Pascal Eigenspaces and Invariant Sequences of the First or Second Kind

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Abstract

An infinite real sequence \( \{a_n\} \) is called an invariant sequence of the first (resp., second) kind if \( a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \) (resp., \( a_n = \sum_{k=n}^{\infty} \binom{n}{k} (-1)^k a_k \)). We review and investigate invariant sequences of the first and second kind, and study their relationships using similarities of Pascal-type matrices and their eigenspaces.

Keywords: Invariant sequence, Pascal matrix, Eigenvalue, Eigenvector

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1. Introduction

Inverse relations play an important role in combinatorics. The binomial inversion formula, which states that for sequences \( \{a_n\} \) and \( \{b_n\} \) \((n = 0, 1, 2, \ldots)\),

\[
a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k,
\]

is a typical inverse relation considered in [6, 8, 10, 14, 15, 16]. Specifically, (1.1) motivated Sun [14] to investigate the following sequences.

**Definition 1.1.** Let \( \{a_n\} \) \((n = 0, 1, 2, \ldots)\) be a sequence such that

\[
(-1)^{s-1} a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k; \quad s = 1 \quad \text{or} \quad s = 2.
\]

We refer to \( \{a_n\} \) as an invariant sequence (when \( s = 1 \)) or an inverse invariant sequence (when \( s = 2 \)) of the first kind.
Several examples of invariant sequences of the first kind can be found in [14], including
\[ \{1/2n\}, \{nF_{n-1}\}, \{L_n\}, \{(-1)^nB_n\} \ (n \geq 0), \]
where \(F_{-1} = 0\) and \(\{F_n\}, \{L_n\}, \{B_n\}\) are the Fibonacci sequence, Lucas sequence, and Bernoulli numbers [7], respectively. In this paper, we will establish (see Lemma 2.1) the modified binomial inversion formula, that is,
\[ a_n = \sum_{k=n}^{\infty} \binom{k}{n} (-1)^k b_k \quad \text{if and only if} \quad b_n = \sum_{k=n}^{\infty} \binom{k}{n} (-1)^k a_k. \quad (1.3) \]

Motivated by (1.3), we will introduce and consider the following sequences.

**Definition 1.2.** Let \(\{a_n\} (n = 0, 1, 2, \ldots)\) be a sequence such that
\[ (-1)^{s-1} a_n = \sum_{k=n}^{\infty} \binom{k}{n} (-1)^k a_k; \quad s = 1 \text{ or } s = 2. \quad (1.4) \]

We refer to \(\{a_n\}\) as an invariant sequence (when \(s = 1\)) or an inverse invariant sequence (when \(s = 2\)) of the second kind.

Naturally arising are the questions of existence, identification, and construction of (inverse) invariant sequences of the second kind, as well as the problem of characterizing such sequences and examining their relationship to their counterparts of the first kind. Invariant sequences, which are also called self-inverse sequences in [15], have indeed been studied by several authors [6, 8, 14, 15]. They are naturally connected to involutory (also known as involution or self-invertible) matrices [9] and to Riordan involutions [5]. Involutory matrices find use in numerical methods for differential equations [2, 9]. They are also useful in cryptography, information theory, and computer security by providing convenient encryption and decryption methods [1]. Motivated by Shapiro’s open questions [12], Riordan involutions have been intensely investigated as a combinatorial concept [4, 5]. In this paper, we investigate invariant sequences by means of the eigenspaces of \(PD\) and \(P^T D\), where \(P\) is the Pascal matrix and \(D\) an infinite diagonal matrix with alternating diagonal entries in \(\{1, -1\}\) (see Sections 2, 3). In fact, \(PD\) and \(P^T D\) are involutory matrices and \(PD\) is a Riordan involution. Our investigation follows the ideas and connections of invariant sequences to the eigenspaces of \(PD\) and \(P^T D\) developed in Choi et al. [6]. This will allow us to associate (inverse) invariant sequences of the first and second kind, as well as identify and construct such sequences (Section 4).

2. Notation and preliminaries

The following notation and conventions are used throughout the manuscript.

- The infinite matrices in this paper have infinite numbers of rows \(i\) and columns \(j\), with \(i, j \in \{0, 1, 2, \ldots\}\).
\( \mathbf{E}_\lambda(A) \) denotes the eigenspace of a (finite or infinite) matrix \( A \) corresponding to its eigenvalue \( \lambda \).

For a matrix \( A \) with columns \( A_j \ (j = 0, 1, 2, \ldots) \) and with \( \mathbf{0}_j \) denoting the vector of zeros in \( \mathbb{R}^j \), \( A^j \) denotes the matrix whose \( j \)th column is \( \begin{bmatrix} \mathbf{0}_j \\ A_j \end{bmatrix} \), where \( \mathbf{0}_0 \) is by convention vacuous.

For a matrix \( A \), its (possibly infinite) row and column index sets are \( J \) and \( K \), respectively. For \( J_0 \subseteq J, K_0 \subseteq K \), let \( A(J_0|K_0) \) denote the matrix obtained from \( A \) by deleting rows in \( J_0 \) and columns in \( K_0 \), and let \( A[J_0|K_0] \) denote the matrix \( A(J_0|K_0) \), where \( J_0 = J \setminus J_0, K_0 = K \setminus K_0 \).

For brevity, write \( A(J_0) \) and \( A(J_0|\cdot) \) in place of \( A(\emptyset|K_0) \) and \( A(J_0|\emptyset) \), respectively. Further, for \( m, n \in \{0, 1, 2, \ldots\} \), we let \( A_{m,n} = A([0, 1, 2, \ldots, m]|[0, 1, 2, \ldots, n]) \); \( A_{m,m} \) is abbreviated by \( A_m \).

The binomial coefficient (“i choose j”) is denoted by \( \binom{i}{j} \) with the convention that it equals 0 when \( i < j \) or \( j < 0 \).

\( P = \binom{0}{i} \) \((i,j = 0, 1, 2, \ldots)\) denotes the (infinite) Pascal matrix.

\( D = \text{diag}(1, -1, 1, -1, \ldots) \).

Infinite real sequences \( \{x_n\} \) are identified with the infinite dimensional real vector space \( \mathbb{R}^\infty \) consisting of column vectors \( x = [x_0, x_1, x_2, \ldots]^T \).

