Inverse relations in Shapiro’s open questions✩

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A R T I C L E   I N F O

Article history:
Received 27 September 2016
Received in revised form 9 November 2017
Accepted 14 November 2017

Keywords:
Riordan matrix
Semi-Riordan matrix
Catalan number
Motzkin number

A B S T R A C T

As an inverse relation, involution with an invariant sequence plays a key role in combinatorics and features prominently in some of Shapiro’s open questions (Shapiro, 2001). In this paper, invariant sequences are used to provide answers to some of these questions about the Fibonacci matrix and Riordan involutions.

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1. Introduction

Inverse relations play a pivotal role in many research topics in combinatorics [11]. Among the open questions posed by Shapiro [12], Q2, Q8, and Q8.1 regard involutions as a trait of inverse relations and have been the research focus of several authors [2–4, 12]. The concept of a Riordan matrix and generalizations of the Pascal, Catalan, and Motzkin triangles [15] allow us in this paper to obtain in-depth answers to Shapiro’s open questions above; our answers are naturally related to the Fibonacci matrix and Riordan involutions.

Definition 1.1. An infinite lower triangular matrix \( R = (g(x), f(x)) \) with entries in \( \mathbb{R} \), the real numbers, is called a Riordan matrix provided that the generating function of the \( i \)th column of \( R \) is \( g(x)f(x)^i \) for \( i = 0, 1, 2, \ldots \), where \( g(x) = g_0 + g_1x + g_2x^2 + \cdots \) and \( f(x) = f_1x + f_2x^2 + \cdots \) with \( g_0 \neq 0 \) and \( f_1 \neq 0 \). Moreover, a Riordan matrix \( R \) is called a Riordan involution if \( R^2 = (1, x) \), and a Riordan pseudo involution if \((RD)^2 = (1, x)\), where \( D = \text{diag}(1, -1, 1, -1, \ldots)\) [4].

The definition of Riordan (pseudo) involutions above is indeed facilitated by the fact that the set of all Riordan matrices is a group under matrix multiplication, referred to as the Riordan group [14], where multiplication amounts to

\[
(g(x), f(x))(h(x), l(x)) = (g(x)h(f(x)), l(f(x))).
\]

(1.1)

Shapiro’s aforementioned open questions have long been of great interest in the investigation of Riordan (pseudo) involutions, as well as the Riordan group. We restate them below, recalling that \( D = \text{diag}(1, -1, 1, -1, \ldots) \).

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✩ Research supported by Daegu University Research Grant 20150223.
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Q2: Let

\[
\mathbb{F} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

denote the Fibonacci matrix \([12]\). Is there a combinatorial connection between \(\mathbb{F}\) and \(DF^{-1}D\)?

Q8: Can every Riordan involution \(R\) be written as \(R = DBB^{-1}\) for some element \(B\) in the Riordan group?

Q8.1: If a Riordan involution \(R\) is of some particular combinatorial significance, can we find a \(B\) in the Riordan group, which has a related combinatorial significance and \(R = DBB^{-1}\)?

In \([13]\), several subgroups of the Riordan group are introduced. For example, \(((g(x), c\lambda)|c \neq 0)\) and \(((g(x), c\lambda g(x))|c \neq 0)\) are called the \(c\)-Appell and \(c\)-Bell subgroups, respectively. In the case of \(c = 1\), these are simply called the Appell and Bell subgroups, respectively. In \([3]\), Cheon and Kim showed the existence of Riordan matrices \(B\) as affirmative answers to Q8 and Q8.1 by using an antisymmetric function and by adopting the Bell subgroup, respectively. In \([4]\), Cheon et al. presented a pseudo involution \(R_n = (n = 0, 1, \ldots)\) as a generalization of the RNA triangle, where for any generating function \(G(x)\) with \(G(0) \neq 0\) and for each nonnegative integer \(n\), \(R_n = \left(\frac{G(x)}{x^{n+1}}\right)^n xg(x)\). The latter equality is a generalized form of Cameron and Nkwanta’s example in \([2]\) for \(G(x) = 1 - x\), that is, \(W_n = \left(xg(x)\left(\frac{1-x}{1-xg(x)}\right)^n xg(x)\right)\) with \(W_n = A^{-n}W_0A^n\); the latter can be thought of as a partial answer to Q8, where \(g(x) = \frac{(1-x+x^2) - \sqrt{(1-x+x^2)^2 - 4x^2}}{2x^2}\) and \(A^n = \left(\frac{1}{1-xg(x)}x\right)\).

The shared notion of involutions as self-inverse relations in Q8 and Q8.1, as well as the Fibonacci and Catalan numbers, have been extensively studied. Little attention has been paid, however, to Q2. In fact, the row sums of \(\mathbb{F}\) and \(DF^{-1}D\) are the Fibonacci and Catalan numbers, respectively. The objectives of our research are to present plausible answers to Q2, and also to give answers related to invariant sequences as self-inverse relations \([9–11, 17]\) to Q8 and Q8.1. More specifically, in this paper, we investigate the structure of entries in \(DF^{-1}D\) and the role of \(DF^{-1}D\) in transforming invariant sequences, giving rise to answers for Q2. We also provide a method for constructing invariant sequences shown in \([10]\) by means of Riordan (pseudo) involutions, which allows us to answer both Q8 and Q8.1.

2. Notation and preliminaries

The following notation and conventions are used throughout the paper:

- Infinite matrices have infinite numbers of rows \(i\) and columns \(j\), with \(i, j \in \{0, 1, 2, \ldots\}\).
- Infinite real sequences \(\{x_n\}\) are identified with the infinite dimensional real vector space \(\mathbb{R}^\infty\) consisting of column vectors \(x = [x_0, x_1, \ldots]^T\).
- \(\mathbb{E}_\lambda(A)\) denotes the eigenspace of a (finite or infinite) matrix \(A\) corresponding to its eigenvalue \(\lambda\).
- The column vector of zeros in \(\mathbb{R}^j\) is denoted by \(\mathbf{0}_j\), with \(\mathbf{0}_0\) being by convention vacuous.
- For a matrix \(A = [a_{ij}]\) with columns \(A_j\), \(A^i\) denotes the matrix whose column \(j\) is \(a_{ij}\); that is,

\[
A^i = \begin{bmatrix}
a_{00} & 0 & 0 & 0 & \cdots \\
a_{10} & a_{01} & 0 & 0 & \cdots \\
a_{20} & a_{11} & a_{02} & 0 & \cdots \\
a_{30} & a_{21} & a_{12} & a_{03} & 0 & \cdots \\
a_{40} & a_{31} & a_{22} & a_{13} & a_{04} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.
\]

