithm 1.1) for any } \alpha \in \alpha^{n,n}\}.$$

It is obvious that $\mathcal{J}_n \subseteq \tilde{\mathcal{J}}_n$. ¿From Theorem 4.2 in [13] we know that $\mathcal{H}_n \subseteq \mathcal{J}_n$ and hence $\mathcal{H}_n \subseteq \tilde{\mathcal{J}}_n$. Messaoudi [23] posed the question whether $\tilde{\mathcal{J}}_n \subseteq \mathcal{H}_n$. We have observed here that $\tilde{\mathcal{J}}_2 \subseteq \mathcal{H}_2$. However, the general inclusion $\tilde{\mathcal{J}}_n \subseteq \mathcal{H}_n$ is not true for all $n > 2$, as the following example shows.

Example 5.4.1 *Let*

$$A = \begin{bmatrix} 2 & -1+i & -1 \\ 1 & 3 & -1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Since $det(\mathcal{M}(A)) = -1.0711 < 0$, $\mathcal{M}(A)$ is not a nonsingular M-matrix (cf [6]) and hence A is not an H-matrix. It is tedious to verify the fact that A admits a convergent incomplete point LU-factorizations for any $\alpha \in \alpha^{3,3}$. Therefore, we put the details of this verification in Appendix II. In Appendix I, we have written two Matlab functions, one for Algorithm 1.1, the other for Algorithm 5.1. The testing table in Appendix II can be obtained from either of those two algorithms. The above example can be extended to the case where $n > 3$ by taking

$$\hat{A} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & I_{n-3} \end{bmatrix}.$$

As $\hat{A}$ is the direct sum of the identity and a matrix in $\tilde{J}_3$ but not in $\mathcal{H}_3$, it follows that $\hat{A}$ is in $\tilde{\mathcal{J}}_n$ but not in $\mathcal{H}_n$.

Chapter 6

CONCLUSION

This thesis is devoted to the study of generalizations of diagonal dominance, which include double diagonal dominance and generalized diagonal dominance.

Our study of double diagonal dominance is motivated by Pang's work [26]. Using the directed graph of a matrix, we characterized H-matrices in $\mathbf{G}_2^{n,n}$ (cf Theorem 3.2.4). We also extended a well-known result on diagonal dominance to double diagonal dominance, that is, Schur complements of a doubly diagonally dominant matrix are also doubly diagonally dominant (cf Theorem 3.3.7). We discussed subclasses of H-matrices , each of which contains $\mathbf{G}_1^{n,n}$ as a subclass, and studied their relationships. We especially obtained two characterizations of H-matrices in $\mathbf{G}^{n,n}$, and corrected an inaccurate claim in [26].

Our interests in studying H-matrices include searching for new algorithms and criteria. By Theorem 1.3.1, $A \in \mathcal{H}_n$ iff there is a positive vector x such that $\mathcal{M}(A)x > 0$ or iff (1.3.3) holds. However, for a given matrix, it is not easy to find such a vector x or show that one does not exist. In Chapter 2, we gave Algorithm $\mathbb{H}$ and proved that it is efficient and computationally convenient. Some numerical examples, given in Chapter 2, show that in certain cases, Algorithm $\mathbb{H}$ requires less operations than a direct method. Since we did not obtain an upper bound on the number of iterations, we could not estimate the number of operations required by Algorithm $\mathbb{H}$.

Finally, as an application of H-matrices in the study of iterative solutions of linear systems, we considered the *block* OBV factorizations of an H-matrix. We showed that under certain conditions the construction of an OBV factorization of an H-matrix is at least as "stable" as the construction of the OBV-factorization of its comparison matrix. We also obtained some new characterizations of H-matrices in terms of the sets $\mathcal{F}_n, \mathcal{T}_n, \mathcal{J}_n$ and $\mathcal{K}_n$. All these results extend the corresponding results in [23]. Lastly we showed by a counterexample that a matrix which admits a convergent incomplete point LU-factorization for any $\alpha \in \alpha^{n,n}$ is not necessarily an H-matrix, which answers a question posed in [23].