Notice that as a consequence of the binomial inversion formula \([1, 1]\), we have that

\[
P^{-1} = DPD = \begin{bmatrix} (-1)^{i-j} \binom{i}{j} \end{bmatrix} (i, j = 0, 1, 2, \ldots).
\]

Thus \( (PD)^{-1} = PD \) and \([1, 1]\) can be converted \([3]\) into a vector equation for \( x = [a_0, a_1, a_2, \ldots]^T \) and \( y = [b_0, b_1, b_2, \ldots]^T \in \mathbb{R}^\infty \), as follows:

\[
PDx = y \quad \text{if and only if} \quad PDy = x.
\]  \(2.1\)

\begin{lemma}
Let \( P \) and \( D \) be the Pascal matrix and the diagonal matrix defined above, and let \( x, y \in \mathbb{R}^\infty \). Then

\[
P^T Dx = y \quad \text{if and only if} \quad P^T Dy = x,
\]  \(2.2\)

and the modified binomial inversion formula \([1, 3]\) holds.
\end{lemma}

\begin{proof}
As \( P^{-1} = DPD \), we have \( (P^T D)^{-1} = D(P^T )^{-1} = P^T D \). As a consequence, \(2.2\) holds.

Let \( x = [a_0, a_1, a_2, \ldots]^T \) and \( y = [b_0, b_1, b_2, \ldots]^T \) denote the vectors in \( \mathbb{R}^\infty \) whose entries are the members of the Fibonacci and Lucas sequences, respectively; that is

\[
F_0 = 0, \ F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2),
\]

\[
L_0 = 2, \ L_1 = 1, \ L_n = L_{n-1} + L_{n-2} \quad (n \geq 2).
\]
\end{proof}
The generating functions of $F$ and $L$ are $h_1(x) = \frac{1-x}{1-x^2}$ and $h_2(x) = \frac{2-x}{1-x^2}$, respectively \[3\] \[7\].

The following fact is known, however, we include a proof for completeness.

**Lemma 2.2.** $PD\,F = -F$ and $PD\,L = L$.

**Proof.** Let $g(x) = \frac{1}{1-x}$ and $f(x) = \frac{x}{1-x}$. For $j = 0, 1, 2, \ldots$, the generating function of the $j$th column of $PD$ is $g(x)f(x)^j$ \[13\]. Thus the generating function of $PD\,F$ is

$$
[g(x), g(x)f(x), g(x)f(x)^2, \ldots][F_0, F_1, F_2, \ldots]^T = g(x)(F_0 + F_1f(x) + F_2f(x)^2 + \cdots)
$$

$$= g(x)h_1(f(x)) = -h_1(x),$$

which implies that $PD\,F = -F$. The proof of $PD\,L = L$ is similar. \[\blacksquare\]

That is, $-1$ and $1$ are eigenvalues of $PD$ and consequently of $P^TD$. In fact, these are the only eigenvalues of $PD$ and $P^TD$; see \[6\]. The corresponding eigenspaces are infinite dimensional. Indeed, if we consider the Pascal-type matrices $P^\uparrow$ and $Q^\downarrow$ constructed via the Pascal matrix $P$ and the matrix

$$Q = P + \begin{bmatrix} 1 & 0^T \\ 0 & P \end{bmatrix},$$

then, as shown in \[6\], the columns of

$$\begin{bmatrix} 0^T \\ P^i \end{bmatrix}$$

and

$$Q^\downarrow$$

form bases for $E_{-1}(PD)$ and $E_1(PD)$, respectively. The following observation follows directly from the definitions and properties mentioned above:

**Observation 2.3.** The entries of $x \in \mathbb{R}^\infty$ form

- an invariant sequence of the first kind if and only if $x \in E_1(PD)$;
- an inverse invariant sequence of the first kind if and only if $x \in E_{-1}(PD)$;
- an invariant sequence of the second kind if and only if $x \in E_1(P^TD)$;
- an inverse invariant sequence of the second kind if and only if $x \in E_{-1}(P^TD)$.

Based on Observation \[2.3\], our goal is to study the eigenspaces $E_{\lambda}(PD)$ and $E_{\lambda}(P^TD)$ ($\lambda \in \{1, -1\}$) and discover their relationships. Our approach entails showing the existence of an infinite invertible matrix $N$ such that

$$N(P^TD)N^{-1} = (P_1^TD_1) \bigoplus (P_1^TD_1) \bigoplus \cdots \bigoplus (P_1^TD_1) \bigoplus \cdots$$

(2.3)

and

$$D(N^{-1})^T D(P^T) D N T D = (P_1^TD_1) \bigoplus (P_1^TD_1) \bigoplus \cdots \bigoplus (P_1^TD_1) \bigoplus \cdots,$$

(2.4)

which are infinite direct sums of copies of $P_1^TD_1$ and $P_1D_1$, respectively. This result will be applied to characterize $E_{\lambda}(PD)$ and $E_{\lambda}(P^TD)$. Extending the work in \[6\], we will also show that the columns of $P^T$ and $Q^\downarrow(0|0)$ form bases for $E_{\lambda}(P^TD)$. This will indeed allow us to investigate the relationships between invariant sequences of the first and second kind.
3. The Eigenspaces of $P^TD$ and $PD$

Let $B = [b_{ij}]$ $(i,j = 0,1,2,\ldots)$ be the matrix defined by

$$b_{ij} = \begin{cases} (-1)^{j-i}, & \text{if } i \leq j, \\ 0, & \text{if } i > j, \end{cases}$$

and let $J(a)$ denote the infinite Jordan block of the form

$$\begin{bmatrix} a & 1 \\ & a & 1 \\ & & \ddots & \ddots \\ & & & a & 1 \\ & & & & O \end{bmatrix}.$$  

(3.2)

It readily follows that $B^{-1} = J(1)$.

In the next two lemmas, we will construct an infinite matrix $N$ and its inverse $M = N^{-1}$, which will give rise to similarity transformations of $P^TD$ and $PD$ into direct sums as in (2.3) and (2.4).

