Algorithmic Detection and Construction of N-matrices

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Abstract

N-matrices are real $n \times n$ matrices all of whose principal minors are negative. We provide (i) an $O(2^n)$ test to detect whether or not a given matrix is an N-matrix, and (ii) a characterization of N-matrices, leading to the recursive construction of every N-matrix.

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1 Introduction and Motivation

This work concerns N-matrices, that is, real $n \times n$ matrices, $A \in \mathbb{R}^{n \times n}$, all of whose principal minors are negative.

In prior discussions of N-matrices, their resemblance to P-matrices, which are matrices all of whose principal minors are positive, invariably comes up first. Indeed, P-matrices are widely studied since they contain many classes of matrices, such as the positive definite matrices and the M-matrices; they find applications in mathematical programming, the study of univalence and complexity theory (see e.g., [1, 13]). N-matrices find similar applications and possess properties analogous to P-matrices; they were introduced in [10] and have been studied in [14] and [17].

Among the motivating factors for studying N-matrices is their connection to univalence (injectivity of differential maps in $\mathbb{R}^n$) and their role in the Linear Complementarity Problem. In addition, as it is evident in the existing theory of N-matrices and will be reinforced by the results herein, it is illuminating to identify and compare the effects of having signed principal minors in the two cases of N-matrices and P-matrices. There are similarities, distinctions, but also some unexpected connections between the two classes. Such instances will surface in our study of how to (i) detect N-matrices efficiently (Section 3), and (ii) construct all the N-matrices (Section 4). Some background material and basic properties of N-matrices are reviewed in Section 2, which will help us develop and appropriately frame the results. Matlab implementations of algorithms for the detection of N-matrices and P-matrices are included in Section 5 for the reader’s convenience.

2 Background, Notation and Context

For a positive integer $n$, let $\langle n \rangle = \{1, 2, \ldots, n\}$. For $\alpha \subseteq \langle n \rangle$, $|\alpha|$ denotes the cardinality of $\alpha$ and $\alpha^c = \langle n \rangle \setminus \alpha$. For $\alpha \subseteq \langle n \rangle$ with $|\alpha| = k$ and its elements arranged in ascending order, we let $x[\alpha]$ denote the vector in $\mathbb{R}^k$ obtained from the entries of $x \in \mathbb{R}^n$ indexed by $\alpha$. Moreover, we let $A[\alpha, \beta]$ denote the submatrix of $A \in \mathbb{R}^{n \times n}$ whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively; the elements of $\alpha, \beta$ are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1. We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$ and refer to it as a principal submatrix of $A$ and its determinant as a principal minor of $A$.

Given $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \langle n \rangle$ such that $A[\alpha]$ is invertible, $A/A[\alpha]$ denotes the
Schur complement of $A[\alpha]$ in $A$, that is,

$$
A/A[\alpha] = A[\overline{\alpha}] - A[\overline{\alpha}, \alpha]A[\alpha]^{-1}A[\alpha, \overline{\alpha}].
$$

**Definition 2.1.** Matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is

- an *N-matrix* if $\det A[\alpha] < 0$ for all nonempty $\alpha \subseteq \langle n \rangle$;
- a *P-matrix* if $\det A[\alpha] > 0$ for all nonempty $\alpha \subseteq \langle n \rangle$;
- an *almost P-matrix* if $\det A[\alpha] > 0$ for all nonempty proper $\alpha \subseteq \langle n \rangle$ and $\det A < 0$.

We further classify an N-matrix $A \in \mathbb{R}^{n \times n}$ as being

- *of the first category* if there exist $i, j \in \langle n \rangle$ such that $a_{ij} > 0$; or
- *of the second category* if $a_{ij} < 0$ for all $i, j \in \langle n \rangle$.

For an array $X$, we denote $X \geq 0 \ (X \leq 0)$ to signify that all the entries of $X$ are nonnegative (nonpositive). Similarly, $X > 0 \ (X < 0)$ means all the entries of $X$ are positive (negative).

For reference and context needed in our further considerations, we gather below some analogous properties of N-matrices and P-matrices. First we recall two basic definitions:

For $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the *Linear Complementarity Problem*, LCP$(A, q)$, is to find, if possible, $x \in \mathbb{R}^n$ such that

$$
x \geq 0, \quad y = Ax + q \geq 0 \quad \text{and} \quad x^Ty = 0.
$$

For details and background on the Linear Complementarity Problem, see [4].