- Let \(M\) and \(N\) be the (possibly infinite) row and column index sets of a matrix \(A\), respectively. For subsets \(M_0 \subseteq M\), \(N_0 \subseteq N\), \(A(M_0|N_0)\) denotes the matrix obtained from \(A\) by deleting all rows indexed by \(M_0\) and columns indexed by \(N_0\). For brevity, we write \(M(i)|j)\) in place of \(M(\{i\}|j)\) for \(i, j \in \{0, 1, 2, \ldots\}\).
- \(P = \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]\) \((i, j = 0, 1, 2, \ldots)\) denotes the (infinite) Pascal matrix.
• For a constant $a$, $J(a)$ denotes the infinite Jordan block of the form

$$\begin{bmatrix} a & 1 \\ a & 1 & \ddots \\ & & \ddots & a \\ O & & & \ddots \\ & & & & O \end{bmatrix}.$$

Now, we present a slight extension of the notion of a Riordan matrix in Definition 1.1.

**Definition 2.1.** An infinite lower triangular matrix $R$ with entries in $\mathbb{R}$, the real numbers, is called a semi-Riordan matrix if the generating function of the $i$th column of $R$ is $g(x)f(x)^i$ for $i = 0, 1, \ldots$, where $g(x) = g_0 + g_1x + g_2x^2 + g_3x^3 + \cdots$ and $f(x) = f_1x + f_2x^2 + f_3x^3 + \cdots$. For convenience, we denote it by $R = (g(x), f(x))$ similarly to the Riordan matrix abbreviation.

Note that the definition of semi-Riordan matrices can be obtained by excluding the conditions $g_0 \neq 0$ and $f_1 \neq 0$ from that of Riordan matrices in Definition 1.1. For two semi-Riordan matrices $(g(x), f(x))$ and $(h(x), l(x))$, matrix multiplication can be defined only when $f(x)$ and $l(x)$ are inverse $R$-invariant sequence of the first kind and $g(x)$ and $h(x)$ are inverse $R$-invariant sequence of the second kind, respectively; see [10]. It can be proven easily that

$$(g(x) + h(x), f(x)) = (g(x) + h(x), f(x)),$$

where $u(x)$ is the generating function of a column vector in $\mathbb{R}^\infty$. We generalize the notions of invariant and inverse invariant sequences of the first or second kind in [10] as follows:

**Definition 2.2.** For a Riordan involution (resp., pseudo involution) $R$, $x \in \mathbb{R}^\infty$ is called

(a) an $R$-invariant sequence of the first kind if $x \in \mathbb{E}_1(R)$ (resp., $x \in \mathbb{E}_1(RD)$).

(b) an inverse $R$-invariant sequence of the first kind if $x \in \mathbb{E}_{-1}(R)$ (resp., $x \in \mathbb{E}_{-1}(RD)$).

(c) an $R$-invariant sequence of the second kind if $x \in \mathbb{E}_1(R^T)$ (resp., $x \in \mathbb{E}_1(R^TD)$).

(d) an inverse $R$-invariant sequence of the second kind if $x \in \mathbb{E}_{-1}(R^T)$ (resp., $x \in \mathbb{E}_{-1}(R^TD)$).

Let $F = [F_0, F_1, F_2, \ldots]^T$ and $L = [L_0, L_1, L_2, \ldots]^T$ denote the vectors in $\mathbb{R}^\infty$ whose entries are the members of the Fibonacci and Lucas sequences, respectively; that is

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2),$$

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad (n \geq 2).$$

$F$ and $L$ are inverse $P$-invariant and $P$-invariant sequences of the first kind, respectively, for a Riordan pseudo involution $P$ [5,10]. It can be proven easily that $J(0)F$ and $J(0)L$ are $P$-invariant and inverse $P$-invariant sequences of the second kind, respectively; see [10].

The following result leads us to investigate the relationships between the Fibonacci matrix $F$ and $DF^{-1}D$ [10]; the last two clauses appeared in [5]. Note that, by utilizing (1.1) and (2.1), we have expressed the result in terms of the semi-Riordan matrices.

**Lemma 2.3.** Let $P = \left( \frac{1}{1-x}, \frac{x}{1-x} \right)$, $D = (1, -x)$, and $Q = \left( \frac{1-x}{x}, \frac{1}{x} \right)$. Then the following hold:

(a) The columns of $P^\updownarrow_1 = (1, x(1 + x))$ form a basis for $\mathbb{E}_1(P^TD)$.

(b) The columns of $Q^\updownarrow_1(0,0) = (1 + 2x, x(1 + x))$ form a basis for $\mathbb{E}_{-1}(P^TD)$.

(c) The columns of $Q^\updownarrow_{1 \uparrow} = \left( \frac{1}{x}, \frac{x}{1-x} \right)$ form a basis for $\mathbb{E}_{-1}(PD)$.

(d) The columns of $Q^\downarrow_1 = \left( \frac{x}{1-x}, \frac{1-x}{x} \right)$ form a basis for $\mathbb{E}_1(PD)$.

Surprisingly, $P^\updownarrow_1$ is equal to the Fibonacci matrix $F$, the columns of which form a basis for $\mathbb{E}_1(P^TD)$.

**Definition 2.4.** We call the four special matrices $P^\updownarrow_1$, $Q^\updownarrow_1(0,0)$, $\begin{bmatrix} 0^\updownarrow_1 \\ \updownarrow_1 \end{bmatrix}$, and $Q^\downarrow_1$ the Fibonacci matrix of the second kind, the Lucas matrix of the second kind, the Fibonacci matrix of the first kind, and the Lucas matrix of the first kind, and we denote them by $F^{(2)}$, $L^{(2)}$, $F^{(1)}$, and $L^{(1)}$, respectively.
For each Riordan matrix $R = (g(x), f(x))$, we consider its inverse matrix given by

$$R^{-1} = (1/g(\overline{f}(x)), \overline{f}(x)), \quad (2.2)$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$, i.e., $f(\overline{f}(x)) = x$ [14].