**Lemma 3.1.** Let $m$ be a positive integer and let $N(m) = H(m)H(m-1)\cdots H(1)$ with $H(l) = I_{2l-2} \oplus B$ $(l = 1,2,\ldots,m)$, where $B$ is the matrix defined in (3.1). Then

$$\lim_{m \to \infty} N(m) = N = [n^\infty_{ij}]$$

is the infinite matrix defined by $n^\infty_{00} = 1$, $n^\infty_{0j} = 0$ and $n^\infty_{ij} = 0$ for $i,j = 1,2,\ldots$, and

$$n^\infty_{ij} = \begin{cases} (-1)^{j-i}\left(\frac{j-i}{k} \right), & \text{if } 1 \leq i \leq j, \\ 0, & \text{if } i > j \geq 1. \end{cases}$$

**Proof.** Let $m$ be a positive integer and let $N(m) = H(m)H(m-1)\cdots H(1) = [n^m_{ij}]$ with $n^m_{00} = 1$, $n^m_{0j} = 0$ and $n^m_{ij} = 0$ for $i,j = 1,2,\ldots$. We will prove that

$$n^m_{ij} = \begin{cases} (-1)^{j-i}\left(\frac{k+j-i}{k} \right), & \text{if } i \leq j, \\ 0, & \text{if } i > j \end{cases}$$

by induction on $m$, where

$$k = \begin{cases} \left\lfloor \frac{1-1}{2} \right\rfloor, & \text{if } i = 1,2,\ldots,2m; j = 1,2,\ldots, \\ m-1, & \text{if } i = 2m+1,2m+2,\ldots; j = 1,2,\ldots \end{cases}$$

The claim is clear for $m = 1$. For $m = 2$, by the construction of $H(1)$ and $H(2)$, we have

$$n^2_{ij} = \begin{cases} (-1)^{j-i}\left(\frac{k+j-i}{k} \right), & \text{if } i \leq j, \\ 0, & \text{if } i > j \end{cases}$$
Let now \( m \geq 3 \). Then by the construction of \( H(m) \), we have \( n_{ij}^m = n_{ij}^{m-1} \) for all \( i = 1, 2, \ldots, 2m - 2 \) and all \( j = 1, 2, \ldots \). By the induction hypothesis,

\[
n_{ij}^m = \begin{cases} 
(-1)^{j-i} \binom{k+j-i}{k}, & \text{if } i \leq j, \\
0, & \text{if } i > j,
\end{cases}
\]

where

\[
k = \begin{cases} 
\lfloor \frac{i+j}{2} \rfloor, & \text{if } i \leq 2m, \quad j = 1, 2, \ldots, \\
m - 1, & \text{if } i = 2m + 1, 2m + 2, \ldots; \quad j = 1, 2, \ldots,
\end{cases}
\] (3.3)

because

\[
n_{ij}^m = \sum_{l=i}^{\infty} (-1)^{j-i} (-1)^{j-l} \binom{m - 2 + j - l}{m - 2}
\]

\[
= (-1)^{j-i} \sum_{l=i}^{\infty} \binom{m - 2 + j - l}{m - 2} = (-1)^{j-i} \binom{m - 1 + j - i}{m - 1}
\]

for each pair of indices \( i \) and \( j \) with \( i = 2m - 1, 2m, \ldots \) and \( j = 1, 2, \ldots \). Thus, by (3.3), we obtain

\[
\lim_{m \to \infty} N_{ij}^m = N_{ij}^\infty = \begin{cases} 
(-1)^{j-i} \binom{\lfloor \frac{i+j}{2} \rfloor}{\lfloor \frac{i+j}{2} \rfloor}, & \text{if } i \leq j, \\
0, & \text{if } i > j
\end{cases}
\]

for each \( i,j = 1,2,\ldots \). Clearly, we have \( n_{00}^\infty = 1, n_{0j}^\infty = 0 \) and \( n_{ij}^\infty = 0 \) for \( i,j = 1,2,\ldots \) by the construction of \( H(l) \) \((l = 1,2,\ldots)\), and the proof is complete. \( \blacksquare \)

The difference sequence \( \Delta a = [\Delta a_0, \Delta a_1, \Delta a_2, \ldots]^T \) of a sequence \( a = [a_0, a_1, a_2, \ldots]^T \) is defined by \( \Delta a_i = a_{i+1} - a_i \) for each \( i = 0, 1, 2, \ldots \). Let \( \Delta^k a = [\Delta^k a_0, \Delta^k a_1, \Delta^k a_2, \ldots]^T \) \((k = 0, 1, 2, \ldots)\) be the \( k \)th difference sequence defined inductively by \( \Delta^k a = \Delta(\Delta^{k-1} a) \), where \( \Delta^0 a = a \). The infinite matrix

\[
\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots \\
\Delta a_0 & \Delta a_1 & \Delta a_2 & \cdots \\
\Delta^2 a_0 & \Delta^2 a_1 & \Delta^2 a_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
is called the difference matrix of \( a \). It is well known \(^3\) that for each \( n = 0, 1, 2, \ldots \),

\[
a_n = a_0 \binom{n}{0} + \Delta a_0 \binom{n}{1} + \Delta^2 a_0 \binom{n}{2} + \cdots + \Delta^n a_0 \binom{n}{n},
\]

which is used in the proof of the following lemma.

**Lemma 3.2.** Let \( M = [m_{ij}^\infty] \) be the matrix with \( m_{00}^\infty = 1 \), \( m_{0j}^\infty = 0 \) and \( m_{ij}^\infty = 0 \) for \( i, j = 1, 2, \ldots \), and

\[
m_{ij}^\infty = \begin{cases} \binom{i+j}{j}, & \text{if } i \leq j, \\ 0, & \text{if } i > j \end{cases} \quad \text{for } i, j = 1, 2, \ldots.
\]

Then \( M = N^{-1} \), where \( N \) is the matrix defined in Lemma 3.1.

**Proof.** Let \( n_{ij}^\infty \) denote the \((i, j)\) entry of \( N \) \((i, j = 0, 1, 2, \ldots)\). We would like to show that

\[
\sum_{i=0}^\infty n_{ii}^\infty m_{ij}^\infty = \delta_{ij},
\]

the Kronecker delta. For \( i = 0 \) or \( j = 0 \), there is nothing to show, so let \( i \geq 1 \) and \( j \geq 1 \). If \( i = j \) resp. \( i > j \), then clearly

\[
\sum_{i=0}^\infty n_{ii}^\infty m_{ij}^\infty = \sum_{i=0}^\infty n_{ii}^\infty \binom{i+j}{j} \binom{\frac{i}{2}}{\frac{j}{2}} = 1 \quad \text{resp.} \quad \sum_{i=0}^\infty n_{ii}^\infty m_{ij}^\infty = 0.
\]

