## General Form Linear Program

Note: Any linear program can be reformulated into an equivalent standard form linear program

$$\min_{x \in \mathbb{R}^n} \quad z = c^T x$$

$$\begin{aligned}
\text{s.t.} \quad & a_i^T x \geq b_i & i \in M_1 \\
& a_i^T x \leq b_i & i \in M_2 \\
& a_i^T x = b_i & i \in M_3 \\
& x_j \geq 0 & j \in N_1 \\
& x_j \leq 0 & j \in N_2 \\
& x_j \text{ urs} & j \notin N_1 \text{ and } j \notin N_2
\end{aligned}$$

## Inequality Form

$$\min_{x \in \mathbb{R}^n} \quad z = c^T x \quad \leftarrow \text{linear objective}$$

$$\text{s.t.} \quad Ax \geq b \quad \leftarrow \text{set of linear inequalities}$$

$$x \geq 0 \quad \leftarrow \text{sign restriction}$$

- Typically solved via interior point methods.
- Source of many geometric insights.

## Standard Form

$$\min_{x \in \mathbb{R}^n} \quad z = c^T x \quad \leftarrow \text{linear objective}$$

$$\text{s.t.} \quad Ax = b \quad \leftarrow \text{set of linear equalities}$$

$$x \geq 0 \quad \leftarrow \text{non-negative sign restriction}$$

- First solved by the Simplex Method of Dantzig (1947).
- Default choice of many software solvers even today.
- Useful also for insights inspired by linear algebra.

"Find the smallest value of the objective function over all non-negative solutions to the system of equations $Ax=b$."

## Some Notation

- Decision Variables $x_1, x_2, \cdots, x_n$ are collected in a column vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T,$$

as are other vectors in $\mathbb{R}^n$.

- Matrices and important pieces of them are denoted:

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}}_{\text{elements of } A} = \underbrace{\begin{bmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_n \\ | & | & & | \end{bmatrix}}_{\text{columns of } A} = \underbrace{\begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}}_{\text{rows of } A}$$

- For any vector $x \in \mathbb{R}^n$, $x \geq b$ means $x_k \geq b$ for each $1 \leq k \leq n$.
  $b \in \mathbb{R}$.

- For any two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means $x_k \geq y_k$ for each $1 \leq k \leq n$.

- Systems of linear equations (or inequalities) can also be written in matrix form. For example,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}\right\} \text{ is equivalent to } \left\{ Ax = b \right.$$

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Suppose we are given a set of vectors in $\mathbb{R}^n$ that form a basis, and let $y$ be an arbitrary vector in $\mathbb{R}^n$. Express $y$ as a linear combination of the basis vectors.

Let $\{v_1, v_2, \dots, v_n\}$ be the given basis. We seek scalars $c_1, c_2, \dots, c_n$ such that $y = \sum_{k=1}^{n} c_k v_k$.

If we let $c = [c_1\ c_2 \cdots c_n]^T$ and $V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

then $y = Vc$. Since a basis forms a linearly independent set we know that the columns of $V$ are linearly independent. By the invertible matrix theorem $V$ is invertible. Thus, $c = V^{-1}y$ are the unique scalars that give $y$ as a linear combination of the basis vectors.