For $P \in (1, x(1 + x)) \text{ resp. } L = (1 + 2x, x(1 + x))$, we can obtain by (2.2) the well-known result that $(p(2))^{-1} = (1, \frac{1 - \sqrt{4x + 1}}{2x}, 1)$ and $(L(2))^{-1} = (1, \frac{1 - \sqrt{4x - 1}}{2x}, 1)$. Therefore,

$$(p(2))^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 2 & -2 & 1 & \cdots \\ 0 & -5 & 5 & -3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (L(2))^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & \cdots \\ 6 & -3 & 1 & 0 & \cdots \\ -20 & 10 & -4 & 1 & 0 & \cdots \\ 70 & -35 & 15 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$  

The relationships between these four special matrices and $D(L(2))^{-1}D$, as well as $D(p(2))^{-1}D$, play a key role in our investigation.

Our effort in this paper will proceed as follows: In Section 3, we show that

- $D(p(2))^{-1}D = (1, \nu C(x)) = P \left( \frac{1 - \sqrt{4x}}{2x}, \frac{x + \sqrt{1 - x^2 - 4x^2}}{2x} \right)$,
- $D(L(2))^{-1}D = \left( \frac{1}{1 - 2x(1 + x)}, \nu C(x) \right) = (W(x), x) D(p(2))^{-1}D$

by utilizing the Catalan and Motzkin numbers [8], where $C(x) = \frac{1 - \sqrt{4x}}{2x}$, $M(x) = \frac{1 - x - \sqrt{(1-x)^2 - 4x^2}}{2x}$, and $W(x) = \sum_{n=0}^{\infty} (\frac{2n}{n}) x^n$. In the process, we show that each entry of $D(p(2))^{-1}D$ can be expressed as a linear combination of Catalan numbers with coefficients connected to the Fibonacci sequence, and that each entry of $D(L(2))^{-1}D$ can be expressed as a linear combination of Catalan numbers with coefficients connected to the Fibonacci and Lucas sequences. This amounts to an answer to Q2.

In Section 4, we show that $D(p(2))^{-1}D$ and $D(L(2))^{-1}D$ provide a mechanism (featuring the Catalan numbers) for transforming a $P$-invariant into an inverse $P$-invariant sequence of the second kind, and vice versa. This provides an answer to Q2, as a combinatorial relationship between $p(2)$ and $D(p(2))^{-1}D$. We also present the relationships (featuring the Catalan and Motzkin numbers) between the generating functions of $P$-invariant and inverse $P$-invariant sequences of the second kind. In Section 5, we provide a method for constructing $P$-invariant sequences of the first or second kind by means of the Riordan (pseudo) involution $R$ itself. It follows that for every Riordan involution $R$ in the $(-1)$-Appell subgroup, there exists $B_n (n = 1, 2, \ldots)$ in the Appell subgroup such that $R = B_n DB_n^{-1}$, thus providing answers to Q8 and Q8.1.

3. The structures of $D(p(2))^{-1}D$ and $D(L(2))^{-1}D$

In this section, we investigate the structures of the matrices $D(p(2))^{-1}D$ and $D(L(2))^{-1}D$, using the nth Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ and the Motzkin number $M_n = \sum_{k=0}^{\frac{n}{2}} \binom{n}{k} C_k$ for $n = 0, 1, 2, \ldots$ [8]. This contributes to the investigation of the structures of $D(p(2))^{-1}D$ and $D(L(2))^{-1}D$, leading to answers for Q2. Hereafter, let $C(x)$ and $M(x)$ denote the generating functions of the Catalan numbers and Motzkin numbers, respectively.

**Lemma 3.1.** Let $p(2)$ and $L(2)$ denote the Fibonacci and Lucas matrices of the second kind, respectively. Let $P$ be the Pascal matrix.

Then the following hold:

(a) $D(p(2))^{-1}D = (1, \nu C(x)) = P \left( \frac{1}{1 + x}, \frac{x + \sqrt{\frac{x}{1 + x}}}{1 + x} \right)$.

(b) $D(L(2))^{-1}D = \left( \frac{1}{1 - 2x(1 + x)}, \nu C(x) \right) = (W(x), x) P \left( \frac{1}{1 + x}, \frac{x + \sqrt{\frac{x}{1 + x}}}{1 + x} \right)$.

(c) $D(L(2))^{-1}D = \left( \frac{1}{1 - 2x(1 + x)}, \nu D(p(2))^{-1}D = (W(x), x) D(p(2))^{-1}D = D(p(2))^{-1}D \left( \frac{1}{1 + x}, x \right)$.

where $C(x) = \frac{1 - \sqrt{4x}}{2x}, M(x) = \frac{1 - x - \sqrt{(1-x)^2 - 4x^2}}{2x}$, and $W(x) = \sum_{n=0}^{\infty} (\frac{2n}{n}) x^n$.

**Proof.** It follows directly from (1.1) and (2.2) that $D(p(2))^{-1}D = \left( 1, \frac{1 - \sqrt{4x}}{2x} \right)$ and $D(L(2))^{-1}D = \left( \frac{1}{1 - \sqrt{4x}}, \frac{1 - \sqrt{4x}}{2x} \right)$. Since $T(\frac{x}{1 + x}) = M(x)$, which is the result in [8], we obtain

$$\left( \frac{1}{1 + x}, \frac{x}{1 + x} \right) \left( C(x) - \frac{1}{x} \right) = \left( M(x), \frac{x}{1 + x} \right), \quad (3.1)$$
where $T(f(x)) = \frac{1}{1+x^2}f(\frac{x}{1+x^2})$ is the Euler transformation and

$$M(x) = \frac{1-x - (1-x)^2 - 4x^2}{2x^2}.$$ 

By Newton’s binomial theorem [1, 6], $(1 - 4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (\frac{n}{2}) (-4x)^n$ and $(1 - \frac{1-4x}{4}) = \sum_{n=0}^{\infty} \frac{1}{n+1} (\frac{2n}{n}) x^{n+1}$. Thus, $D^{(2)} = (1, x, C(x))$ by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ [14]. Since $P\left(\frac{1}{1+x^2}, \frac{1}{2x}\right) = (1, x)$ and $\left(\frac{x-1}{x}, xC(x)\right) = (1, xC(x))$, by (3.1) we obtain

$$(1, xC(x)) = P\left(M(x), \frac{x}{1+x}\right)\left(\frac{x}{C(x)-1}, xC(x)\right),$$

which implies (a) from the fact that $\frac{x}{C(x)-1} = 1 - x - xC(x)$, $C(\frac{x}{1+x}) = 1 + xM(x)$, and $M(x) - xM(x) - x^2M(x)^2 = 1$ [8].