So it is enough to show that if \( j = i + r \) with \( r > 0 \), then \( \sum_{i=0}^\infty n_{ii}^\infty m_{ij}^\infty = 0 \). Let \( k = \lfloor \frac{i}{2} \rfloor \) and consider the sequence \( z = \left[ \binom{k+r}{k}, -\binom{k+r-1}{k-1}, \binom{k+r-2}{k-1}, \ldots, (-1)^{r-1} \binom{k+1}{k}, (-1)^r \binom{k}{k}, 0, \ldots \right]^T \). We can construct the difference matrix \( A = [a_{ij}] \) having \( z \) as its first column as follows:

\[
A = \begin{bmatrix}
\binom{k+r}{k} & \binom{k+r-1}{k-1} & \cdots & \binom{r}{0} & \cdots \\
-\binom{k+r-1}{k-1} & -\binom{k+r-2}{k-1} & \cdots & -\binom{r-1}{0} & \cdots \\
\binom{k+r-2}{k-1} & \binom{k+r-3}{k-2} & \cdots & \binom{r-2}{0} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
(-1)^{r-2} \binom{k+2}{k} & (-1)^{r-2} \binom{k+1}{k-1} & \cdots & (-1)^r (\binom{k+1}{k}) & \cdots \\
(-1)^{r-1} \binom{k+1}{k} & (-1)^{r-1} \binom{k}{k-1} & \cdots & (-1)^r (\binom{k}{k}) & \cdots \\
(-1)^r \binom{k}{k} & (-1)^r (\binom{k-1}{k-1}) & \cdots & (-1)^r (\binom{0}{0}) & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots 
\end{bmatrix},
\]
where $Y$ is the $(r+1) \times (r+1)$ matrix given by

\[
Y = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & * & \cdots & * & * & * & * \\
0 & 0 & * & \cdots & * & * & * & * & * \\
0 & * & \cdots & * & * & * & * & * & * \\
* & * & \cdots & * & * & * & * & * & * \\
\end{bmatrix}.
\]

Thus $a_{0j} = 0$ for $j = k + 1, \ldots, k + r$. Since $k + 1 \leq \lfloor \frac{i+r}{2} \rfloor \leq \lfloor \frac{i-1+r+r}{2} \rfloor = k + r$, by using (3.4) we obtain

\[
\sum_{l=0}^{\infty} n_{il}^l m_{lj} = \sum_{l=1}^{j} n_{il}^l m_{lj} = \sum_{l=i}^{i+r} (-1)^{i-l} \binom{k+l-i}{k} \binom{t}{i+r-l} = \sum_{s=0}^{t} (-1)^{r+s} \binom{k+r-s}{k} \binom{t}{s} = (-1)^r a_{0t} = 0,
\]

where $k = \lfloor \frac{i-1}{2} \rfloor$ and $t = \lfloor \frac{i+r}{2} \rfloor$, completing the proof. ■

Let $M_{(m)} = U_{(1)} U_{(2)} \cdots U_{(m)}$, where $U_{(l)} = I_{2l-2} \oplus J(1)$ ($l = 1, 2, \ldots, m$) where $J(1)$ is the matrix defined in (3.2). Then, by Lemmas 3.1 and 3.2 we have $\lim_{m \to \infty} M_{(m)} = M$, which is the matrix defined in Lemma 3.2. In fact we have

\[
N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots \\
0 & 0 & 1 & -1 & 1 & -1 & 1 & \cdots \\
0 & 0 & 0 & 1 & -2 & 3 & -4 & \cdots \\
0 & 0 & 0 & 0 & 1 & -3 & 6 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & -3 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]
and

\[
M = N^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 2 & 1 & 1 & \cdots \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.
\]

We can now state and prove the similarity transformations of $PD$ and $P^TD$ claimed in (2.3) and (2.4).

**Theorem 3.3.** Let $P = \left( \binom{i}{j} \right) (i, j = 0, 1, 2, \ldots)$ and $D = \text{diag}(1, -1, 1, -1, \ldots)$. Then,

(a) $N(P^TD)M = (P_i^TD_1) \bigoplus (P_i^TD_1) \bigoplus \cdots \bigoplus (P_i^TD_1) \bigoplus \cdots,$

(b) $(DM^TD)(PD)(DN^TD) = (P_1D_1) \bigoplus (P_1D_1) \bigoplus \cdots \bigoplus (P_1D_1) \bigoplus \cdots,$

where $P_1$ resp. $D_1$ are the leading $2 \times 2$ submatrices of $P$ resp. $D$, and $N$ resp. $M$ are the matrices defined in Lemmas 3.1 resp. 3.2.

**Proof.** (a) For each $j = 1, 2, \ldots$, let $H(j) = I_{2j-2} \bigoplus B$ and $U(j) = I_{2j-2} \bigoplus J(1)$, where $B$ is the matrix defined in (3.1). Let $m, n$ be arbitrary positive integers with $n = 2m + 1$. First, we will show by induction on $m$ that

\[
(H(m)H(m-1) \cdots H(1))P^TD(U(1)U(2) \cdots U(m)) = Z \bigoplus (P^TD),
\]

where $Z$ is an $n \times n$ matrix such that $Z = (P_i^TD_1) \bigoplus (P_i^TD_1) \bigoplus \cdots \bigoplus (P_i^TD_1)$. When $m = 1$, the first row of $H(1)P^TDU(1)$ is clearly $[1, -1, 0, 0, \ldots]$. Since for a pair of indices $i$ and $k$ with $i, k = 1, 2, \ldots$,

\[
(H(1)P^TD)_{ik} = (-1)^k \binom{k}{i} - \binom{k}{i+1} + \cdots + (-1)^k \binom{k}{k} = (-1)^k \binom{k-1}{i-1},
\]

the second row of $H(1)P^TDU(1)$ is $[0, -1, 1, -1, 1, \ldots]U(1) = [0, -1, 0, 0, \ldots]$. For a pair of indices $i$ and $j$ with $i, j \geq 2$, we have

\[
(H(1)P^TDU(1))_{ij} = (-1)^{j-1} \binom{j-2}{i-1} + (-1)^j \binom{j-1}{i-1} = (-1)^{j-2} \binom{j-2}{i-2},
\]

and so $H(1)P^TDU(1) = (P_1^TD_1) \bigoplus (P^TD)$. By induction on $m$, it follows

\[
N(m)P^TDU(m) = (P_i^TD_1) \bigoplus (P_i^TD_1) \bigoplus \cdots \bigoplus (P_i^TD_1) \bigoplus (P^TD),
\]

\[9\]
where \( N_m = H_m H_{m-1} \cdots H_1 \) and \( M_m = U_1 U_2 \cdots U_m \). Since \( m \) was an arbitrary positive integer,

\[
NP^TDM = (P_1^T D_1) \bigoplus (P_1^T D_1) \bigoplus \cdots (P_1^T D_1) \bigoplus \cdots,
\]

where \( N = \lim_{m \to \infty} N_m \) and \( M = \lim_{m \to \infty} M_m \).

