Math 464 Spring 2018 Midterm Exam
(Take-Home Part)

Instructions: Answer all five questions. The list of allowed resources: the course text, your personal course notes, your homework. Work alone. Do not discuss this exam with anyone until solutions are posted. Unless other arrangements have been made, electronic solution submissions are due by 1PM Sunday March 11. Always show your work.

1. You own a company that assembles computers for retail stores. The production costs per computer during the next four months are \$250, \$400, \$200 and \$350, respectively. The company can make mo more than 85 computers in any month. Your customer requires 50, 65, 100 and 70 computers over the next four months. Your company can store completed computers from one month to the next at a cost of \$25 per computer, but no more than 20 computers can be stored at any time. You estimate that any computers left over after month 4 can be sold in the company showroom for \$500 each. Formulate an integer program that could be used to find the lowest-cost production strategy that meets consumer demand.

2. Solve the given linear program using the Simplex Method used in class. Show that $B = \{3, 4, 5\}$ is a feasible basis and use this as your initial basis.

$$\begin{aligned} \min_{x\in\mathbb{R}^4} \quad & z = x_1 - 2x_2 + x_3 - x_4 \\ \text{s.t.} \quad & 2x_1 - 3x_2 - x_3 + x_4 \le 0 \\ & -x_1 + 2x_2 + 2x_3 - 3x_4 \le 1 \\ & -x_1 + x_2 - 4x_3 + x_4 \le 8 \\ & x \ge 0 \end{aligned}$$

3. How might you use linear programming to solve the following *maximization* problem?

$$\begin{aligned} \max \quad & z = |2x_1 - 3x_2| \\ \text{s.t.} \quad & 4x_1 + x_2 \le 4 \\ & 2x_1 - x_2 \le 1/2 \\ & x \ge 0 \\ & x \in \mathbb{R}^2 \end{aligned}$$

4. Consider the linear program

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & z = d^T x \\ \text{s.t.} \quad & Ax = b \\ & Bx \ge c \end{aligned} \tag{1}$$

(a) Let the feasible region be denoted $\Omega = \{x \in \mathbb{R}^n \mid Ax = b, Bx \geq c\}$. Prove the following statement. If $\Omega \neq \emptyset$ and $\exists\ u \in \mathbb{R}^n$ for which $Au = 0$ and $Bu \geq 0$ then Ω is unbounded. [Hint: One way to show that a set is unbounded is to show that for every $K > 0$ there exist $x, y \in \Omega$ such that $\|x - y\| \geq K$.]

(b) Modify the statement in part (a) to be a test for the unboundedness of linear program (1). Justify. [Hint: One necessary condition for a linear program to be unbounded is that the feasible region be unbounded.]

5. Consider the optimization problem

$$\min_{x \in \Omega \subset \mathbb{R}^n} f(x), \tag{2}$$

where Ω is non-empty, bounded and not necessarily a polyhedron, and $f(x)$ is linear.

(a) Prove that no optimal point x^* can be an interior point of Ω. [Hint: Try a proof by contradiction.]

(b) Give and explain a scenario in which (2) has no optimal solution.

Defs. Consider $S \subset \mathbb{R}^n$. Element $x \in S$ is said to be an interior point of S if there exists $\theta > 0$ such that $x + \theta d \in S$ for every $d \in \mathbb{R}^n$, $\|d\| \leq 1$. The interior of S, denoted $\text{int}(S)$, is the set of all interior points of S.