Clause (b) follows from (a) and $D^{(2)} = D(W(x), x)\{1, xC(x)\}$ since $\sqrt{1-4x} = 1 - 2xC(x)$.

Clause (c) is a consequence of (a), (b), and $D^{(2)} = (1, xC(x))\{1-\frac{1}{2x}\}$. □

The following theorem contains the recurrence relations for the entries of $D^{(2)}$ and $D^{(2)}$.

**Theorem 3.2.** Let $D^{(2)} = [r_{ij}]$ and $D^{(2)} = [q_{ij}]$ for $i$ and $j$ with $i, j = 0, 1, 2, \ldots$ Then

(a) $r_{00} = 1$ and for $i = 1, 2, \ldots; j = 2, 3, \ldots$

$r_{00} = 0, r_{11} = \frac{2i-2}{i-1}$, and $r_{ij} = -r_{i-1,j-2} + r_{i,j-1}$. (3.2)

(b) $q_{00} = \frac{2}{i+1}$ for $i = 0, 1, 2, \ldots; q_{11} = \frac{2}{i+1}$ for $i = 1, 2, \ldots$, and

$q_{ij} = -q_{i-1,j-2} + q_{i,j-1}$ for $i = 1, 2, \ldots; j = 2, 3, \ldots$. (3.3)

**Proof.** Let $g(x)^{j-1}\left(\frac{1-\sqrt{1-4x}}{2}\right)^j$ be the generating function of the $j$th column of $D^{(2)}$ when $s = 1, g(x) = 1$ or $D^{(2)}$ (when $s = 2, g(x) = \frac{1}{1+x}$). For $j \geq 2$,

$$g(x)^{j-1}\left(\frac{1-\sqrt{1-4x}}{2}\right)^j = g(x)^{j-1}\left(\frac{1-\sqrt{1-4x}}{2}\right)^j - xg(x)^{j-1}\left(\frac{1-\sqrt{1-4x}}{2}\right)^j,$$

which, along with Lemma 3.1, implies that (a) and (b) hold. □

To illustrate this theorem, all the entries of $D^{(2)}$ and $D^{(2)}$ can be completely determined by the recurrence relations in Theorem 3.2, once the entries in the first and second columns are determined; indeed, we have

$$D^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 2 & 1 & 1 & \cdots \\ 0 & 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad D^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 6 & 3 & 1 & 0 & \cdots \\ 20 & 10 & 4 & 1 & 0 & \cdots \\ 70 & 35 & 15 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

The following is a result of the row sums of the matrices $D^{(2)}$ and $D^{(2)}$. It immediately follows from the fact that their A-sequences have the same generating function $A(x) = \frac{1}{1-x}$ [9, 15].

**Corollary 3.3.** Let $D^{(2)} = [r_{ij}]$ and $D^{(2)} = [q_{ij}]$ for $i$ and $j$ with $i, j = 0, 1, 2, \ldots$ Then for positive integers $i$ and $j$ with $i \geq j$, the following hold:

(a) $r_{i,j+1} = r_{i-1,j+1} + r_{i-1,j-2} + \cdots + r_{i-1,j-i}$.

(b) $q_{i,j+1} = q_{i-1,j+1} + q_{i-1,j-2} + \cdots + q_{i-1,j-i}$.

Let $e_i$ ($i = 0, 1, 2, \ldots$) denote the $i$th column of the identity matrix $I$, and let $e = e_0 + e_1 + e_2 + \cdots$. Then we directly obtain the following result from Corollary 3.3.
Corollary 3.4. Let $\text{F}^{(2)}$ and $L^{(2)}$ denote the Fibonacci and Lucas matrices of the second kind, respectively. Then the following hold:

(a) $D(\text{F}^{(2)})^{-1} \text{D} = [C_0, C_1, C_2, \ldots, C_n, \ldots]^T$.
(b) $D(L^{(2)})^{-1} \text{D} = [C_0, 3C_1, 5C_2, \ldots, (2n + 1)C_n, \ldots]^T$.

Proof. Let $D(\text{F}^{(2)})^{-1} \text{D} = [r_{ij}]$ and $D(L^{(2)})^{-1} \text{D} = [q_{ij}]$ for $i, j = 0, 1, \ldots$. For positive integers $i$ and $j$ with $i \geq j$, it follows from the case of $j = i$ in Corollary 3.3 that

$$
r_{ij} = r_{i-1, 0} + r_{i-1, 1} + \cdots + r_{i-1, i-1},
q_{ij} = q_{i-1, 0} + q_{i-1, 1} + \cdots + q_{i-1, i-1},
$$

which imply that $r_{ij}$ and $q_{ij}$ are the $(i - 1)$th row sums with $r_{i1} = \frac{1}{7}(2^{i-2})$ and $q_{i1} = \frac{1}{7}(2^i)$. Thus, for $i = 0, 1, 2, \ldots$, the $i$th row sums of $D(\text{F}^{(2)})^{-1} \text{D}$ resp. $D(L^{(2)})^{-1} \text{D}$ are $\frac{1}{7}(2^i)$ resp. $\frac{2^{i+1}+1}{7}(2^i)$, and (a) and (b) are proven. \(\blacksquare\)

We now present one of our main results as an answer to Q2. The results provide a combinatorial relationship between $\text{F}^{(2)}$ and $D(\text{F}^{(2)})^{-1} \text{D}$ and furthermore, between $\text{L}^{(2)}$ and $D(L^{(2)})^{-1} \text{D}$.