(b) It follows directly from (a) that

\[
(DM^T D)PD(DN^T D) = (P_1 D_1) \bigoplus (P_1 D_1) \bigoplus \cdots (P_1 D_1) \bigoplus \cdots,
\]

completing the proof. □

\[\text{Theorem 3.4. Let } N \text{ resp. } M = N^{-1} \text{ be the matrices in Lemma 3.1 resp. Lemma 3.2. Then the following hold:}\]

(a) \( \{Me_0, Me_2, Me_4, \ldots \} \) is a basis for \( E_1(P^T D) \).

(b) \( \{M(e_0 + 2e_1), M(e_2 + 2e_3), \ldots \} \) is a basis for \( E_{-1}(P^T D) \).

(c) \( \{DN^T D(2e_0 + e_1), DN^T D(2e_2 + e_3) \ldots \} \) is a basis for \( E_1(PD) \).

(d) \( \{DN^T De_1, DN^T De_3, DN^T De_5, \ldots \} \) is a basis for \( E_{-1}(PD) \).

Proof. (a) Let \( B_{(1)} = \{e_0, e_2, e_4, \ldots \} \) and \( x = [x_0, x_1, x_2, \ldots]^T \in E_1(NP^T DM) \). Then, by Theorem 3.3 we have

\[
(NP^T DM - I)x = \left( \bigoplus_{i=1}^{\infty} \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} \right) x = 0,
\]

which implies that \( x_i = t_i, x_{i+1} = 0 \) for each \( i = 0, 2, 4, \ldots \) and \( t_i \in \mathbb{R} \). So \( B_{(1)} \) spans \( E_1(NP^T DM) \) and since \( e_0, e_2, e_4, \ldots \) are linearly independent, \( B_{(1)} \) is a basis for \( E_1(NP^T DM) \). Therefore \( \{Me_0, Me_2, Me_4, \ldots \} \) is a basis for \( E_1(P^T D) \).

(b) Let \( B_{(-1)} = \{e_0 + 2e_1, e_2 + 2e_3, \ldots \} \) and \( y = [y_0, y_1, y_2, \ldots]^T \in E_{-1}(NP^T DM) \). Then, by Theorem 3.3 we have

\[
(NP^T DM + I)y = \left( \bigoplus_{i=1}^{\infty} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \right) y = 0,
\]

which implies that \( \begin{bmatrix} y_i \\ y_{i+1} \end{bmatrix} = s_i \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) for each \( i = 0, 2, 4, \ldots \) and \( s_i \in \mathbb{R} \). So \( B_{(-1)} \) spans \( E_{-1}(NP^T DM) \) and since \( e_0 + 2e_1, e_2 + 2e_3, \ldots \) are linearly independent, \( B_{(-1)} \) is a basis for \( E_{-1}(NP^T DM) \). Therefore \( \{M(e_0 + 2e_1), M(e_2 + 2e_3), \ldots \} \) is a basis for \( E_{-1}(P^T D) \).

Clauses (c) and (d) can be proven similarly. □
Consider now
\[
P^T↓ = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 3 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 3 & 4 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & 6 & 5 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

The following is a matrix expression of Theorem 3.4; the last two clauses appeared in [6].

**Corollary 3.5.** Let \( Q = P + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and \( D = \text{diag}((-1)^0, (-1)^1, \ldots) \), where \( P = \begin{bmatrix} \binom{i}{j} \end{bmatrix} \) (\( i, j = 0, 1, 2, \ldots \)). Then the following hold:

(a) The columns of \( P^T↓ \) form a basis for \( E_1(P^T D) \).

(b) The columns of \( Q^T↓(0|0) \) form a basis for \( E_{-1}(P^T D) \).

(c) The columns of \( \begin{bmatrix} 0^T \\ P_i \end{bmatrix} \) form a basis for \( E_{-1}(PD) \).

(d) The columns of \( Q^↓ \) form a basis for \( E_1(PD) \).

**Proof.** (a) Let \((i, j)\) be a pair of indices \( i \) and \( j \) with \( i, j \geq 0 \). The \( i \)th component of \( M e_{2j} \) equals \( \binom{j}{i-j} \) when \( i \leq 2j \), and equals 0, otherwise; \( M e_{2j} = \begin{bmatrix} 0, \ldots, 0, (\binom{j}{0}), (\binom{j}{1}), \ldots, (\binom{j}{j}), 0, 0, \ldots \end{bmatrix}^T \) is the \( j \)th column of \( P^T↓ \), which implies (a).

(b) The \( i \)th component of \( M(e_{2j} + 2e_{2j+1}) \) equals \( \binom{j}{i-j} + 2 \binom{j}{i-j-1} = \binom{j+1}{i-j} + \binom{j}{i-j-1} \), when \( i \leq 2j+1 \), and equals 0 otherwise;

\[
M(e_{2j} + 2e_{2j+1}) = \begin{bmatrix}
\binom{j}{0}, \ldots, 0, \\
\binom{j}{1}, \ldots, \binom{j}{j}, 0, 0, \ldots
\end{bmatrix}^T
\]

is the \( j \)th column of \( P^T↓(0|0) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) \( P^T \) \( (0|0) = Q^T↓(0|0) \), which implies (b). Using the fact that for a pair of indices \( i \) and \( j \) with \( i \geq j \geq 1 \),

\[
(DN^T D)_{ij} = (-1)^i(-1)^{i-j} \binom{j-i}{\frac{j-i}{2}} (-1)^j,
\]

clauses (c) and (d) can be proven similarly. \( \blacksquare \)


4. Invariant sequences of two kinds: Relations and examples

We begin by some basic examples of (inverse) invariant sequences.

Example 4.1. It follows from Corollary 3.5 (c) and (d) that the Fibonacci sequence $F$ is an inverse invariant sequence of the first kind and the Lucas sequence $L$ is an invariant sequence of the first kind.