Given a nonempty $\alpha \subseteq \langle n \rangle$ and provided that $A[\alpha]$ is invertible, the *Principal Pivot Transform* of $A \in \mathbb{R}^{n \times n}$ relative to $\alpha$ is defined as the matrix $\text{ppt} (A, \alpha)$ obtained from $A$ by replacing

- $A[\alpha]$ by $A[\alpha]^{-1}$,
- $A[\alpha, \overline{\alpha}]$ by $-A[\alpha]^{-1}A[\alpha, \overline{\alpha}]$,
- $A[\overline{\alpha}, \alpha]$ by $A[\overline{\alpha}, \alpha]A[\alpha]^{-1}$
- $A[\overline{\alpha}]$ by $A/A[\alpha]$.

For its properties and details on the Principal Pivot Transform, see [19].

**N-matrices:**
• [N1] $A \in \mathbb{R}^{n \times n}$ is an N-matrix if and only if $A^{-1}$ is an almost P-matrix [16].

• [N2] $A \in \mathbb{R}^{n \times n}$ is an N-matrix of the first category if and only if $A$ satisfies the following conditions: $A$ has at least one positive entry in each column, $LCP(A, q)$ has a unique solution for all $q \not\geq 0$, exactly three solutions for all $q > 0$, and at most two solutions for any other $q \in \mathbb{R}_+^n$ (nonnegative orthant) [15].

• [N3] $A \in \mathbb{R}^{n \times n}$ is an N-matrix of the second category if and only if $A < 0$ and for every $q > 0$, $LCP(A, q)$ has exactly 2 solutions [17].

• [N4] $A \in \mathbb{R}^{n \times n}$ is an N-matrix of the second category if and only if $A < 0$ and $A$ does not reverse the sign of any nonzero, unsigned vector $x = [x_i] \in \mathbb{R}^n$; i.e., $(Ax)_i x_i \leq 0$ for all $i \in \langle n \rangle$ implies $x \geq 0$ or $x \leq 0$ [17].

• [N5] If $A \in \mathbb{R}^{n \times n}$ is an N-matrix, then $A/A[\alpha]$ is a P-matrix for all proper subsets $\alpha$ of $\langle n \rangle$ [18].

• [N6] Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix, $\alpha$ be a proper subset of $\langle n \rangle$ and $B = \text{ppt}(A, \alpha)$. Then $\det(B[\alpha]) < 0$ and all other principal minors of $B$ are positive [18].

• [N7] N-matrices have exactly one real negative eigenvalue [17].

**P-matrices:** See [13, Chapter 3] or [20] for a treatment of P-matrices.

• [P1] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if $A^{-1}$ is a P-matrix.

• [P2] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if for every $q \in \mathbb{R}^n$, $LCP(A, q)$ has a unique solution.

• [P3] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if $A$ does not reverse the sign of any nonzero vector $x = [x_i] \in \mathbb{R}^n$; i.e., $(Ax)_i x_i \leq 0$ for all $i \in \langle n \rangle$ implies $x = 0$.

• [P4] If $A \in \mathbb{R}^{n \times n}$ is a P-matrix, then $A/A[\alpha]$ is a P-matrix for all $\alpha \subseteq \langle n \rangle$.

• [P5] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if $\text{ppt}(A, \alpha)$ is a P-matrix for any (and thus all) $\alpha \subseteq \langle n \rangle$.

• [P6] P-matrices have no real negative eigenvalues.
3 Detecting N-matrices

The problem of detecting P-matrices is known to be co-NP-complete [5]. The computation of the principal minors of $A \in \mathbb{R}^{n \times n}$ via row reduction leads to an $O(n^3 2^n)$ effort. A more efficient, recursive algorithm to detect P-matrices of $O(2^n)$ time complexity is developed in [21] and is based on the following theorem.

**Theorem 3.1.** [21, Theorem 3.1] Let $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \langle n \rangle$ with $|\alpha| = 1$. Then $A$ is a P-matrix if and only if $A[\alpha]$, $A[\bar{\alpha}]$ and $A/A[\alpha]$ are P-matrices.

We can extend the theorem above into the following characterization of N-matrices.

**Theorem 3.2.** Let $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \langle n \rangle$ with $|\alpha| = 1$. Then $A$ is an N-matrix if and only if $A[\alpha]$, $A[\bar{\alpha}]$ are N-matrices and $A/A[\alpha]$ is a P-matrix.