Theorem 3.5. Let $\text{F}^{(2)}$ and $L^{(2)}$ denote the Fibonacci and Lucas matrices of the second kind, respectively. Then the following hold:

(a) $D(\text{F}^{(2)})^{-1} \text{D}_{0} = [C_0, 0, 0, \ldots]^T$ if $j \geq 1$, then $D(\text{F}^{(2)})^{-1} \text{D}_{j} = [r_{0j}, r_{1j}, \ldots, r_{nj}, \ldots]^T$, where $r_{0j} = \binom{j-1}{0}C_{n-1} - \binom{j-2}{1}C_{n-2} + \cdots + \binom{-1}\binom{j-1}{\frac{j}{2}}C_{n-\frac{j}{2}}$ if $n \geq j$.
(b) $D(L^{(2)})^{-1} \text{D}_{0} = [C_0, 2C_1, 3C_2, \ldots]^T$ if $j \geq 1$, then $D(L^{(2)})^{-1} \text{D}_{j} = [q_{0j}, q_{1j}, \ldots, q_{nj}, \ldots]^T$, where for $n \geq 1$, $q_{n1} = (2n - 1)C_{n-1}$ and for $j \geq 2$ with $n \geq j$,

$$
q_{nj} = (n-1)C_{n-1} - \left[(n-2)\left(\begin{array}{c} j-2 \\ 1 \end{array}\right) + \left(\begin{array}{c} j-3 \\ 0 \end{array}\right)\right]C_{n-2} + \left[(n-3)\left(\begin{array}{c} j-3 \\ 2 \end{array}\right) + \left(\begin{array}{c} j-4 \\ 1 \end{array}\right)\right]C_{n-3} + \cdots
+ \left(-1\right)^{\frac{j-1}{2}}\left[(n-\frac{j}{2})\left(\begin{array}{c} \left\lfloor \frac{j-1}{2} \right\rfloor \\ \left\lfloor \frac{j-1}{2} \right\rfloor \end{array}\right) + \left(\begin{array}{c} \left\lfloor \frac{j-1}{2} \right\rfloor - 1 \\ \left\lfloor \frac{j-1}{2} \right\rfloor \end{array}\right)\right]C_{n-\frac{j}{2}}. \tag{3.4}
$$

Proof. (a) The proof proceeds by induction on $j$ with $j \geq 1$. For $j = 1$ and 2, we know that the generating function for the $j$th column of $D(\text{F}^{(2)})^{-1} \text{D}$ is $xC(x)$ if $j = 1$ and $x^2C(x)^2$ if $j = 2$ (see Lemma 3.1(a)). Since $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$, for $n \geq 0$ [1],

$$
x^2C(x)^2 = x^2 \sum_{n=0}^{\infty} (C_0C_n + C_1C_{n-1} + \cdots + C_nC_0)x^n = x^2 \sum_{n=0}^{\infty} C_{n+1}x^n = \sum_{n=2}^{\infty} C_{n-1}x^n.
$$

Thus we have $r_{n1} = C_{n-1}$ for $n \geq 1$ and $r_{n2} = C_{n-1}$ for $n \geq 2$, and we can commence the induction. Let $j \geq 3$. If $j$ is odd, then by the induction hypothesis, we obtain:

$$
r_{n-1,j-2} = \left(\begin{array}{c} j-3 \\ 0 \end{array}\right)C_{n-2} + \cdots + \left(-1\right)^{\frac{j-3}{2}}\left(\begin{array}{c} \frac{j-3}{2} \\ -1 \end{array}\right)C_{n-\frac{j-3}{2}} + \left(-1\right)^{\frac{j-3}{2}}\left(\begin{array}{c} \frac{j-3}{2} \\ -1 \end{array}\right)C_{n-\frac{j-3}{2}},
\tag{3.5}
$$

$$
r_{nj} = \left(\begin{array}{c} j-2 \\ 0 \end{array}\right)C_{n-1} - \left(\begin{array}{c} j-3 \\ 1 \end{array}\right)C_{n-2} + \cdots + \left(-1\right)^{\frac{j-3}{2}}\left(\begin{array}{c} \frac{j-3}{2} \\ -1 \end{array}\right)C_{n-\frac{j-3}{2}}.
$$

By (3.2) applied to (3.5),

$$
r_{nj} = \left(\begin{array}{c} j-1 \\ 0 \end{array}\right)C_{n-1} - \left(\begin{array}{c} j-2 \\ 1 \end{array}\right)C_{n-2} + \cdots + \left(-1\right)^{\frac{j-1}{2}}\left(\begin{array}{c} \frac{j-1}{2} \\ -1 \end{array}\right)C_{n-\frac{j-1}{2}},
$$

from which the result follows. If $j$ is even, the result can be proven similarly.

(b) For each positive integer $n \geq 1$, by Theorem 3.2 we obtain:

$$
q_{n1} = \frac{1}{2} \left(\begin{array}{c} 2n \\ n \end{array}\right) = \frac{2n(2n-1)}{n(n-1)!} = (2n-1)C_{n-1}.
$$

For each $j \geq 2$ with $n \geq j$, we prove the result by induction on $j$. By (3.3),

$$
q_{n2} = (2n-1)C_{n-1} - nC_{n-1} = (n-1)C_{n-1} \text{ and } q_{n3} = (n-1)C_{n-1} - (2n-3)C_{n-2},
$$

respectively.
which is the result (3.4) for \( j = 2 \) and 3 with \( n \geq j \). The induction commences: Assume that \( j \geq 4 \) is even. Then by the induction hypothesis,

\[
q_{n-1,j-2} = (n-2)C_{n-2} - \left[ (n-3)\left( \frac{j}{2} - 1 \right) + \frac{j-5}{1} \right] C_{n-3} + \left[ (n-4)\left( \frac{j}{2} - 1 \right) + \frac{j-6}{1} \right] C_{n-4} + \cdots
\]

\[
+ (-1)^{\frac{j-4}{4}} \left[ \left( \frac{n-j}{2} \right) \left( \frac{j-2}{2} - 1 \right) + \frac{j-4}{1} \right] C_{n-\frac{j}{2}},
\]

\[(3.6)\]

\[
q_{n,j-1} = (n-1)C_{n-1} - \left[ (n-2)\left( \frac{j}{2} - 1 \right) + \frac{j-4}{1} \right] C_{n-2} + \left[ (n-3)\left( \frac{j}{2} - 1 \right) + \frac{j-5}{1} \right] C_{n-3} + \cdots
\]

\[
+ (-1)^{\frac{j-2}{4}} \left[ \left( \frac{n-j}{2} \right) \left( \frac{j-2}{2} - 1 \right) + \frac{j-4}{1} \right] C_{n-\frac{j}{2}},
\]

\[(3.7)\]