The $i$th row of $P^T$ is

$$\left[\begin{array}{c}
\left\lfloor \frac{i}{2} \right\rfloor \\
0, \ldots, 0, \left(\left\lfloor \frac{i}{2} \right\rfloor \right), \ldots, \left(\frac{i-2}{2}\right), \left(\frac{i-1}{2}\right), \left(\frac{i}{2}\right), 0, 0, \ldots
\end{array}\right],$$

from which we can get that $J_0F$ is an invariant sequence of the second kind and $J_0L$ is an inverse invariant sequence of the second kind. Recall that $J_0$ is the infinite Jordan block with 0 in the main diagonal, which is the matrix defined in (3.2).

In the following theorem, we provide a general mechanism for transforming invariant sequences into inverse invariant sequences, and vice versa.

Theorem 4.2. Let $J(\lambda)$ denote the matrix defined in (3.2). Then the following hold:

(a) If $x$ is an invariant sequence of the second kind, then $(J(1) + J(0))T x$ is an inverse invariant sequence of the second kind.

(b) If $x$ is an inverse invariant sequence of the second kind, then $J(2)^{-1}J(0)x$ is an invariant sequence of the second kind.

(c) If $x$ is an invariant sequence of the first kind, then $(-J(0)J(2)^{-1})^T x$ is an inverse invariant sequence of the first kind.

(d) If $x$ is an inverse invariant sequence of the first kind, then $(J(-1) + J(0))x$ is an invariant sequence of the first kind.

Proof. (a) If $x$ is an invariant sequence of the second kind, then by Corollary 3.5 (a), there exists $b \in \mathbb{R}^\infty$ such that $P^T b = x$. Let $L = [l_{ij}]$ be the infinite lower triangular matrix defined by

$$l_{ij} = \begin{cases} (-1)^{i+j}, & \text{if } i \geq j, \\ 0, & \text{if } i < j. \end{cases}$$

Then $LP^T(0) = P^T$ because for a pair of indices $i$ and $j$ with $i, j \geq 0$, the $(i,j)$ entry of $LP^T(0)$ equals

$$(-1)^{i+j}\left(\begin{array}{c}
\frac{j+1}{i+j} \\
0
\end{array}\right) + \ldots + (-1)^{i+j+i-j}\left(\begin{array}{c}
\frac{j+1}{i} \\
\frac{i-j}{i-j}
\end{array}\right) = \left(\begin{array}{c}
\frac{j}{i-j}
\end{array}\right).$$
when \( i \geq j \), and equals 0 otherwise, which coincides with the \((i, j)\) entry of \(P^T\). Thus,

\[
LQ^T(0)0 = P^T + L\left[ \begin{array}{c} 0^T \\ P^T \end{array} \right].
\]

(4.2)

From the fact that \( L^{-1} = J(1)^T \) and by Corollary 3.5 (b), it follows that

\[
Q^T(0)0b = J(1)^TP^Tb + \left[ \begin{array}{c} 0^T \\ P^T \end{array} \right]b = J(1)^Tx + [0, x^T]^T = (J(1) + J(0))^Tx
\]

is an inverse invariant sequence of the second kind.

(b) Let \( x \) be an inverse invariant sequence of the second kind. From (4.2) and \( J(0)J(1)^T = J(1) \), it readily follows that

\[
J(2)^{-1}J(0)Q^T(0)0 = P^T
\]

and so by Corollary 3.5 (b), we conclude that \( J(2)^{-1}J(0)x \) is an invariant sequence of the second kind.

(c) Since \( Q^i = P^i + \left[ \begin{array}{cc} 1 & 0^T \\ 0 & 0^T \\ 0 & P^i \end{array} \right] \), we obtain \( J(0)^TQ^i = J(0)^TP^i + \left[ \begin{array}{cc} 0 & 0^T \\ 1 & 0^T \\ 0 & 0^T \\ 0 & P^i \end{array} \right] \). For nonnegative integers \( i \) and \( k \) with \( i, k = 0, 1, 2, \ldots \), since \( \binom{i}{j} + \binom{i}{j+1} + \cdots + \binom{i}{i+k} = \binom{i+k}{j} \), it implies that

\[
\Omega (\left[ \begin{array}{c} 0^T \\ Q^i \\ P^i \end{array} \right] - \left[ \begin{array}{c} 0^T \\ P^i \end{array} \right]) = \left[ \begin{array}{c} 0^T \\ P^i \end{array} \right],
\]

where \( \Omega \) is the infinite \((0,1)\)-matrix with 1’s everywhere on and below its main diagonal. Since \( -J(-1)^T = \Omega^{-1} \), we get \( (I - J(-1)^T)^{-1} \left[ \begin{array}{c} 0^T \\ Q^i \\ P^i \end{array} \right] = \left[ \begin{array}{c} 0^T \\ P^i \end{array} \right] \), which implies that \( (-J(0)J(-2)^{-1})^T x \) is an inverse invariant sequence of the first kind.

Clause (d) easily follows from (c) similarly. ■

Let \( \tau_1 = \frac{1+\sqrt{5}}{2} \) and \( \tau_2 = \frac{1-\sqrt{5}}{2} \). It is well known that \( F_n = \frac{1}{\sqrt{5}}\tau_1^n - \frac{1}{\sqrt{5}}\tau_2^n \) and \( L_n = \tau_1^n + \tau_2^n \) where \( F_n \) and \( L_n \) are the \( n \)th terms of \( F \) and \( L \), respectively \( (n = 0, 1, 2, \ldots) \). Since \( J(0)F \) is an invariant sequence of the second kind by (4.1), it follows from Theorem 4.2 (a) that

\[
J(1)^TJ(0)F + J(0)^TJ(0)F = J(1)^TJ(0)F + F
\]

is an inverse invariant sequence of the second kind. In fact, the \( n \)th term \((n = 0, 1, 2, \ldots)\) of \( J(1)^TJ(0)F + F \) is

\[
\frac{1}{\sqrt{5}}\tau_1^{n+2} - \frac{1}{\sqrt{5}}\tau_2^{n+2} + \frac{1}{\sqrt{5}}\tau_1^n - \frac{1}{\sqrt{5}}\tau_2^n = L_{n+1},
\]

which implies that \( J(1)^TJ(0)F + F = J(0)L \). On the other hand, since for a pair of nonnegative integers \( i \) and \( j \) with \( i, j = 0, 1, 2, \ldots \),

\[
(J(2)^{-1})_{ij} = \begin{cases} (-1)^{j-i}(\frac{1}{2})^{j-i+1}, & \text{if } i \leq j, \\ 0, & \text{if } i > j, \end{cases}
\]

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and the $n$-th term of $J(2)^{-1}J(0)^2L$ is
\[
\sum_{k=0}^{\infty} (-1)^k(1/2)^{k+1}(r_1^{k+2+n} + r_2^{k+2+n}) = \frac{1}{\sqrt{5}}r_1^{n+1} - \frac{1}{\sqrt{5}}r_2^{n+1} = F_{n+1},
\]
we have $J(2)^{-1}J(0)^2L = J(0)F$, which is an invariant sequence of the second kind by Theorem 4.2 (b), as stated in Example 1.1.

