Solve the following optimization problem using the Simplex Method.

\[
\begin{align*}
\text{max} & \quad x_1 - x_2 + x_3 \\
\text{s.t} & \quad x_1 + x_2 + x_3 \leq 2 \\
& \quad 2x_1 + 2x_2 - 6x_3 \geq 6 \\
& \quad x_1 - x_2 - x_3 \geq 0 \\
& \quad x \geq 0 \\
& \quad x \in \mathbb{R}^3
\end{align*}
\]

**Solution.** We begin by transforming the LP into standard form. We must include three positive slack variables, one for each inequality constraint.

\[
\begin{align*}
\text{max} & \quad z = x_1 - x_2 + x_3 \\
\text{s.t} & \quad x_1 + x_2 + x_3 + x_4 = 2 \\
& \quad 2x_1 + 2x_2 - 6x_3 - x_5 = 6 \\
& \quad -x_1 + x_2 + x_3 + x_6 = 0 \\
& \quad x \geq 0 \\
& \quad x \in \mathbb{R}^6
\end{align*}
\]

For this problem, there is no immediate initial basis. The basis \(\{4, 5, 6\}\) is an infeasible basis with basic variable values \(x_4 = 2, x_5 = -6, x_6 = 0\). So, we seek an initial basis using the Two-Phase Method. We first construct the auxiliary problem by adding one more positive slack variable \((x_7)\) and using an objective function \(w = -x_7\).

\[
\begin{align*}
\text{max} & \quad w = -x_7 \\
\text{s.t} & \quad x_1 + x_2 + x_3 + x_4 = 2 \\
& \quad 2x_1 + 2x_2 - 6x_3 - x_5 + x_7 = 6 \\
& \quad -x_1 + x_2 + x_3 + x_6 = 0 \\
& \quad x \geq 0 \\
& \quad x \in \mathbb{R}^7
\end{align*}
\]

An initial feasible basis for this problem is \(\{4, 7, 6\}\). We solve this problem. If \(x_7\) becomes non-basic at an optimal solution, then the optimal basis is a feasible basis for the original standard form problem. If \(x_7\) cannot be non-basic at the optimal solution, then the original problem is infeasible. We construct the initial tableau of the auxiliary problem.

\[
\begin{array}{cccccccc}
-\quad w & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-\quad w & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
x_4 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
x_7 & 6 & 2 & 2 & -6 & 0 & -1 & 0 & 1 \\
x_6 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\
\end{array}
\]
Notice that this tableau does not have three basic columns. This is because we did not write $w$ in terms of the non-basic variables. To fix this situation, add the $x_6$ row to the objective row to obtain:

$$\begin{array}{cccccc}
-w &=& x_1 & x_2 & x_3 & x_4 \\
6 &=& 2 & 2 & -6 & 0 \\
2 &=& 1 & 1 & 1 & 1 \\
6 &=& 2 & 2 & -6 & 0 \\
0 &=& -1 & 1 & 1 & 0 \\
\hline
0 &=& 0 & 0 & -2 & 0 \\
0 &=& 0 & 0 & 0 & 1 \\
0 &=& 0 & 0 & 0 & 0 \\
\end{array}$$

Next, we solve for the optimal solution to this auxiliary problem. We can enter $x_1$ and then $x_4$ must exit. The result of this pivot is:

$$\begin{array}{cccccc}
-w &=& x_1 & x_2 & x_3 & x_4 \\
2 &=& 0 & 0 & -4 & -2 \\
2 &=& 1 & 1 & 1 & 1 \\
2 &=& 0 & 0 & -6 & -2 \\
2 &=& 0 & 2 & 2 & 1 \\
\hline
2 &=& 2 & -2 & -1 & 0 \\
2 &=& 0 & 0 & 0 & 1 \\
2 &=& 2 & 0 & 1 & 0 \\
\end{array}$$

This tableau is optimal. Because $x_7^* = 2$, we see that no feasible solution exists for the original standard form constraints (for which $x_7 = 0$). Thus, the original problem is infeasible and no solution exists.