**Proof.** Without loss of generality, let $\alpha = \{1\}$; otherwise, our considerations apply to a permutation similarity of $A$. Suppose that $A$ is an N-matrix. By definition, $A[\alpha]$ and $A[\bar{\alpha}]$ are N-matrices. By [N5], $A/A[\alpha]$ is a P-matrix.

Conversely, suppose $A[\alpha]$ and $A[\bar{\alpha}]$ are N-matrices and $A/A[\alpha]$ is a P-matrix. The determinant of any principal submatrix of $A$ without any entries from the first column is a principal minor of $A[\bar{\alpha}]$ and it is thus negative. Let $B$ be any principal submatrix of $A$ with entries from the first row of $A$. Then $C = A[\alpha]$ is a principal submatrix (diagonal entry) of $B$, and $B/C$ is a principal submatrix of $A/A[\alpha]$. Thus $\det(B/C) > 0$ and so $\det(B) = A[\alpha] \det(B/C) < 0$. Hence $A$ is an N-matrix.

Theorem 3.2 suggests the following recursive algorithm for detecting N-matrices.

**ALGORITHM N(A)**

1. Input $A = [a_{ij}] \in \mathbb{R}^{n \times n}$
2. If $a_{11} \geq 0$, output “$A$ is not an N-matrix” stop
3. Compute $A/a_{11}$
4. If $A/a_{11}$ is not P-matrix output “$A$ is not an N-matrix” stop
5. Call $N(A[\{1\}])$
6. Output “$A$ is an N-matrix”
A Matlab implementation of algorithm $N(A)$ is found in Section 5 (NTEST). The algorithm needed in (step 4) of NTEST to detect a P-matrix is based on Theorem 3.1 and also provided in Section 5 (PTEST).

4 Constructing All N-matrices

Examples of N-matrices, even of special structure and form, are not as easy to generate as is for examples of P-matrices. Some possibilities include the types of N-matrices considered in [7] and [11, 12], as well as the totally negative matrices (all minors are negative) in [2, 6]. In [3, Theorem 7.12], some necessary conditions are presented on the signs of the entries of an N-matrix of the first category.

In this section, using a recursion based on rank-one perturbations of N-matrices, we can reverse the steps of the recursive algorithm $N(A)$ that detects N-matrices and thus construct every N-matrix of either category. This approach is based on the following corollary of Theorem 3.2.

Corollary 4.1. Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix of the second category, $a \in \mathbb{R}$ and let $x, y \in \mathbb{R}^n$. Then the following are equivalent:

(i) $U = \begin{bmatrix} A & x \\ y^T & a \end{bmatrix}$ is an N-matrix of the second category.

(ii) $a, x, y < 0$ and $A - \frac{1}{a} xy^T$ is a P-matrix.

Corollary 4.1 allows us to recursively construct $n \times n$ ($n \geq 2$) N-matrices of the second category as follows.

ALGORITHM NCON2

1. Choose $A_1 < 0$

2. For $i = 1 : n - 1$, given the $i \times i$ N-matrix of the second category $A_i$,

   (a) choose $a_i < 0$ and $x^{(i)}, y^{(i)} \in \mathbb{R}^i$ such that $x^{(i)}, y^{(i)} < 0$ and $A_i - \frac{1}{a_i} x^{(i)} y^{(i)T}$ is a P-matrix

   (b) construct the $(i + 1) \times (i + 1)$ matrix $A_{i+1} = \begin{bmatrix} A_i & x^{(i)} \\ y^{(i)T} & a_i \end{bmatrix}$

3. Output “$A = A_n$ is an N-matrix of the second category”
Theorem 4.1. Every matrix constructed by NCON2 is an N-matrix of the second category. Conversely, every N-matrix of the second category can be constructed by NCON2.

Proof. By Corollary 4.1, the sequence of matrices $A_{i+1}$ ($i = 1, \ldots, n-1$) constructed by NCON2, including $A_1$, are N-matrices of the second category. To prove the converse, we proceed by induction on the order of matrices. The statement is trivial for $n = 1$. Let $n \geq 2$ and suppose that every N-matrix of the second category of order smaller than $n$ can be constructed by NCON2. Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix of the second category. Then $A$ can be partitioned as

$$A = \begin{bmatrix} A_{n-1} & u \\ v^T & a \end{bmatrix},$$

where $A_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)}$ is N-matrix of the second category, $u, v \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}$. By inductive hypothesis, $A_{n-1}$ can be constructed by NCON2. Since $A$ is N-matrix of the second category, by Corollary 4.1, $A_{n-1}/a = A_{n-1} - \frac{1}{a}uv^T$ is a P-matrix, $a, x, y < 0$. Thus $A_n = A$ can be constructed by NCON2 with the following choices:

$$a_{n-1} = a, \; x^{(n-1)} = u \quad \text{and} \quad y^{(n-1)} = v.$$

To extend our construction methodology to N-matrices of the first category, we recall the following results.

