Math 464 QUIZ April 12 Name: ______________________________

Definition. A set $E \subset \mathbb{R}^n$ of the form

$$E = E(z, D) = \left\{ x \in \mathbb{R}^n \ : \ (x - z)^T D^{-1} (x - z) \leq 1 \right\},$$

where D is an $n \times n$ positive definite symmetric matrix, is called an ellipsoid with center $z \in \mathbb{R}^n$.

Task. Using the definition above, show that for any $y \in \mathbb{R}^n$, $y > 0$, and $\beta \in (0, 1)$, the set

$$S = \left\{ x \in \mathbb{R}^n \ : \ \sum_{k=1}^{n} \frac{(x_k - y_k)^2}{y_k^2} \leq \beta^2 \right\}$$

is an ellipsoid. First, state as clearly as possible *what* details you need to show.

Solution. We must show that set S can be written in the form of an ellipsoid given in the definition. In particular, we must find an $n \times n$ positive definite matrix D and vector $z \in \mathbb{R}^n$ such that

$$\sum_{k=1}^{n} \frac{(x_k - y_k)^2}{y_k^2} \leq \beta^2 \quad \text{is equivalent to} \quad (x - z)^T D^{-1} (x - z) \leq 1.$$

Because $0 < \beta < 1$,

$$\sum_{k=1}^{n} \frac{(x_k - y_k)^2}{\beta^2 y_k^2} \leq 1.$$

We then note that

$$\begin{aligned} \sum_{k=1}^{n} \frac{(x_k - y_k)^2}{\beta^2 y_k^2} &= [x_1 - y_1, \ \dots, x_n - y_n] \begin{bmatrix} \frac{1}{\beta^2 y_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\beta^2 y_n^2} \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix} \\ &= (x - z)^T D^{-1} (x - z), \end{aligned}$$

where $z = y$ and

$$D = \begin{bmatrix} \beta^2 y_1^2 & & 0 \\ & \ddots & \\ 0 & & \beta^2 y_n^2 \end{bmatrix}.$$

Note that D is an $n \times n$ diagonal matrix with positive diagonal entries and is therefore positive definite.