APPENDIX I: MATLAB FUNCTIONS

I.1 Matlab Function For Algorithm ℍ

```
function [diagonal,m] = hmat(a, theta, maxit)
% INPUT: a=square matrix, theta=parameter of re-distribution
% maxit=maximum number of iterations allowed
% OUTPUT: m=number of iterations performed,
% diagonal=diagonal matrix d so that ad is strictly diag.  dominant
%        =[ ] if a is not an H-matrix)
n= size(a,1); diagonal=eye(n); m=1; one=ones(1,n); stoppage=0;
if (nargin==1); theta=.9; maxit=100; end
if (nargin==2) maxit=100; end
if (1-all(diag(a)))
  stoppage=1; diagonal=[ ]; m=m-1; 'Input is NOT an H-matrix',
end
while (stoppage==0 & m<maxit+1)
     for i=1:n
         r(i)=sum(abs(a(i,1:n)))-abs(a(i,i));
         if (abs(a(i,i))>r(i))
             d(i)=((1-theta)*a(i,i)+theta*r(i))/(abs(a(i,i)));
         else
             d(i)=1;
         end
     end
     if (d==one)
         stoppage=1; diagonal=[ ]; 'Input is NOT an H-matrix',
     elseif (d<one)
         stoppage=1; 'Input IS an H-matrix',
     else
         for i=1:n
```

```
            diagonal(i,i)=diagonal(i,i)*d(i);
        end
        a=a*diag(d); m=m+1;
    end
end
if (m==maxit+1 & stoppage==0)
    diagonal=[ ]; m=m-1;
    'Inconclusive:  Increase "theta in (0,1)" or increase "maxit"',
end
```

I.2 Matlab Function For Algorithm 1.1

```
function [l,u,r]=LUF(A,V,X)
% This function implements the incomplete block
% LU factorizaton of A.
% The matrix X is a (0,1) matrix, the (i,j)-th
% block is nonzero iff
% the nonzero block entry in (i,j) position is accepted
% thoughout Gaussian elimination.
% V is a vector of dimension m and V(i)
% is the order of the i-th diagonal
% block entry of A. r is the spectral radius of
% the matrix (l*u)^-1*N,
% where N = l*u - A.
l = eye(size(A)); n = size(A,1);
m = length(V);
b = A;
A = A .* X; % Hadamard product.
for i = 1:m
    v(i) = 0;
    for k = 1:i
        v(i) = v(i) + V(k);
    end
end
for r = 1:m-1 % Gaussian elimination at r-th step.
    if r > 1
        ir = v(r-1) + (1:V(r)); % Notice the brackets (1:V(r)).
    else
    ir = 1:V(r);
        end
```

```
    for i = r+1:m
        ix = v(i-1) + (1:V(i));
        if any(any(X(ix,ir)))
            l(ix,ir) = A(ix,ir) * inv(A(ir,ir));
            for j = r+1:m
                iy = v(j-1) + (1:V(j));
                if any(any(X(ix,iy))) & any(any(X(ir,iy)))
                    A(ix,iy) = A(ix,iy) - l(ix,ir)*A(ir,iy);
                end
            end
        end
    end
end
for i=1:m
    if i > 1
        ix = v(i-1) + (1:V(i));
    else
        ix = 1:v(1);
    end
        for j = i:m
            if j > 1
                iy = v(j-1) +(1:V(j));
            else
                iy = 1:v(1);
            end
            u(ix,iy) = A(ix,iy);
        end
end
N = l*u -b;
T = inv(l*u);
r=max(abs(eig(T*N)));
```