It directly follows from Theorem 4.2 (d) that $(J(0)+J(-1))F$, namely $L$, is an invariant sequence of the first kind. Since for a pair of nonnegative integers $i$ and $j$ with $i, j = 0, 1, 2, \ldots$, $(-J(-2)^{-1})_{ij} = \left\{\begin{array}{ll} (1/2)^{j-i+1}, & \text{if } i \leq j, \\ 0, & \text{if } i > j, \end{array}\right.$ we obtain $(-J(-2)^{-1})^T [0, L^T]^T = F$, which is an inverse invariant sequence of the first kind by Theorem 4.2 (c), because for $n = 0, 1, 2, \ldots$, $\sum_{k=0}^{n-1} (1/2)^{n-k}L_k = (1/2)^n \sum_{k=0}^{n-1} ((2r_1)^k + (2r_2)^k) = \frac{1}{\sqrt{5}}r_1^n - \frac{1}{\sqrt{5}}r_2^n = F_n$.

The sequence $B = (B_0, B_1, B_2, \ldots)^T$ defined by $B_0 = 1$ and $\sum_{k=0}^{n} \binom{n+1}{k}B_k = 0$ ($n \geq 1$) comprises the Bernoulli numbers and $DB$ is an invariant sequence of the first kind [11], which also follows directly from the fact that $PDDB = DB$. A new inverse invariant sequence of the first kind from the Bernoulli numbers $B$ is provided next. See Table 1 for explicit members of these sequences.

**Corollary 4.3.** Let $B = (B_0, B_1, B_2, \ldots)^T$ be the Bernoulli numbers. Then the sequence $K = (K_0, K_1, K_2, \ldots)^T$ defined by
\[
K_0 = 0, \quad K_n = \sum_{k=0}^{n-1} (1/2)^{n-k}(-1)^kB_k \quad (n = 1, 2, \ldots)
\]
is an inverse invariant sequence of the first kind.

**Proof.** It follows from Theorem 4.2 (c) that
\[
(-J(-2)^{-1})^T J(0)^T DB = (-J(-2)^{-1})^T [0, DB^T]^T
\]
is an invariant sequence of the first kind. Notice now that the sequence $K$ in the statement is indeed equal to $(-J(-2)^{-1})^T [0, DB^T]^T$. ■

By Theorem 4.2 (d), we get $DB = (J(0)+J(-1))K$, since the first component of $(J(0)+J(-1))K$ is clearly $(-1)^0B_0$, and for each positive integer $i$ with $i = 1, 2, \ldots$, $i$th component of $(J(0)+J(-1))K$ is
\[
-\sum_{k=0}^{i-1} (1/2)^{i-k}(-1)^kB_k + 2 \sum_{k=0}^{i} (1/2)^{i+1-k}(-1)^kB_k = (-1)^iB_i.
\]
By Corollary 3.5 and Theorem 4.2 we can directly get more (inverse) invariant sequences of the first and second kind as follows:
Corollary 4.4. For a positive integer \( n \), let \( \Phi_n = S_{l_0} S_{l_1} \ldots S_{l_{n-1}} S_{l_2} S_{l_1} \), where

\[
S_{l_i} = \begin{cases} P^{T_i}, & \text{if } l_i = 1, \\ J(1)^T + J(0)^T, & \text{if } l_i = -1,
\end{cases}
\]

and let \( \Phi_n = \tilde{S}_{l_0} \tilde{S}_{l_1} \ldots \tilde{S}_{l_{n-1}} \tilde{S}_{l_2} \tilde{S}_{l_1} \), where

\[
\tilde{S}_{l_i} = \begin{cases} J(2)^{-1} J(0), & \text{if } l_i = 1, \\ Q^{T_i}(0)^T, & \text{if } l_i = -1
\end{cases}
\]

for \( i = 1, 2, \ldots, n \). Then for \( x \in \mathbb{R}^\infty \), we have the following:

(a) If \( n \) is odd (even) and \( l_i = (-1)^{i+1} \) for \( i = 1, 2, \ldots, n \), then \( \Phi_n x \) is an invariant (inverse invariant) sequence of the second kind.

(b) If \( n \) is odd (even) and \( l_i = (-1)^i \) for \( i = 1, 2, \ldots, n \), then \( \tilde{\Phi}_n x \) is an invariant (inverse invariant) sequence of the second kind.

Example 4.5. It follows from (4.1) and Corollary 4.4(a) that for \( x = [x_0, x_1, x_2, \ldots]^T \in \mathbb{R}^\infty \), \( \Phi_2 x = y \) is an inverse invariant sequence of the second kind, where \( y = [y_0, y_1, y_2, \ldots]^T \) with

\[
y_i = \sum_{t=\lceil \frac{i}{2} \rceil}^i \left( \binom{t}{i-1-t} + \binom{t+1}{i-t} \right) x_t \quad (i = 0, 1, 2, \ldots).
\]

For example, let \( x = [0, 0, 0, 0, 0, 0, 1, 0, 0, \ldots]^T \). This results to the inverse invariant sequence of the second kind \( y = [y_0, y_1, y_2, \ldots]^T \), where

\[
y_i = \left( \frac{7}{i-1-7} \right) + \left( \frac{8}{i-7} \right) \quad (i = 0, 1, 2, \ldots).
\]

That is, \( y_0 = \cdots = y_6 = 0 \), \( y_7 = y_8, \ldots, y_{15} \); e.g., \( y_{10} = \left( \frac{7}{10-1-7} \right) + \left( \frac{8}{10-7} \right) = 77 \).