Theorem 4.2. [3, Theorem 7.12] Let $A = [a_{ij}]$ be an N-matrix of the first category. Then the following hold:

(i) All the entries of $A$ are nonzero.

(ii) Each row and column of $A$ has at least one positive entry.

(iii) Both $a_{ij}$ and $a_{ji}$ have the same sign.

(iv) If $a_{ij}, a_{ik} > 0$, then $a_{jk} < 0$.

The next result gives another necessary condition on the signs of the entries of an N-matrix of the first category.

Theorem 4.3. Let $A = [a_{ij}]$ be an N-matrix of the first category. If $a_{ij} > 0$ and $a_{ik} < 0$, then $a_{jk} > 0$. 

Proof. Suppose that \( a_{ij} > 0 \) and \( a_{ik} < 0 \). Let \( a_{jk} < 0 \). By (ii) of Theorem 4.2, \( a_{ji} > 0 \) and \( a_{ki}, a_{kj} < 0 \). Consider the 3 \( \times \) 3 principal submatrix

\[
A[\{i, j, k\}] = \begin{bmatrix}
  a_{ii} & a_{ij} & a_{ik} \\
  a_{ji} & a_{jj} & a_{jk} \\
  a_{ki} & a_{kj} & a_{kk}
\end{bmatrix}.
\]

Then

\[
\det([\{i, j, k\}]) = a_{ii}(a_{jj}a_{kk} - a_{jk}a_{kj}) - a_{ij}(a_{ji}a_{kk} - a_{ki}a_{jk}) > 0,
\]

a contradiction. Hence \( a_{jk} > 0 \). \( \square \)

The following result gives a nice partition of N-matrix of the first category.

**Theorem 4.4.** [15, Theorem 4.3] Let \( A \) be an N-matrix of the first category. Then there exists a permutation matrix \( P \) such that

\[
PAP^T = \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix},
\]

(4.1)

where \( A_{11}, A_{22} < 0 \) are square matrices and \( A_{12}, A_{21} > 0 \).

By Theorem 4.4, in order to construct all N-matrices of the first category of size \( n \geq 2 \), it is sufficient to construct them in the form (4.1), where \( A_{11} \in \mathbb{R}^{k \times k} \) \((k < n)\). This can be achieved using the following corollary of Theorem 3.2.

**Corollary 4.2.** Let \( A = \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix} \in \mathbb{R}^{n \times n} \) \((n \geq 2)\) be an N-matrix, where

\( A_{11} \in \mathbb{R}^{k \times k} \) \((k \leq n)\), \( A_{22} \in \mathbb{R}^{(n-k) \times (n-k)} \) with \( A_{11}, A_{22} < 0 \) and \( A_{12}, A_{21} > 0 \). Let \( a \in \mathbb{R} \) and let \( x, y \in \mathbb{R}^n \). Then the following are equivalent:

(i) \( U = \begin{bmatrix}
  A \\
  y^T \\
  a
\end{bmatrix} \) is an N-matrix of the first category.

(ii) \( a < 0, A - \frac{1}{a}xy^T \) is a P-matrix and either \( x[\langle k \rangle], y[\langle k \rangle] > 0, x[\langle k \rangle], y[\langle k \rangle] < 0 \) or \( x[\langle k \rangle], y[\langle k \rangle] < 0, x[\langle k \rangle], y[\langle k \rangle] > 0 \).