I.3 Matlab Function For Algorithm 5.1

```
function [L,U,r] = LUFF(A,V,X)
% This function is used to obtain the block OBV
% factorization of a matrix A.
% V is a vector of dimension m whose i-th component is the order
% of the i-th diagonal block entry of A.
```

```
% It returns a triplet (L, U, r) where L is a lower block matrix
% with all diagonal blocks equal to I and U is an upper block
triangular
% matrix.  r is the spectral radius of matrix (L*U)^{-1}*N, where
% N=L*U-A.
n = size(A,1);
m = length(V);
B = A;
A = X .* A;
ir = 1:V(1);  P(ir,ir) = A(ir,ir);  L(ir,ir) = A(ir,ir);  U(ir,ir)
= A(ir,ir);
for i = 1:m
     v(i) = 0;
     for k = 1:i
         v(i) = v(i) + V(k);
     end
end
for j = 2:m
     jy = v(j-1) + (1:V(j));
     U(1:v(1),jy) = A(1:v(1),jy);
     L(jy,1:v(1)) = A(jy,1:v(1));
end
for i = 2:m
     ix = v(i-1) +(1:V(i));
     Q(ix,ix) = A(ix,ix);
     for s=1:i-1
         if s == 1
             ss = 1:v(1);
         else
             ss = v(s-1) + (1:V(s));
         end
         Q(ix,ix) = Q(ix,ix) - L(ix,ss)*inv(P(ss,ss))*U(ss,ix);
         % Assume p(ss,ss) is nonsingular.
     end
     P(ix,ix) = X(ix,ix) .* Q(ix,ix);
     L(ix,ix) = X(ix,ix) .* Q(ix,ix);
     U(ix,ix) = X(ix,ix) .* Q(ix,ix);
     for j = i+1:m
         jy = v(j-1) + (1:V(j));
         U(ix,jy) = A(ix,jy);
```

```
        L(jy,ix) = A(jy,ix);
        for s=1:i-1
            if s == 1
                ss = 1:v(1);
            else
                ss = v(s-1) + (1:V(s));
            end
            U(ix,jy) = U(ix,jy) -L(ix,ss) *inv(P(ss,ss))* U(ss,jy);
            L(jy,ix) = L(jy,ix) -L(jy,ss) *inv(P(ss,ss))* U(ss,ix);
        end
        U(ix,jy) = X(ix,jy) .* U(ix,jy);
        L(jy,ix) = X(jy,ix) .* L(jy,ix);
    end
end
L = L * inv(P);
T = L * U;
N = T - B;
r = max(abs(eig(inv(T)*N)));
```

APPENDIX II
TEST TABLE

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.9252
2	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2.0 & -1.0+1.0i & 0 \\ 0 & 3.0 & 0 \\ 0 & 0 & 3.0 \end{bmatrix}$	0.9131
3	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.6350

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
4	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.9051
5	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.7035
6	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.8782
7	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.7196
8	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.5270

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
9	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.8511
10	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.7387
11	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.5270
12	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.6334
13	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.7382

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
14	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.6497
15	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.8176
16	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.6934
17	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.5012
18	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.5987

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
19	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.5430
20	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.6841
21	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.7099
22	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.6497
23	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.9757

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
24	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.4030
25	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.4413
26	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.4657
27	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.4752
28	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.7957

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
29	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.5599
30	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -0.5 \\ 0 & 0 & 3 \end{bmatrix}$	0.4657
31	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$	0.4072
32	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.6032
33	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.4657

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
34	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.4030
35	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.4413
36	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.9757
37	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.56+0.8i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.6326
38	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.6895

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
39	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.6806
40	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1-0.33i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.9757
41	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.5590
42	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.4648
43	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.4680

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
44	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$	0.6609
45	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & 0.5 \\ 0 & 0 & 3 \end{bmatrix}$	0.4807
46	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.3344
47	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1-0.33i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.3584
48	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.56+0.8i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.3850