Once more, by (3.3), as well as by (3.6) and (3.7), it follows that

\[
q_{nj} = (n-1)C_{n-1} - \left[ (n-2)\left( \frac{j}{2} - 1 \right) + \frac{j-3}{0} \right] C_{n-2} + \left[ (n-3)\left( \frac{j}{2} - 1 \right) + \frac{j-4}{0} \right] C_{n-3} + \cdots
\]

\[
+ (-1)^{\frac{j-2}{4}} \left[ \left( \frac{n-j}{2} \right) \left( \frac{j-2}{2} - 1 \right) + \frac{j-4}{0} \right] C_{n-\frac{j}{2}},
\]

which is the desired result (3.4). The case of odd \( j \) can be proven similarly. 

It follows from Theorem 3.5 that \( D(P^{(2)})^{-1}D \) and \( D(L^{(2)})^{-1}D \) can be expressed in terms of the Catalan numbers such that each entry of the matrices entails the Fibonacci or Lucas sequences:

\[
D(P^{(2)})^{-1}D = \begin{bmatrix}
C_0 & 0 & 0 & 0 & \cdots \\
0 & C_0 & 0 & 0 & \cdots \\
0 & C_1 & C_1 & 0 & \cdots \\
0 & C_2 & C_2 - C_1 & 0 & \cdots \\
0 & C_3 & C_3 - C_2 & C_3 - 2C_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
and
\]

\[
D(L^{(2)})^{-1}D = \begin{bmatrix}
C_0 & 0 & 0 & 0 & \cdots \\
2C_1 & C_0 & 0 & 0 & \cdots \\
3C_2 & 3C_1 & C_1 & 0 & \cdots \\
4C_3 & 5C_2 & 2C_2 & 2C_2 - 3C_1 & 0 & \cdots \\
5C_4 & 7C_3 & 3C_3 & 3C_3 - 5C_2 & 3C_3 - 7C_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

4. The role of \( D(P^{(2)})^{-1}D \) and \( D(L^{(2)})^{-1}D \) in \( P \)-invariant sequences

This section focuses on the role of \( D(P^{(2)})^{-1}D \) and \( D(L^{(2)})^{-1}D \) in transforming (inverse) \( P \)-invariant sequences. This allows us to examine combinatorial relationships between \( P^{(2)} \) and \( D(P^{(2)})^{-1}D \). We begin with a known result from Sun [16], which is needed in our subsequent discussion and can be proven readily by taking advantage of the Riordan matrix multiplication.

**Lemma 4.1.** Let \( P \) denote the Pascal matrix. Then \( g(x) \) is the generating function of a \( P \)-invariant or an inverse \( P \)-invariant sequence of the first kind if and only if \( \frac{1}{1-x} g \left( \frac{x^2}{1-x} \right) = \pm g(x) \).
Proof. Let \( g(x) = \sum_{n=0}^{\infty} a_n x^n \). Then \( g(x) \) is the generating function of a \( P \)-invariant or an inverse \( P \)-invariant sequence of the first kind if and only if \( \left( \frac{1}{1-x}, \frac{x}{1-x} \right) [a_0, a_1, a_2, \ldots]^T = \pm [a_0, a_1, a_2, \ldots]^T \), which stands for \( \frac{1}{1-x} \mathcal{G} \left( \frac{x}{1-x} \right) = \pm g(x) \) because \( PD = (\frac{1}{1-x}, \frac{x}{1-x}) \) where \( D = (1, -x) \). ■

The following result follows from Lemmas 2.3 and 4.1.

**Corollary 4.2.** Let \( P \) denote the Pascal matrix. Then \( g(x) \) is the generating function of a \( P \)-invariant sequence of the first kind if and only if \( \frac{1}{1-x} g(x) \) is the generating function of an inverse \( P \)-invariant sequence of the first kind.

Proof. Let \((l(x), h(x))\) denote a semi-Riordan matrix such that \( \mathcal{P}^{(1)} = (l(x), h(x)) \mathcal{L}^{(1)} \). Then by Lemma 2.3(c) and (d), \( h(x) \) satisfies \((1-x)h(x)^2 + x^2h(x) - x^3 = 0 \). So \( h(x) = \frac{-x^2 \pm \sqrt{x^4 + 4(1-x)^2}}{2(1-x)} \), which coincides with \( \frac{x}{1-x} \) or \( x \). Thus, we obtain \((l(x), h(x)) = \left( \frac{1}{1-x}(2-x), \frac{x}{1-x} \right) \) or \( \left( \frac{x}{1-x}, x \right) \). Let \( w \) be a \( P \)-invariant sequence of the first kind with its generating function \( g(x) \). Then \( w = \mathcal{L}^{(1)} u \) for some \( u \in \mathbb{R}^\infty \) and we have \((l(x), h(x)) w = \mathcal{P}^{(1)} u \). Thus, by Lemma 2.3(c), \((l(x), h(x)) w \) is an inverse \( P \)-invariant sequence of the first kind with its generating function \( \frac{x}{1-x} g(x) \) because \( \frac{x}{1-x} g(x) = \frac{x}{1-x} \mathcal{G} \left( \frac{x}{1-x} \right) = \frac{x}{2-x} g(x) \) by Lemma 4.1. The other direction can be proven similarly. ■

Next, we examine the role of \( D(\mathcal{P}^{(2)})^{-1} D \) and \( D(\mathcal{L}^{(2)})^{-1} D \) with the Catalan numbers for converting a \( P \)-invariant to an inverse \( P \)-invariant sequence of the first kind, and vice versa.