**Proof.** Suppose that \( U = [u_{ij}] = \begin{bmatrix}
  A \\
  y^T \\
  a
\end{bmatrix} \) is an N-matrix of the first category and let \( A_{11} = [a_{ij}^{11}], A_{12} = [a_{ij}^{12}], A_{21} = [a_{ij}^{21}] \) and \( A_{22} = [a_{ij}^{22}] \). By Theorem 3.2, \( a < 0 \) and \( A - \frac{1}{a}xy^T \) is a P-matrix. Now we show that if \( x[\langle k \rangle] \) contains a positive entry then \( x[\langle k \rangle] > 0 \) and \( x[\langle k \rangle] < 0 \). Without loss of generality assume
that \( x_1 = u_{1n+1} > 0 \). Since \( u_{1i} = a_{11}^{ii} < 0 \) for \( i \in \langle k \rangle \setminus \{1\} \), by Theorem 4.3, 
\( u_{i,n+1} = x_i > 0 \) for \( i \in \langle k \rangle \setminus \{1\} \). For \( j \in \langle k \rangle \), \( u_{j1} = a_{j,k+1}^{21} > 0 \). By (iv) of 
Theorem 4.2, \( u_{jn+1} = x_j < 0 \) for \( j \in \langle k \rangle \). Thus \( x[\langle k \rangle] > 0 \) and \( x[\langle k \rangle] < 0 \). By (iii) of 
Theorem 4.2, \( y[\langle k \rangle] > 0 \) and \( y[\langle k \rangle] < 0 \). Similarly one can show that if \( x[\langle k \rangle] \) contains a 
negative entry then \( x[\langle k \rangle] > 0 \) and \( x[\langle k \rangle] < 0 \). By (iii) 
of Theorem 4.2, \( y[\langle k \rangle] > 0 \) and \( y[\langle k \rangle] < 0 \). Similarly one can show that if \( x[\langle k \rangle] \) contains a 
negative entry then \( x[\langle k \rangle], y[\langle k \rangle] > 0 \) and \( x[\langle k \rangle], y[\langle k \rangle] < 0 \).

The converse follows from Theorem 3.2.

Remark 4.1. Let \( U = \begin{bmatrix} A & x \\ y^T & a \end{bmatrix} \) be an N-matrix of the first category as defined in 
Corollary 4.2 such that \( x[\langle k \rangle], y[\langle k \rangle] < 0 \) and \( x[\langle k \rangle], y[\langle k \rangle] > 0 \). Then interchanging 
the \((k+1)\)-st and \((n+1)\)-st columns of \( U \) and subsequently interchanging the 
\((k+1)\)-st and \((n+1)\)-st rows, we can write it in the form (4.1).

Using Corollary 4.2, we can now recursively construct \( n \times n \) \((n \geq 2)\) N-matrices 
of the first category in the form (4.1) as follows.

**ALGORITHM NCON1**

1. Construct \( A_k = A_{11} \) using algorithm NCON2

2. For \( i = k : n - 1 \), given the \( i \times i \) matrix \( A_i \),
   
   (a) choose \( a_i < 0 \), \( x^{(i)}, y^{(i)} \in \mathbb{R}^i \) such that \( x[\langle k \rangle], y[\langle k \rangle] > 0 \), \( x[\langle k \rangle], y[\langle k \rangle] < 0 \), and \( A_i - \frac{1}{a_i} x^{(i)} y^{(i)^T} \) is a P-matrix

   (b) construct the \((i + 1) \times (i + 1)\) matrix \( A_{i+1} = \begin{bmatrix} A_i & x^{(i)} \\ y^{(i)^T} & a_i \end{bmatrix} \)

3. Output “\( A = A_n \) is an N-matrix of the first category”

**Theorem 4.5.** Every matrix constructed by NCON1 is an N-matrix of the first category. Conversely, every N-matrix of the first category can be constructed as a permutational similarity of a matrix constructed by NCON1.

**Proof.** By Corollary 4.2, the sequence of matrices \( A_{i+1} \) \((i = 1, \ldots, n - 1)\) constructed by NCON1, are N-matrices of the first category. We use induction on the order of matrices to prove the converse. The base case \( n = 2 \) is obvious. Let \( n > 2 \) and suppose that every N-matrix of the first category of order smaller than \( n \) can be constructed as a permutational similarity of a matrix constructed by NCON1. Let \( A \in \mathbb{R}^{n \times n} \) be an N-matrix of the first category. Then there exists a permutation matrix \( P \) such that
\[ PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \]

where \( A_{11} \in \mathbb{R}^{k \times k} (k < n), A_{22} \in \mathbb{R}^{(n-k) \times (n-k)} \) with \( A_{11}, A_{22} < 0 \) and \( A_{12}, A_{21} > 0 \).