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
49	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1-0.33i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 4-0.33i \end{bmatrix}$	0.5378
50	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.56+0.08i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & -1 \\ 0 & 0 & 3.56+0.08i \end{bmatrix}$	0.5294
51	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & -1 \\ 0 & 0 & 3 \end{bmatrix}$	0.8124
52	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 4.67 \end{bmatrix}$	0.4237
53	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 3.67 \end{bmatrix}$	0.4680

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
54	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.88-0.16i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 3 \end{bmatrix}$	0.4807
55	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.667 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.2819
56	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -0.5 \\ 0 & 0 & 3.33 \end{bmatrix}$	0.3983
57	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -0.5 \\ 0 & 0 & 4 \end{bmatrix}$	0.3584
58	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & -0.5 \\ 0 & 0 & 4 \end{bmatrix}$	0.1118

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
59	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.56-0.08i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & -0.5 \\ 0 & 0 & 3.28+0.04i \end{bmatrix}$	0.3555
60	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1-0.33i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 5-0.33i \end{bmatrix}$	0.1052
61	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.88-0.16i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & 0 \\ 0 & 0 & 4 \end{bmatrix}$	0.1118
62	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.88-0.16i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & 0 \\ 0 & 3.5-0.5i & -1 \\ 0 & 0 & 3.88-0.16i \end{bmatrix}$	0.1457
63	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.67 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -0.5 \\ 0 & 0 & 4.33 \end{bmatrix}$	0.2720

(Continued)

#	$\alpha \in \alpha^{3,3}$	L	U	$\rho((LU)^{-1}N)$
64	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1 & 0.88-0.16i & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & -1+i & -1 \\ 0 & 3.5-0.5i & -0.5 \\ 0 & 0 & 4.44-0.08i \end{bmatrix}$	0.0000

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LIST OF SYMBOLS

$\triangleq$ denotes a definition

$A = [a_{ij}] \in \mathbf{C}^{n,n}$: an $n \times n$ complex matrix with entries a_{ij}

$\langle n \rangle = \{1, \cdots, n\}$

$\mathbf{Z}^{n,n}$: the $n \times n$ real matrices with nonpositive off diagonal entries

$\sigma(A)$ is the spectrum of A

$\rho(A)$ is the spectral radius of A

$A \geq 0$ denotes an entrywise nonnegative array A

$R_i(A) = \sum_{k \neq i} |a_{ik}|$

$|A| = [|a_{ik}|]$

$\mathbf{G}^{n,n}$: the class of $n \times n$ doubly diagonally dominant matrices

$\mathbf{G}_1^{n,n}$: the class of $n \times n$ strictly doubly diagonally dominant matrices

$\mathbf{G}_2^{n,n}$: the class of $n \times n$ irreducibly doubly diagonally dominant matrices

$\mathcal{M}(A)$ is the comparison matrix of A

e is the column vector all of whose entries are ones

E is the matrix all of whose entries are ones

$diag(x)$: diagonal matrix with diagonal entries equal to the entries of x

$diag(A)$: diagonal matrix with diagonal entries equal to the diagonal entries of A

$\mathcal{H}_n$: the class of $n \times n$ H-matrices

$\mathcal{D}_A$: diagonal matrices D such that AD is strictly diagonally dominant

$A * B$ is the Hadamard (entrywise) product of A and B

$\mathbf{N}_1(X) = \{i \in \langle n \rangle \ : \ |x_{ii}| > R_i(X)\}$

$\mathbf{N}_2(X) = \langle n \rangle \setminus \mathbf{N}_1(X)$

$\alpha^{n,n}$: the class of $n \times n$ $(0,1)$ matrices with diagonal entries equal to 1.

$\beta^{n,n}$: the class of $n \times n$ $(0,1)$ matrices

$\Omega^d(A) = \{B = [B_{ij}] \in \mathbf{C}^{n,n} : diag(|B_{ii}|) = diag(|A_{ii}|) \text{ and } |B_{ij}| \leq |A_{ij}| \quad i, j \in \langle m \rangle\}$