**Theorem 4.3.** For \( v \in \mathbb{R}^\infty \), let \( w = \mathcal{P}^{(1)} v \) and \( x = \mathcal{L}^{(1)} v \). Then the following hold:

- (a) \( \left( \frac{2-xC(x)}{xC(x)}, xC(x) \right) w = D(\mathcal{P}^{(2)})^{-1} Dx \),
- (b) \( \left( \frac{2-xC(x)}{xC(x)-2x^2C(x)^2}, xC(x) \right) w = D(\mathcal{L}^{(2)})^{-1} Dx \),
- (c) \( \left( \frac{xC(x)}{xC(x)-2x^2C(x)^2}, xC(x) \right) x = D(\mathcal{P}^{(2)})^{-1} Dw \),
- (d) \( \left( \frac{xC(x)}{(1-2xC(x))(2-xC(x))}, xC(x) \right) x = D(\mathcal{L}^{(2)})^{-1} Dw \),

where \( C(x) = \frac{1-\sqrt{1-4x}}{2x} \).

Proof. We already know that \( \mathcal{P}^{(1)} = \left( \frac{x}{2-x}, x \right) \mathcal{L}^{(1)} \) or \( \mathcal{P}^{(1)} = \left( \frac{x}{1-x(2-x)}, \frac{x}{1-x} \right) \mathcal{L}^{(1)} \) as in the proof of Corollary 4.2. Since

\[
\begin{pmatrix}
-1 + \frac{2}{xC(x)} \\
xC(x) - 1
\end{pmatrix} = \begin{pmatrix}
x \left( \frac{1}{1-x(2-x)} \right) \\
-x \left( \frac{1}{1-x} \right)
\end{pmatrix} = \begin{pmatrix} 2 & -1 \\ xC(x) - 1 & x \end{pmatrix} \left( \frac{x}{2-x} \right)
\]

clauses (a) and (b) directly follow from Lemma 3.1. Clauses (c) and (d) can be proven similarly. ■

For \( v \in \mathbb{R}^\infty \), let \( GF(v) \) denote the generating function of \( v \). In the following corollary, the Catalan and Motzkin numbers play a critical role in transforming the generating functions of \( P \)-invariant and inverse \( P \)-invariant sequences of the first kind.

**Corollary 4.4.** For \( v \in \mathbb{R}^\infty \), let \( g_v(x) \) resp. \( g_v(x) \) denote \( GF(\mathcal{P}^{(1)} v) \) resp. \( GF(\mathcal{L}^{(1)} v) \). Then we have the following:

- (a) \( g_v(xC(x)) = (C(x) - \frac{2}{x}) g_v(-xC(x)^2) \),
- (b) \( g_v(-xC(x)^2) = \frac{2-xC(x)}{xC(x)+1} g_v(xC(x)) \),
- (c) \( g_v(x + \frac{x}{xC(x)+1} M(\frac{x}{1-x})) = (\frac{x}{2-x} + \frac{1}{1-x} M(\frac{x}{1-x})) g_v(\frac{x}{xC(x)+1} M(\frac{x}{1-x})) \),
- (d) \( g_v(x C(x)+1) M(\frac{x}{1-x}) = (\frac{2-xC(x)}{xC(x)+1} g_v(x + \frac{x}{xC(x)+1} M(\frac{x}{1-x}))) \),

where \( C(x) = \frac{1-\sqrt{1-4x}}{2x} \) and \( M(x) = \frac{1-x}{1-x} \sqrt{1-x^2} \).

Proof. By Lemma 4.1 and Theorem 4.3(a) and (c), we have \( g_v(t) = \frac{2-t}{t} g_v(-t) = -\frac{2}{1-t} g_v(xC(x)^2) \) and \( g_v(t) = \frac{t}{2-t} g_v(t) = \frac{t}{2-t} g_v(\frac{x}{1-x}) \) where \( t = xC(x) \). Thus (a) and (b) follow from the known fact that \( C(x) = 1 + xC(x)^2 \) [14]. By using the Euler transformation in the proof of Lemma 3.1, we obtain

\[
C(x) = 1 + xC(x)^2 \frac{x}{1-x}.
\]

Applying (4.1) to (a) yields (c) and applying (4.1) and the known fact \( M(\frac{x}{1-x}) = 1 + xC(x)^2 \) [7] to (b) yields (d). ■
In the next theorem, we present one of our main results, namely an answer to Q2, featuring a combinatorial relationship between $\mathcal{P}(2)$ and $D(\mathbb{L}^{(2)})^{-1}D$ and furthermore, between $\mathbb{L}^{(2)}$ and $D(\mathbb{L}^{(2)})^{-1}D$. The result reveals that $D(\mathbb{P}(2))^{-1}D$ and $D(\mathbb{L}^{(2)})^{-1}D$ provide a mechanism (that features the Catalan numbers) for transforming a $P$-invariant into an inverse $P$-invariant sequence of the second kind, and vice versa.

**Theorem 4.5.** For $v \in \mathbb{R}^{\infty}$, let $y = \mathbb{P}(2)v$, $z = \mathbb{L}^{(2)}v$. Then the following hold:

(a) \( \left( \frac{1}{1+2xC(x)} , xC(x) \right) z = D(\mathbb{L}^{(2)})^{-1}Dy \),

(b) \( \left( \frac{1}{1-4xC(x)} , xC(x) \right) z = D(\mathbb{L}^{(2)})^{-1}Dy \),

(c) \((1 + 2xC(x), xC(x)) y = D(\mathbb{P}(2))^{-1}Dz\),

(d) \( (1 + 2xC(x), xC(x)) y = D(\mathbb{L}^{(2)})^{-1}Dz\),

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

**Proof.** (a) From Lemma 2.3(a) and (b), it can be derived easily that $\mathbb{L}^{(2)} = (1 + 2x, x) \mathbb{P}(2)$. Since \( \left( \frac{1}{1+2xC(x)} , xC(x) \right) (1+2x,x) = (1,xC(x)) \), (a) follows directly from Lemma 3.1(a). Clauses (b), (c), and (d) can be proven similarly. □

We conclude this section with the relationships (involving the Catalan and Motzkin numbers) between the generating functions of $P$-invariant and inverse $P$-invariant sequences of the second kind.

**Corollary 4.6.** For $v \in \mathbb{R}^{\infty}$, let $h_{\cdot}(x)$ resp. $h_{-\cdot}(x)$ denote $GF(\mathbb{P}(2)v)$ resp. $GF(\mathbb{L}^{(2)}v)$. Then we have the following:

(a) \( h_{\cdot}(xC(x)) = \frac{1}{1+2xC(x)} h_{-\cdot}(xC(x)) \),

(b) \( h_{-\cdot}(x + \frac{x^2}{1-x} M(\frac{x}{1-x}) ) = (1 + 2xC(x)) h_{\cdot}(x + \frac{x^2}{1-x} M(\frac{x}{1-x}) ) \),

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ and $M(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x}$.