Let \[ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{n-1} & u \\ v & a \end{bmatrix}, \]

where \( A_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)} \) is N-matrix, \( a < 0 \), \( u, v \in \mathbb{R}^{n-1} \) with \( u[k], v[k] > 0 \) and \( u[k], v[k] < 0 \). Now, either \( A_{n-1} < 0 \) or \( A_{n-1} \) is of the form (4.1). By inductive hypothesis, \( A_{n-1} \) can be constructed using \textbf{NCON1}. Since \( A \) is N-matrix of the first category, by Corollary 4.2, \( A_{n-1}/a = A_{n-1} - \frac{1}{a} uv^T \) is a P-matrix. Thus \( A_n = PAP^T \) can be constructed by \textbf{NCON1} with the following choices:

\[ a_{n-1} = a, \quad x^{(n-1)} = u \text{ and } y^{(n-1)} = v. \]

\[ \Box \]

\textbf{Remark 4.2.}

(1) The implementation of step 2(a) in algorithms \textbf{NCON1} and \textbf{NCON2} can be done via random choice of the appropriately signed vectors \( x^{(i)} \) and \( y^{(i)} \) and judicious choice of the diagonal entries \( a_i \). The process of choosing \( a_i \) so that \( A_i - \frac{1}{a_i} x^{(i)} y^{(i)T} \) is a P-matrix is developed and its effects explained in the recursive construction of all P-matrices presented in [22, Section 4].

(2) In light of [N1] in Section 2, Algorithms \textbf{NCON1} and \textbf{NCON2} may also be viewed as methods to construct almost P-matrices via inversion.

We proceed with two illustrative examples of N-matrices constructed using \textbf{NCON1} and \textbf{NCON2}.

\textbf{Example 4.1.} We construct \( 3 \times 3 \) N-matrix of the first category. Let \( A_1 = -1, a_1 = -1, x^{(1)} = [2] \text{ and } y^{(1)} = [2] \). Then \( A_1 - \frac{1}{a_1} x^{(1)} y^{(1)T} = [3] \) is a P-matrix. By \textbf{NCON1}, \( A_2 = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \) is N-matrix of the first category. Now, let \( a_2 = -1, x^{(2)} = [2 \ -1]^T \text{ and } y^{(2)} = [2 \ -2]^T \). Then \( A_2 - \frac{1}{a_2} x^{(2)} y^{(2)T} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \) is a P-matrix. Again, by \textbf{NCON1}, \( A_3 = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & -1 \\ 2 & -2 & -1 \end{bmatrix} \) is N-matrix of the first category.
Example 4.2. In this example, we construct a $3 \times 3$ N-matrix of the second category by NCON2. Let $A_1 = -1, a_1 = -1, x^{(1)} = [-1]$ and $y^{(1)} = [-2]$. Then $A_1 - \frac{1}{a_1} x^{(1)} y^{(1)}^T = [1]$ is a P-matrix. By NCON2, $A_2 = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$ is N-matrix of the second category. Now, we take $a_2 = -1, x^{(2)} = [-2, -1]^T$ and $y^{(2)} = [-3, -2]^T$. Then $A_2 - \frac{1}{a_2} x^{(2)} y^{(2)}^T = \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ is a P-matrix. Hence, by NCON2, $A_3 = \begin{bmatrix} -1 & -1 & -2 \\ -2 & -1 & -1 \\ -3 & -2 & -1 \end{bmatrix}$ is N-matrix of the second category.

5 NTEST and PTEST

We include Matlab code for the detection of P-matrices and N-matrices.

**PTEST** (detects P-matrices)

function [r] = ptest(A)
% Return r=1 if 'A' is a P-matrix (r=0 otherwise).

n = length(A);
if ~((A(1,1)>0), r = 0;
elseif n==1, r = 1;
else
    b = A(2:n,2:n);
    d = A(2:n,1)/A(1,1);
    c = b - d*A(1,2:n);
    r = ptest(b) & ptest(c);
end

**NTEST** (detects N-matrices)

function [r] = ntest(A)
% Return r=1 if 'A' is a N-matrix (r=0 otherwise).

n = length(A);
if ~((A(1,1)<0), r = 0;
elseif n==1, r = 1;
else
    b = A(2:n,2:n);
    d = A(2:n,1)/A(1,1);
    c = b - d*A(1,2:n);
    r = ntest(b) & ptest(c);
Note that the time complexity of PTEST is $O(2^n)$ [21], and so this must also be the case for NTEST as the same binary tree of matrices (of order reduced by 1) is being recursively created by the two algorithms.

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