Corollary 4.6. For a positive integer \( n \), let \( \Psi_n = T_{l_0} T_{l_1} \ldots T_{l_{n-1}} T_{l_2} T_{l_1} \), where

\[
T_{l_i} = \begin{cases} Q^{T_i}, & \text{if } l_i = 1, \\ (-J(-2)^{-1})^T J(0)^T, & \text{if } l_i = -1
\end{cases}
\]

\[
|X_{2n+1}| = |X_{0,1,2, \ldots, n}|
\]

Table 1: (Inverse) invariant sequences of the first kind associated with the Bernoulli numbers.

<table>
<thead>
<tr>
<th>n</th>
<th>B</th>
<th>DB</th>
<th>K</th>
<th>0 1 2 3 4 5 6 7 8 9 10 11 12 ⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>−1/2 1/6 0 −1/30 0 1/42 0 −1/30 0 5/66 0 −691/2730 ⋯</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>−1/6 1/6 0 −1/30 0 1/42 0 −1/30 0 5/66 0 −691/2730 ⋯</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1/2 1/6 1/6 1/2 3/10 1/5 1/6 1/6 1/5 1/2 3/10 −1/10 5/210 −2/14 −263/9399 ⋯</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
and let $\tilde{\Psi}_n = \tilde{T}_n \tilde{T}_{n-1} \cdots \tilde{T}_3 \tilde{T}_2 \tilde{T}_1$, where

$$
\tilde{T}_i = \begin{cases} 
J(0) + J(-1), & \text{if } l_i = 1, \\
\begin{bmatrix} 0^T \\ P^\perp \end{bmatrix}, & \text{if } l_i = -1 
\end{cases}
$$

for $i = 1, 2, \ldots, n$. Then for $x \in \mathbb{R}^\infty$, we have the following:

(a) If $n$ is odd (even) and $l_i = (-1)^{i+1}$ for $i = 1, 2, \ldots, n$, then $\Psi_n x$ is an invariant (inverse invariant) sequence of the first kind.

(b) If $n$ is odd (even) and $l_i = (-1)^i$ for $i = 1, 2, \ldots, n$, then $\tilde{\Psi}_n x$ is an inverse invariant (invariant) sequence of the first kind.

Example 4.7. For $i = 1, 2, \ldots$, the $i$th row of $\begin{bmatrix} 0^T \\ P^\perp \end{bmatrix}$ is $[(i-1), (i-2), \ldots, \left(\begin{array}{c} \frac{i+1}{2} \\ \frac{i-1}{2} \end{array}\right), 0, 0, \ldots]$. It follows that for $x = [x_0, x_1, x_2, \ldots]^T \in \mathbb{R}^\infty$, $\tilde{\Psi}_2 x = y$ is an invariant sequence of the first kind, where by Corollary 4.6 (b), $y = (y_0, y_1, y_2, \ldots)^T$ satisfies

$$
y_i = \sum_{t=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \left( \begin{array}{c} i-1-t \\ t-1 \end{array} \right) + \left( \begin{array}{c} i-t \\ t \end{array} \right) x_t.
$$

For example, let $x = [0, 0, 0, 0, 1, 0, 0, \ldots]^T$. This results to the invariant sequence of the first kind $y = [y_0, y_1, y_2, \ldots]^T$, where

$$
y_i = \left( \begin{array}{c} i-1-7 \\ 6 \end{array} \right) + \left( \begin{array}{c} i-7 \\ 7 \end{array} \right) (i = 1, 2, \ldots).
$$

Thus $y_0 = y_1 = \cdots = y_{13} = 0$ and one can find by direct calculation, the nonzero components $y_{14}, y_{15}, \ldots$; e.g., $y_{14} = \left( \begin{array}{c} 6 \\ 7 \end{array} \right) = 2$.

In the next two theorems, we obtain direct relationships among (inverse) invariant sequences of the first kind and (inverse) invariant sequences of the second kind.

Theorem 4.8. Let $x$ and $y$ be, respectively, either

(i) an invariant sequence of the first kind and an inverse invariant sequence of the second kind,

or

(ii) an inverse invariant sequence of the first kind and an invariant sequence of the second kind.

Then $x^T D y = 0$.

Proof. From $PDx = \lambda x$ and $P^T D y = -\lambda y$, we have $x^T D P^T D y = \lambda x^T D y$, which implies that $2\lambda x^T D y = 0$ for $\lambda \in \{1, -1\}$. ■

The following theorem is a useful tool for proving the final theorem.

Theorem 4.9. For every positive integer $n$,
The columns of \((P + D)^n\) are invariant sequences of the first kind, and the columns of \((P - D)^n\) are inverse invariant sequences of the first kind.

(b) the columns of \((P^T + D)^n\) are invariant sequences of the second kind, and the columns of \((P^T - D)^n\) are inverse invariant sequences of the second kind.

**Proof.** Let \(n\) be a positive integer. Then

\[
PD(P + D)^n = PD(P + D)(P + D)^{n-1} = (P + D)^n
\]

and

\[
P^T D(P^T - D)^n = P^T D(P^T - D)(P^T - D)^{n-1} = -(P^T - D)^n,
\]

which respectively imply that each column of \((P + D)^n\) is an invariant sequence of the first kind, and each column of \((P^T - D)^n\) is an inverse invariant sequence of the second kind. The other assertions of the theorem follow similarly. ■

The following theorem is a form of converse of Theorem 4.8.

**Theorem 4.10.** Let \(x_i \in \mathbb{R}^\infty (i = 0, 1, 2, \ldots), y \in \mathbb{R}^\infty \setminus \{0\},\) and let \(X = [x_0, x_1, x_2, \ldots].\) Then for each \(i = 0, 1, 2, \ldots,\)

(a) if \(x_i^T Dy = 0\) and \(X = P + D (X = P - D),\) then \(y\) is an inverse invariant (invariant) sequence of the second kind, and \(x_i\) is an invariant (inverse invariant) sequence of the first kind.

(b) if \(x_i^T Dy = 0\) and \(X = P^T + D (X = P^T - D),\) then \(y\) is an inverse invariant (invariant) sequence of the first kind and \(x_i\) is an inverse invariant (invariant) sequence of the second kind.

**Proof.** If \(x_i^T Dy = 0\) and \(X = P + D,\) then since \(X^T Dy = (P^T + D)Dy = 0,\) we have \(P^T Dy = -y\) and by Theorem 4.9 \(PDx_i = x_i.\) Thus \(y\) is an inverse invariant sequence of the second kind and \(x_i\) is an invariant sequence of the first kind. This proves the first case of part (a). For the case of \(X = P - D\) and part (b), the results can be shown similarly. ■

**References**