**Proof.** Clause (a) follows from Lemma 3.1 and Theorem 4.5 and (b) is due to (a) and (4.1). □

5. A connection between Riordan involutions and invariant sequences

In this section, the Riordan (pseudo) involution $R$ is applied to construct $R$-invariant sequences of the first or second kind, as in Definition 2.2. From this, we show that for every Riordan involution $R$ in the $(-1)$-Appell subgroup, there exists $B_n (n = 1, 2, \ldots)$ in the Appell subgroup such that $R = B_n DB_n^{-1}$. Interestingly, for each $n = 1, 2, \ldots$, such a $B_n$ can be directly constructed from $R$ itself and has a related combinatorial significance for $R$-invariant or inverse $R$-invariant sequences of the first kind.

The following lemma provides a method for constructing $R$-invariant sequences of the first or second kind by means of the Riordan (pseudo) involution $R$ itself.

**Lemma 5.1.** Let $U = RD$ and $V = DR$, where $D = (1, -x)$. For each positive integer $n$, let $R$ be a Riordan (pseudo) involution. Then the following hold:

(a) The columns of $(U + D)^n ((R + D)^n)$ are $R$-invariant sequences of the first kind.

(b) The columns of $(U - D)^n ((R - D)^n)$ are inverse $R$-invariant sequences of the first kind.

(c) The columns of $(V^T + D)^n ((R^T + D)^n)$ are $R$-invariant sequences of the second kind.

(d) The columns of $(V^T - D)^n ((R^T - D)^n)$ are inverse $R$-invariant sequences of the second kind.

**Proof.** Let $U = RD$ and $V = DR$, where $R$ is a Riordan involution and $D = (1, -x)$. Then for each positive integer $n$, since $R$ is a Riordan involution, we obtain

\[ R(U + D)^n = (RRD + U)(U + D)^{n-1} = (U + D)^n \]

resp.

\[ R^T(V^T - D)^n = (R^T R D - V^T)(V^T - D)^{n-1} = -(V^T - D)^n, \]

which imply that each column of $(U + D)^n$ is an $R$-invariant sequence of the first kind resp. each column of $(V^T - D)^n$ is an inverse $R$-invariant sequence of the second kind. The two clauses (b) and (c) are also proven similarly. □

The following theorem is an affirmative answer to Q8 for every Riordan involution $R$ in the $(-1)$-Appell subgroup. In particular, for each positive integer $n = 1, 2, \ldots$, the matrix $B_n$, which is a Riordan matrix in the Appell subgroup and satisfies the condition $R = B_n DB_n^{-1}$, can be obtained directly from the Riordan involution $R$. 

---

Theorem 5.2. Let $R = (g(x), -x)$ be a Riordan involution such that the diagonal entries of $RD$ are positive, where $D = (1, -x)$. Then there exists a Riordan matrix $B_n = [(U + D)^n]_0, (U - D)^n_1, (U + D)^n_2, (U - D)^n_3, \ldots \} (n = 1, 2, \ldots)$ such that

$$B_n = ((g(x) + 1)(g(x) + g(-x))^{n-1}, x)$$

(5.1)

and

$$R = B_nDB_n^{-1},$$

(5.2)

where $U = RD$ and $(U + D)^n$ resp. $(U - D)^n$ are the $i$th resp. $j$th columns of $(U + D)^n$ resp. $(U - D)^n$ for nonnegative integers $i$ and $j$ with $i, j = 0, 1, 2, \ldots$

Proof. We prove (5.1) by induction on $n$ for $n \geq 1$. Since $(U + D)^{n-1}_{2i} = (g(x) + 1)x^{2i}$ and $(U - D)^{n-1}_{2i+1} = (g(x) + 1)x^{2i+1}$ for a nonnegative integer $i$, it immediately follows that $B_1 = (g(x) + 1, x).$ Let $n \geq 2$. Then for a nonnegative integer $i$, by the induction hypothesis and the fact that $g(x)g(-x) = 1$, we obtain

$$(U + D)^{n-1}_{2i} = (U + D)(U + D)^{n-1}_{2i-2} = ((g(x)x + (1, -x)(g(x) + 1)(g(x) + g(-x))^{n-2})x^{2i} = (g(x) + 1)(g(x) + g(-x))^{n-1}x^{2i},$$

since $g(x)^2 + g(x) + g(-x) + 1 = (g(x) + 1)(g(x) + g(-x)).$ Similarly, we can obtain $(U - D)^{n-1}_{2i+1} = (g(x) + 1)(g(x) + g(-x))^{n-1}x^{2i+1}$ for a nonnegative integer $i$, which implies the result (5.1). By using multiplication in the Riordan group, we obtain the following:

$$B_nDB_n^{-1} = \begin{pmatrix} (g(x) + 1)(g(x) + g(-x))^{n-1} \\ (g(-x) + 1)(g(x) + g(-x))^{n-1} \end{pmatrix}, \begin{pmatrix} -x \\ -x \end{pmatrix} = (g(x), -x),$$

which implies (5.2), and the proof is complete. ■

Finally, we claim an answer to Q8.1, namely, that for a Riordan involution $R$ in the $(-1)$-Appell subgroup, $B_n(n = 1, 2, \ldots)$ has a related combinatorial significance.

Theorem 5.3. Let $R = (g(x), -x)$ be a Riordan involution such that the diagonal entries of $RD$ are positive, where $D = (1, -x)$. Then there exists a Riordan matrix $B_n(n = 1, 2, \ldots)$ such that the columns of $B_n$ are $R$-invariant or inverse $R$-invariant sequences of the first kind.

Proof. It follows directly from Lemma 5.1 and Theorem 5.2. ■

Acknowledgments

The authors would like to express gratitude to the anonymous referees for their valuable comments and suggestions, which lead to significant improvements of the paper. The first author expresses his appreciation to the Department of Mathematics and Statistics at Washington State University for supporting his stay on his sabbatical leave.

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