This document illustrates the level of detail and explanation expected in good solutions to problems assigned in MATH 464. This one solution cannot cover all ground as you will be assigned various types of problems throughout the semester. The selected example problem is Exercise 1.12 from Bertsimas and Tsitsiklis.

Exercise 1.12 (Chebychev center) Consider a set P described by linear inequality constraints, that is, $P = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i,\ i = 1, 2, \ldots, m\}$. A ball with center y and radius r is defined as the set of all points within (Euclidean) distance r from y. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set P. (*The center of such a ball is called the Chebychev center of P.*) Provide a linear programming formulation of this problem.

Solution. Without loss of generality, we consider $\|a_i\| = 1,\ i = 1, 2, \ldots, m$. Next, we notice that $y \in P$ if, and only if, $a_i^T y \leq b_i,\ i = 1, 2, \ldots, m$. Suppose $y \in P$, then distance from y to the i^{th} constraint boundary is $b_i - a_i^T y$. As long as this distance is not greater than r for each constraint, the ball is in P. Thus, for the ball to be in P we have the set of constraints

$$a_i^T y + r \leq b_i, \quad i = 1, 2, \ldots, m.$$

The objective is to maximize r, so we have the linear program:

$$\boxed{\begin{array}{cl} \max\limits_{\substack{y \in \mathbb{R}^n \\ r \in \mathbb{R}}} & f(y, r) = r \\ s.t. & a_i^T y + r \leq b_i, \quad i = 1, 2, \ldots, m \end{array}}$$

This linear program is unbounded whenever P is not bounded. If P is bounded then there exists at least one optimal solution, say (y^*, r^*), where y^* is a Chebychev center of P with y^* a distance at least r^* from every constraint boundary. It is also possible for the optimal solution radius $r^* < 0$. In this case, the feasible region is empty and we expect y^* to be the center of the *smallest* ball that touches all constraint boundaries *but is not contained in P*. We could formulate the linear program:

$$\boxed{\begin{array}{cl} \max\limits_{\substack{y \in \mathbb{R}^n \\ r \in \mathbb{R}}} & f(y, r) = r \\ s.t. & a_i^T y + r \leq b_i, \quad i = 1, 2, \ldots, m \\ & r \geq 0 \end{array}}$$

which forces any optimal solution to describe a feasible ball. If this linear program has no solution, we conclude that $P = \emptyset$.